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ON THE CONVEXITY OF INTERACTING PARTICLE SYSTEMS IN THE PLANE

In this article we consider a system of n interacting points in $\mathbb{R}^2$ with dynamics defined by a linear term and bounded C^1 perturbations with bounded Jacobian matrix. We study the asymptotic behavior of the configuration after recentering and rescaling by the linear flow. Next, we present the results about the convergence of the rescaled system and discuss the long-term behavior of some geometric properties, e.g. the convexity of the polygon formed by the interacting points. We prove that for almost all initial conditions the time average of the convexity indicator of the evolving polygon equals the convexity indicator of the limiting configuration of the rescaled system.

1. Introduction

We study geometric properties of a system of interacting particles in the plane. Given a configuration of points $x_1, \dots, x_n \in \mathbb{R}^2$, one can associate various geometric properties, such as the area or perimeter of the convex hull, or the convexity of the polygon formed by the particles. In this article we focus on the convexity property and study its asymptotic behavior.

The dynamics are governed by the system of differential equations:

$$\frac{dx_i}{dt} = \frac{A}{n} \sum_{j=1}^{n} (x_i - x_j) + \phi_i(x_1, \dots, x_n),$$

where A is a diagonal 2×2 matrix with positive eigenvalues λ_1, λ_2, and $\phi_i : \mathbb{R}^{2n} \to \mathbb{R}^2$ is a bounded C^1 perturbation with bounded Jacobian matrix. Let $C \coloneqq \sup_{x \in \mathbb{R}^{2n}} \|\phi_i(x)\|$ and $L \coloneqq \sup_{x \in \mathbb{R}^{2n}} \|D\phi_i(x)\|$ denote these bounds. Also, let $\mu \coloneqq \min\{\lambda_1, \lambda_2\}$ denote the smallest eigenvalue of A and $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ denote the center of mass of the system.

Our main result establishes that the particles settle into a non-degenerate asymptotic shape in rescaled coordinates. Furthermore, we prove that the time average of the polygon's convexity is determined entirely by this limiting configuration.

Theorem 1.1. *The limits $k_i = \lim_{t\to\infty} e^{-At}(x_i - \bar{x})$ exist for all $i = \overline{1, \dots, n}$. Moreover, if $\mu > 4nL$, then for almost every initial condition $(x_1(0), \dots, x_n(0))$, the long-time average of the convexity indicator satisfies:*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}(x_1(t) \dots x_n(t) \text{ is convex})\, dt = \mathbb{1}(k_1 \dots k_n \text{ is convex}).$$

Before analyzing the asymptotic behavior, we present an example of a perturbation satisfying the assumptions above. For $x_i = (x_i^{(1)}, x_i^{(2)}) \in \mathbb{R}^2$, consider the following Kuramoto-type perturbations.

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Example 1.2. The Kuramoto-type perturbations [6] $\phi_i : \mathbb{R}^{2n} \to \mathbb{R}^2$ are defined as averages of pairwise interactions:

$$\phi_i(x_1, \ldots, x_n) = \frac{1}{n} \sum_{j=1}^{n} \begin{bmatrix} K_{ij} \sin\left(x_i^{(1)} - x_j^{(1)}\right) \\ L_{ij} \sin\left(x_i^{(2)} - x_j^{(2)}\right) \end{bmatrix}.$$

To ensure convergence of the rescaled dynamics, we require that the perturbations and their derivatives are bounded. For the Kuramoto-type perturbations, these bounds follow immediately from the boundedness of sin and cos.

Our example uses sin interaction terms just like the classical Kuramoto model, but it produces a completely different result. In the standard Kuramoto model, the particles move in a bounded space, so strong interactions force them to synchronize [6]. In our model, the linear flow pushes the particles apart in the unbounded space $\mathbb{R}^2$. This expansion overpowers the interactions and prevents clustering. Because the particles do not synchronize, we focus on other geometric properties, such as the convexity of the polygon $x_1(t) \ldots x_n(t)$.

2. Asymptotic Behavior and Convergence

In this section, we establish the existence of the limiting configuration $k_1, \ldots, k_n$ and prove that the time average of the convexity indicator converges if $k_1, \ldots, k_n$ is non-degenerate as stated in Theorem 1.1.

