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THE INTERSECTIONS OF IMAGES OF TRAJECTORIES FOR INDEPENDENT BROWNIAN MOTIONS ON CARNOT GROUPS

In this paper we present the conditions for the existence of intersections for functions of several independent Brownian motions on Carnot groups. These conditions are geometrical and local in nature, in the sense that they give a simple relation between the local geometry of a manifold, related to intersections, at a given point, with the structure of Lie algebra of left-invariant vector fields in a neighbourhood of the same point. We prove that these conditions are necessary and sufficient for the existence of intersections, if we exlcude from the consideration the points of a nowhere dense set on the manifold, related to intersections.

1. Introduction

Suppose that $X_1, X_2, \dots, X_n$ are independent stochastic processes with values in $\mathbb{R}^d$. It is a well-known problem to find conditions that are necessary and sufficient for the existence of intersections of these processes, meaning the existence of such $t_1, t_2, \dots, t_n$ that $X_1(t_1) = X_2(t_2) = \dots = X_n(t_n)$. This problem is also closely related to the problem of self-intersections of the process X: find conditions that are necessary and sufficient for the existence of such different $t_1, t_2, \dots, t_n$ that $X(t_1) = X(t_2) = \dots = X(t_n)$.

The first result of this type was proven in [7]: if $n = 2$ and X is a standard Brownian motion then the self-intersections of X exist with probability 1 if $d = 1, 2, 3$ and do not exist with probability 1 if $d \geqslant 4$. Also in [6] it was shown that if $d = 2$ and all X_i are standard Brownian motions, then self-intersections exist with probability 1. In [4] for a class of 3-dimensional hypoelliptic processes it was shown that self-intersections do not exist with probability 1. A sufficient condition for a large class of Levy processes to have self-intersections was proposed in [8] and in [11]. Later in [9] it was proven that the condition from [8] is also necessary for the existence of self-intersections. All the results above can also be formulated for intersections of independent processes. In fact the case of intersections of independent processes is often used as a part of the proof for the case of self-intersections (for example, it was done in [4]).

In this paper we consider the following generalization of this problem: instead of intersections of given independent processes, we consider intersections of images of these processes under some infinitely differentiable functions $f_1, f_2, \dots, f_n$. In other words we study conditions related to the existence of such $t_1, t_2, \dots, t_n$ that $f_1(X_1(t_1)) = f_2(X_2(t_2)) = \dots = f_n(X_n(t_n))$. Moreover we are also considering the case when the last condition is replaced by $F(X_1(t_1), X_2(t_2), \dots, X_n(t_n)) = 0$ for some infinitely differentiable function F. However in this latter case the corresponding conditions, proposed in this paper, are more complicated and harder to check.

We restrict our investigation to the processes, which belong to a class of diffusions called Brownian motions in Lie group (see [10]) in the case when the corresponding Lie

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group is stratified. Such processes, which we call Brownian motions on Carnot groups, were considered previously in [14, 16]. The significance of such processes was discussed in [1]. Also it is known that the differential operators related to these Markov processes can be used as a local approximation for differential operators related to an arbitrary smooth diffusion, as shown in [13].

The main idea behind the proofs is to use the results of [14] and [16]. In [14] the conditions were proposed for the existence of "weighted" local times related to intersections and self-intersections. It is not hard to check that the existence of non-zero local time related to intersections or self-intersections as a limit of certain type of approximations provides the existence of such intersections or self-intersections. Therefore it is not surprising that our main conditions for existence of intersections and self-intersections are derived from those proposed in [14] for the existence of the corresponding local times. At the same time in [16] another condition was proposed for the non-existence of visits to a given set by a joint trajectory of several independent Brownian motions on Carnot groups. And the key result of our paper, the estimate proven in Theorem 4, allows us to link the proposed geometric conditions, appearing originally in [14], with the asymptotic behaviour of surface measure on H needed to satisfy the condition in [16]. In this way we can prove that our conditions are both necessary and sufficient if we exclude from the consideration the points of some subset of H. The exclusion of some points is required because, unfortunately, we could not completely solve the problems related to "singular" points in the set $H = \{x|F(x) = 0\}$. As it turns out the relation of the Carnot group structure with the structure of the manifold H at a given point can be non-trivial, and in order to prove our results we have to restrict ourselves to such points where this relation is "typical", meaning it is in some sense the same as for all points in a small neighbourhood of this point in H (see Definition 1).