To analyze the relative geometry of the particles as they move apart, we recenter the system and rescale by the linear flow. Let $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ be the center of mass. The evolution of each particle is governed by:

$$\frac{dx_i}{dt} = \frac{A}{n} \sum_{j=1}^{n} (x_i - x_j) + \phi_i = A(x_i - \bar{x}) + \phi_i.$$

Summing the equations over $i = \overline{1, \ldots, n}$ yields $\frac{d\bar{x}}{dt} = \bar{\phi}$, where $\bar{\phi} = \frac{1}{n} \sum_{i=1}^{n} \phi_i$. We define the rescaled variables $u_i = e^{-At}(x_i - \bar{x})$. Differentiating with respect to time yields:

$$\begin{aligned} \frac{du_i}{dt} &= -Ae^{-At}(x_i - \bar{x}) + e^{-At} \frac{d}{dt}(x_i - \bar{x}) \\ &= -Ae^{-At}(x_i - \bar{x}) + e^{-At} \left(A(x_i - \bar{x}) + \phi_i - \bar{\phi} \right) \\ &= e^{-At} \left(\phi_i - \bar{\phi} \right). \end{aligned}$$

The following lemmas demonstrate that the trajectories $u_i(t)$ have finite total variation, which implies their convergence.

Lemma 2.1. *The derivative of the rescaled variable is absolutely integrable:*

$$\int_0^\infty \left\| \frac{du_i(t)}{dt} \right\| dt \le \frac{2CK}{\mu} < \infty,$$

Proof. Since $C \coloneqq \sup_{x \in \mathbb{R}^{2n}} \|\phi_i(x)\| < \infty$, we have $\left\|\phi_i - \bar{\phi}\right\| \le 2C$. Using the exponential bound $\left\|e^{-At}\right\| \le Ke^{-\mu t}$ [5, p.29] for some $K > 0$, we obtain:

$$\left\| \frac{du_i}{dt} \right\| = \left\| e^{-At} \left(\phi_i - \bar{\phi} \right) \right\| \le \left\| e^{-At} \right\| \left\| \phi_i - \bar{\phi} \right\| \le 2CKe^{-\mu t}.$$

Integrating both sides from 0 to ∞ yields the desired bound. □

Lemma 2.2. *The limit $k_i \coloneqq \lim_{t\to\infty} u_i(t)$ exists.*

Proof. Let $\varepsilon > 0$. By Lemma 2.1, there exists $T \geq 0$ such that for all $t \geq T$,

$$\int_T^\infty \left\| \frac{du_i(t)}{dt} \right\| dt < \varepsilon.$$

Take arbitrary $t'' > t' \geq T$. Then

$$\begin{aligned} u_i(t'') - u_i(t') &= \int_{t'}^{t''} \frac{du_i(t)}{dt} dt \\ \|u_i(t'') - u_i(t')\| &\leq \int_{t'}^{t''} \left\| \frac{du_i(t)}{dt} \right\| dt \leq \int_T^\infty \left\| \frac{du_i(t)}{dt} \right\| dt < \varepsilon. \end{aligned}$$

It remains to prove the convergence of $u_i(t)$, which can be done in a standard way. □

We now relate the convexity of the evolving polygon $x_1 \cdots x_n$ to that of the limiting configuration $k_1 \cdots k_n$. For any fixed t, the map $x_i \mapsto u_i$ is an affine transformation (a translation followed by a linear scaling). Since convexity is invariant under affine transformations, it follows that

$$\mathbb{1}(x_1 \cdots x_n \text{ is convex}) = \mathbb{1}(u_1 \cdots u_n \text{ is convex}).$$

Definition 2.3. A configuration $v_1, \ldots, v_n$ is called *degenerate* if there exist three distinct vertices that are collinear.

A configuration that is not degenerate is called *non-degenerate*.

The set of convex non-degenerate configurations is an open subset of $\mathbb{R}^{2n}$. If the limiting configuration $k_1, \ldots, k_n$ is convex and non-degenerate, then there exists a time T_0 such that for all $t > T_0$, the configuration $u_1(t), \ldots, u_n(t)$ is also convex.

Under this non-degeneracy condition, we have:

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}(x_1 x_2 \cdots x_n \text{ is convex})\, dt \\ =&\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}(u_1 u_2 \cdots u_n \text{ is convex})\, dt \\ =&\,\mathbb{1}(k_1 k_2 \cdots k_n \text{ is convex}). \end{aligned}$$

3. The Measure of Degenerate Limit Configurations

As established in the previous section, the convergence of the time average indicator depends on the non-degeneracy of the limiting configuration $k = (k_1, \ldots, k_n)$. In this section, we prove that the set of initial conditions leading to such degenerate states is negligible.

Let $\Phi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote the flow in the rescaled variables $u(t) = (u_1(t), \ldots, u_n(t))$, such that $\Phi_t(u_0) = u(t)$ for initial data $u_0 \in \mathbb{R}^{2n}$. From Lemma 2.2, we define the limit map:

$$\Phi(u_0) \coloneqq \lim_{t\to\infty} \Phi_t(u_0) = (k_1, \ldots, k_n).$$

Let $\mathcal{D} \subset \mathbb{R}^{2n}$ denote the set of degenerate polygons. A configuration k belongs to $\mathcal{D}$ if and only if there exists at least one triple of indices (i, j, ℓ) such that the vertices k_i, k_j, k_ℓ are collinear. This condition is equivalent to the vanishing of the determinant:

$$f_{i,j,\ell}(k) \coloneqq \det(k_i - k_j, k_\ell - k_j) = 0.$$

Since $f_{i,j,\ell}$ is a polynomial, $\mathcal{D}$ is a finite union of algebraic sets [2]. Since any proper algebraic subset of $\mathbb{R}^m$ has m-dimensional Lebesgue measure zero [3], it follows that $\mu_{2n}(\mathcal{D}) = 0$. Our goal is to show that $\mu_{2n}(\Phi^{-1}(\mathcal{D})) = 0$. Suppose for the sake of contradiction that $\mu_{2n}(\Phi^{-1}(\mathcal{D})) > 0$. By the regularity of the Lebesgue measure, there exists a compact subset $B \subset \Phi^{-1}(\mathcal{D})$ with $\mu_{2n}(B) > 0$.

To control the measure of the inverse image, we analyze the Jacobian matrix of the flow:

Theorem 3.1. *For any initial condition $u_0 \in \mathbb{R}^{2n}$, the Jacobian of the flow satisfies*

$$\det(D\Phi_t(u_0)) \geq e^{-4nLt} > 0$$

for all $t \geq 0$.

Proof. Let $J(t) = D\Phi_t(u_0)$. The evolution of the rescaled variables is given by $\frac{du}{dt} = F(u,t)$, where $F_i(u,t) = e^{-At}(\phi_i(e^{At}u) - \bar{\phi}(e^{At}u))$. Differentiating the flow equation with respect to u_0 yields the variational equation:

$$\frac{d}{dt}J(t) = \mathcal{A}(t)J(t), \quad \text{where } \mathcal{A}(t) = D_u F(\Phi_t(u_0), t).$$

By Liouville's formula [4, p.134], the Jacobian is:

$$\det(J(t)) = \exp\left(\int_0^t \operatorname{tr}(\mathcal{A}(s))\ ds\right).$$

Although $\mathcal{A}(t)$ is a full $2n \times 2n$ matrix, its trace is simply the sum of the traces of its 2×2 diagonal blocks. By the chain rule, these diagonal blocks are given by:

$$\frac{\partial F_i}{\partial u_i} = e^{-At} D_{x_i}\phi_i(e^{At}u)e^{At} - e^{-At} D_{x_i}\bar{\phi}(e^{At}u)e^{At},$$

where D_{x_i} denotes the 2×2 partial Jacobian matrix with respect to the i-th coordinate pair in $\mathbb{R}^2$. By the cyclic property of the trace, $\operatorname{tr}\big(e^{-At}Me^{At}\big) = \operatorname{tr}\big(Me^{At}e^{-At}\big) = \operatorname{tr}(M)$. Thus, the trace of each diagonal block simplifies to:

$$\operatorname{tr}\left(\frac{\partial F_i}{\partial u_i}\right) = \operatorname{tr}(D_{x_i}\phi_i) - \operatorname{tr}\big(D_{x_i}\bar{\phi}\big).$$

Summing over all $i = 1, \ldots, n$, the total trace of $\mathcal{A}(t)$ becomes:

$$\operatorname{tr}(\mathcal{A}(t)) = \sum_{i=1}^{n} \operatorname{tr}(D_{x_i}\phi_i) - \sum_{i=1}^{n} \operatorname{tr}\big(D_{x_i}\bar{\phi}\big).$$

By definition, $\bar{\phi} = \frac{1}{n}\sum_{j=1}^{n}\phi_j$, which implies that $\sum_{i=1}^{n} D_{x_i}\bar{\phi} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} D_{x_i}\phi_j$. Since $L := \sup \|D\phi_i(x)\| < \infty$, the operator norm of any individual 2×2 partial Jacobian block is bounded by L. Using the fact that the trace of a 2×2 matrix is bounded by twice its operator norm, we obtain:

$$|\operatorname{tr}(\mathcal{A}(t))| \leq \sum_{i=1}^{n} |\operatorname{tr}(D_{x_i}\phi_i)| + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} |\operatorname{tr}(D_{x_i}\phi_j)| \leq 2nL + \frac{1}{n}(n^2 \cdot 2L) = 4nL.$$

It follows that $\operatorname{tr}(\mathcal{A}(t)) \geq -4nL$. Substituting this uniform lower bound into Liouville's formula yields:

$$\det(D\Phi_t(u_0)) \geq \exp\left(\int_0^t -4nL\, ds\right) = e^{-4nLt}.$$

□

To bound the measure of B, we establish three supporting lemmas regarding the volume evolution, the convergence rate of the trajectories, and the geometry of the degenerate set.