In Section 2 we formulate our main result. In Section 3 we consider the relation of the conditions for general F with the conditions for the case of intersections of $f_1(X_1(t_1)), f_2(X_2(t_2)), \ldots, f_n(X_n(t_n))$. In Section 4 we prove Theorem 4 needed to prove the non-existence of intersections using the results from [16]. In Section 5 we show that it is impossible to have a finite but nonempty set of intersection points with positive probability. This is also needed to apply the results of [16]. In Section 6 we prove our main result. In Section 7 we consider the case of self-intersections. In Section 8 we provide several examples of specific processes with or without the existence of intersections or self-intersections.

2. The main result

We fix $n \geqslant 2$ and for each $i = 1, \ldots, n$ consider Carnot groups G_i, which are also Lie groups with the common base space $\mathbb{R}^d$. On each group G_i we define the corresponding basis in the Lie algebra of left-invariant vector fields: L_{ij}, $j = 1, \ldots, d$, with the corresponding homogeneous degrees $p_i : \{1, \ldots, d\} \to \mathbb{N}$, $i = 1, \ldots, n$.

To clarify the meaning of the above notations we recall the definition of Carnot group G_i given in [3]. This definition consists of the following statements, which should hold for any fixed $i = 1, \ldots, n$:

1. Suppose that $(x, y) \mapsto x \bullet y$ is a group operation on G_i, $(x)_j$ is a coordinate j of $x \in G_i$ in a given fixed Euclidean coordinate system on $G_i = \mathbb{R}^d$ and

$$(\delta_\lambda(x))_j = \lambda^{p_i(j)}(x)_j, \lambda > 0, x \in G_i.$$

Then for all $\lambda > 0$, $x \in G_i$, $y \in G_i$:

$$\delta_\lambda(x \bullet y) = \delta_\lambda(x) \bullet \delta_\lambda(y).$$

2. Suppose that the set of vector fields L_{ij} is defined as follows:

$$L_{ij}(x) = \frac{\partial}{\partial (y)_j}(x \bullet y)|_{y=0}, x \in G_i, y \in G_i, j = 1, \dots, d. \tag{1}$$

Then there exists a positive integer l, such that for all $x \in G_i$

$$\dim(l.sp.(\bigcup_{k=1}^{l} S_k(x)) = d,$$

where $S_1 = \{L_{ij} | p_i(j) = 1, j = 1, \dots, d\}$, $S_{k+1} = \{[A, B] = AB - BA | A \in S_1, B \in S_k\}$, $k = 1, 2, \dots$, and $S_k(x)$ is a set of vectors of the form $A(x)$ (vector field A evaluated at x) where $A \in S_k$.

Similarly to the definitions in [16], here we also assume that all families of degrees p_i are arranged in the order of their values, meaning that $p_i(j) \leqslant p_i(k), j < k$. We also always assume that L_{ij} are defined exactly as shown above on the corresponding G_i. Additionally, when dealing with only one Carnot group G (either G_i or the product $G_1 \times \ldots \times G_n$), the notation $x \bullet y$ will be used to define the group addition when needed, and the notation x^{-1} will be used to define the group inverse of $x \in G$. We also denote as Q_i the homogeneous dimension of G_i: $Q_i = \sum\limits_{j=1}^{d} p_i(j)$ and denote as ρ_i the Carnot-Caratheodory distance on G_i. For more detailed information about the definition of Carnot group and related objects see [16] and [3].