Lemma 3.2. $\mu_{2n}(B) \leq e^{4nLt}\mu_{2n}(\Phi_t(B))$.

Proof. By the theorem on the smoothness of the flow [1, p.281], Φ_t is a C^1-diffeomorphism. Using the change of variables formula and the lower bound on the Jacobian from Theorem 3.1, we have:

$$\mu_{2n}(\Phi_t(B)) = \int_B |\det(D\Phi_t(x))| \, d\mu_{2n} \geq e^{-4nLt} \int_B 1 \, d\mu_{2n} = e^{-4nLt} \mu_{2n}(B).$$

□

Lemma 3.3. *There exists a constant $R > 0$ such that for all $t \geq 0$, the set $\Phi_t(B)$ is contained in a shrinking neighborhood of $\mathcal{D}$:*

$$\Phi_t(B) \subset T(\mathcal{D}, \delta_t),$$

where $\delta_t = Re^{-\mu t}$ and $T(\mathcal{D}, \delta)$ denotes the δ-neighborhood of $\mathcal{D}$.

Proof. By Lemma 2.2, the trajectories $u(t)$ converge to their limits exponentially at the rate μ, driven by the linear expansion of the system. Since B is compact, we can choose a global constant $R > 0$ such that $\|\Phi_t(u_0) - \Phi(u_0)\| \leq Re^{-\mu t}$ holds uniformly for all $u_0 \in B$. Since $\Phi(u_0) \in \mathcal{D}$ for all initial conditions in B, the distance from any point in $\Phi_t(B)$ to $\mathcal{D}$ is bounded by $Re^{-\mu t}$. □

Lemma 3.4. *Let $K \subset \mathbb{R}^{2n}$ be compact. Then there exists a constant $M_K > 0$ such that for all sufficiently small $\delta > 0$,*

$$\mu_{2n}\big(T(\mathcal{D}, \delta) \cap K\big) \leq M_K \delta.$$

Proof. Since $\mathcal{D}$ is defined by the vanishing of the non-trivial polynomial $\det(k_i - k_j, k_\ell - k_j) = 0$, it is a real algebraic set [2]. By the standard result on the volumes of tubular neighborhoods of algebraic varieties [7], the volume of a δ-tube around a *compact* subset of such a variety scales proportionally to δ^c, where c is the codimension. Applied to the compact set $\mathcal{D} \cap K$, this yields

$$\mu_{2n}\big(T(\mathcal{D}, \delta) \cap K\big) \leq M_K \delta$$

for all sufficiently small $\delta > 0$, since $\operatorname{codim}(\mathcal{D}) \geq 1$. □

Because $\Phi_t(B)$ converges uniformly to $\Phi(B) \subset \mathcal{D}$ as $t \to \infty$, the trajectories eventually enter and remain in any neighborhood of $\Phi(B)$. Let $K \subset \mathbb{R}^{2n}$ be a fixed compact neighborhood of $\Phi(B)$ (for instance, a closed ε-neighborhood). Then, for all sufficiently large t, we have $\Phi_t(B) \subset K$.

From the bounds derived in lemmas 3.2, 3.3, and 3.4, we can directly constrain the initial measure of B. By Lemma 3.2, we have:

$$\mu_{2n}(B) \leq e^{4nLt} \mu_{2n}(\Phi_t(B)).$$

By Lemma 3.3, the set $\Phi_t(B)$ is entirely contained within $T(\mathcal{D}, Re^{-\mu t})$. Since $\Phi_t(B) \subset K$, we have

$$\Phi_t(B) \subset T(\mathcal{D}, Re^{-\mu t}) \cap K.$$

Applying Lemma 3.4 gives, for all sufficiently large t,

$$\mu_{2n}(\Phi_t(B)) \leq \mu_{2n}\big(T(\mathcal{D}, Re^{-\mu t}) \cap K\big) \leq M_K Re^{-\mu t}.$$

Substituting this estimate into the previous inequality yields

$$\mu_{2n}(B) \leq e^{4nLt}(M_K Re^{-\mu t}) = M_K Re^{(4nL - \mu)t}.$$

Recall that the core assumption of Theorem 1.1 is that $\mu > 4nL$, hence the exponent $(4nL - \mu)$ is strictly negative. It follows from the inequality above that by taking $t \to \infty$, $\mu_{2n}(B)$ must be less than any positive number, which contradicts our assumption that $\mu_{2n}(B) > 0$. Thus, the set of initial conditions leading to a degenerate limit configuration has measure zero.