For any differentiable function $g : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ we write its derivative in the form of $m_2 \times m_1$ matrix g' as follows

$$(g')_{ij}(x) = \frac{\partial g_i}{\partial x_j}(x), i = 1, \dots, m_2, j = 1, \dots, m_1.$$

Consider an infinitely differentiable function $F : G_1 \times \ldots \times G_n \to \mathbb{R}^M$, $M \leqslant nd$, such that $rank\, F'(x) = M$ for all $x \in G_1 \times \ldots \times G_n$. Let us first define points that are in some sense "non-singular" with regard to the interaction between the level sets of F and the structure of Carnot groups $G_1, \dots, G_n$.

We will use the notation x_i to denote the canonical projection of $x \in G_1 \times \ldots \times G_n$ on G_i, and write an element $x \in G$ as $x = (x_1, x_2, \dots, x_n)$ in the form of a vector consisting of its corresponding projections. In the following we use the notation $L_{ij}^{x_i}$ to indicate the action of a linear operator defined by the vector field L_{ij} on the variable x_i:

$$L_{ij}^{x_i} F(x) = \sum_{s=1}^{d} (L_{ij}(x_i))_s \frac{\partial}{\partial (x_i)_s} F(x), i = 1, \dots, n, j = 1, \dots, d$$

where the notation $(\cdot)_s$ is used to denote the coordinate number s of a vector.

Definition 1. We denote as $NS(F)$ a subset of $G_1 \times \ldots \times G_n$ such that $x \in NS(F)$ if and only if for any non-negative integers $s_1, \dots, s_n$ the set of vectors

$$\{L_{ij}^{y_i} F(y) | i = 1, \dots, n, j = 1, \dots, d : p_i(j) \leqslant s_i\}$$

defined for each $y = (y_1, \dots, y_n) \in G_1 \times \ldots \times G_n$, has the same rank (the dimension of its linear span) for all y in some neighbourhood of x with $F(y) = F(x)$.

It is easy to check that the condition for $x \in NS(F)$ holds if any family of vectors $L_{ij}^{y_i} F(y)$ is either linearly independent at $y = x$ or linearly dependent for all y in some neighbourhood of x with $F(y) = F(x)$ (but the reverse may not be true).

Denote $H = \{x \in G_1 \times \ldots \times G_n | F(x) = 0\}$. In the following we are going to focus on the points in $NS(F) \cap H$. The following result shows that such restriction does not take away too much, since the points that are being avoided belong to a nowhere dense closed set in H.

Proposition 1. *The set $NS(F) \cap H$ is open in H, and $H \backslash NS(F)$ is nowhere dense in H.*

Proof. It is clear that $NS(F) \cap H$ is open in H by definition, since there are only finite number of possible sets of vectors to consider.

The linear dependence of the family of vectors $L_{ij}^{y_i} F(y)$ is equivalent to some infinitely differentiable function g being equal zero at y. For example $g(y)$ can be chosen as a determinant of a matrix, containing all pairwise scalar products of given vectors.

But if some family of vectors $L_{ij}^{y_i} F(y)$ does not have a constant rank in the neighbourhood of x, then there is some subfamily of this family that are linearly dependent at $y = x$, but linearly independent at least in one point y in any neighbourhood of x. Therefore $H \backslash NS(F)$ is contained in the union of all the sets of the form

$$\{x \in H | g(x) = 0, \forall \delta > 0 \, \exists y \in H, |x - y| < \delta : g(y) \neq 0\}$$

over a set of functions g corresponding to all possible families of vectors $L_{ij}^{y_i} F(y)$. It is obvious that each such set is nowhere dense in H, so the same can be said about $H \backslash NS(F)$.