4. Lower bound on T_0

As established in the previous section, the condition for a configuration to be degenerate is equivalent to the vanishing of the determinant for some three points i, j, and ℓ:

$$f_{i,j,\ell}(k) := \det(k_i - k_j, k_\ell - k_j) = 0.$$

Assume the limiting configuration $k = \Phi(u_0)$ is non-degenerate. We define its margin of convexity as $\nabla := \min_{i,j,\ell} |f_{i,j,\ell}(k)| > 0$. We want to determine the time T_0 such that for all $t > T_0$, the polygon is convex. Recall that the convexity of the rescaled system is equivalent to the convexity of the original one, hence we wish to guarantee the convexity of $u(t) = \Phi_t(u_0)$.

Since $f_{i,j,\ell}$ is polynomial, it is C^1. Thus, for any $a, b \in \mathbb{R}^{2n}$, the mean value theorem gives

$$f_{i,j,\ell}(a) - f_{i,j,\ell}(b) = \nabla f_{i,j,\ell}(\xi) \cdot (a - b)$$

for some point ξ on the segment joining a and b. Hence,

$$|f_{i,j,\ell}(a) - f_{i,j,\ell}(b)| \le \|\nabla f_{i,j,\ell}(\xi)\| \, \|a - b\| .$$

By Lemma 3.3, there exists $R > 0$ such that

$$\|\Phi_t(u_0) - \Phi(u_0)\| \le Re^{-\mu t}$$

for all u_0 in a fixed compact set B. Since $\Phi_t(B)$ converges uniformly to $\Phi(B) \subset \mathcal{D}$, there exists a compact neighborhood K of $\Phi(B)$ such that, for all sufficiently large t, the segment joining $u(t) = \Phi_t(u_0)$ to $k = \Phi(u_0)$ lies in K. Therefore,

$$|f_{i,j,\ell}(u(t)) - f_{i,j,\ell}(k)| \le M \, \|u(t) - k\| \le MRe^{-\mu t},$$

where

$$M := \sup_{x \in K} \|\nabla f_{i,j,\ell}(x)\| .$$

To preserve non-degeneracy, it is enough that this error be smaller than the convexity margin

$$\nabla := \min_{i,j,\ell} |f_{i,j,\ell}(k)| > 0.$$

Thus it suffices to choose T_0 such that

$$MRe^{-\mu T_0} < \nabla,$$

that is,

$$T_0 > \frac{1}{\mu} \ln\left(\frac{MR}{\nabla}\right).$$

For every $t > T_0$, all determinants $f_{i,j,\ell}(u(t))$ remain nonzero, so the configuration stays non-degenerate. Since convexity is equivalent for the rescaled and original systems, this proves that the polygon is convex for all $t > T_0$. A similar bound can be derived for the time after which the configuration remains non-convex.

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References

1. Vladimir I. Arnold, *Ordinary differential equations*, 3rd ed., Springer-Verlag, 1992.
2. Jacek Bochnak, Michel Coste, and Marie-Françoise Roy, *Real algebraic geometry*, Springer, 1998. https://doi.org/10.1007/978-3-662-03718-8
3. Richard Caron, *The zero set of a polynomial*, (2005). https://doi.org/10.13140/RG.2.1.4432.8169
4. Carmen Chicone, *Ordinary differential equations with applications*, 2 ed., Springer-Verlag, New York, 2006.
5. Lars Grüne, *Mathematical control theory*, https://num.math.uni-bayreuth.de/de/team/lars-gruene/skripten/kontrolltheorie/kt_2023_en.pdf, 2023, Lecture Notes.
6. Yoshiki Kuramoto, *Self-entrainment of a population of coupled non-linear oscillators*, International Symposium on Mathematical Problems in Theoretical Physics, Springer, 1975, pp. 420–422.

7. Martin Lotz, *On the volume of tubular neighborhoods of real algebraic varieties*, 2013. https://arxiv.org/abs/1210.3742

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