□

Suppose that $X_i(t)$, $t \geqslant 0$, $i = 1, \ldots, n$ are independent Brownian motions on Carnot groups G_i (see [14, 16] for the definition). Since each X_i for $i = 1, \ldots, n$ is a Markov process, then the corresponding probability measure P_i that describes the distributions of X_i also depends on the starting point of the process, i.e. P_i is also a function of $X_i(0)$. The joint probability measure of all processes will be denoted as P and it is also a function of $X_i(0)$, $i = 1, \ldots, n$. In the following if a statement is made regarding these probabilities it is meant that such statement is true for all possible cases, i.e. for all possible values of $X_i(0)$, $i = 1, \ldots, n$. We assume that our probability space is complete and that all processes X_i are equal to their continuous modifications (in other words their trajectories are continuous a.s.). These assumptions provide, in particular, that for any $\mathfrak{B}(\mathbb{R}^{nd})$-analytic set $B \subset G_1 \times \ldots \times G_n$, and $\mathfrak{B}(\mathbb{R}^n)$-analytic set $A \subset [0, +\infty]^n$ the set $\{\omega | \exists (t_1, \ldots, t_n) \in A, (X_1(t_1), \ldots, X_n(t_n)) \in B\}$ is measurable (since X_i are measurable processes and by the properties of analytic sets, see for example [5]). For any sets $V \subset G_1 \times \ldots \times G_n$, $T \subset [0, +\infty)^n$ denote

$$M(V, T) = \{(x_1, \ldots, x_n) \in V | \exists (t_1, \ldots, t_n) \in T : X_i(t_i) = x_i, i = 1, \ldots, n\}.$$

The following condition first appeared in [14] (Theorem 2 and Theorem 11) in a slightly different form, as a condition for finiteness of an integral. In our case the function inside such integral is related to the product of powers of potentials of the corresponding Brownian motions on Carnot groups, which is what should be expected, according to the results of [9].

Definition 2. We denote as Θ a subset of $G_1 \times \ldots \times G_n$ such that $x \in \Theta$ if and only if for any $\lambda_i \geqslant 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \lambda_i > 0$, there exist $k_i \in \{0, 1, \ldots, d\}$ and, if $k_i \neq 0$, $t_i : \{1, \ldots, k_i\} \to \{1, \ldots, d\}$, $i = 1, \ldots, n$, satisfying the following conditions:

1. the vectors $L_{i, t_i(j)}^{x_i} F(x)$, $j = 1, \ldots, k_i$, $i = 1, \ldots, n$ are linearly independent and $\sum_{i=1}^{n} k_i = M$;
2. $\sum_{i=1}^{n} \lambda_i \left(2 - \sum_{j=1}^{k_i} p_i(t_i(j)) \right) > 0.$

The following theorem states that as long as a point in H is "non-singular" (i.e. belongs to $NS(F)$), then the existence of points in the set $M(H, (0, t]^n)$ in a small neighbourhood of this point with positive probability is equivalent to having a point from $\Theta \cap H$ in the same neighbourhood (note that Θ is an open set).

Theorem 1. *Suppose that $V \subset G_1 \times \ldots \times G_n$ is an open set and denote $U_0 = NS(F) \cap H$. If $U_0 \cap \Theta \cap V$ is nonempty set, then $M(U_0 \cap V, (0,t]^n)$ is infinite with positive probability for all $t > 0$. Otherwise, if $U_0 \cap \Theta \cap V$ is an empty set, then the set $M(U_0 \cap V, (0,+\infty)^n)$ is empty with probability 1.*

Consider infinitely differentiable functions $f_i : \mathbb{R}^d \to \mathbb{R}^m$, $m \leqslant d$, such that $rank\, f_i'(x) = m$ for all $i = 1, \ldots, n$, $x \in \mathbb{R}^d$. In the following Theorem we choose a special form of the function F, related to intersections of processes $f_i(X_i(\cdot))$:

$$F(x) = (f_1(x_1) - f_2(x_2), \ldots, f_{n-1}(x_{n-1}) - f_n(x_n)),$$

In this case H has the following form:

$$H = \{(x_1, \ldots, x_n) \in G_1 \times \ldots \times G_n | f_i(x_i) = f_j(x_j); i = 1, \ldots, n; j = 1, \ldots, n\}$$

and therefore $M(H,T)$ is the set of points on the trajectory of $(X_1(t_1), \ldots, X_n(t_n)), t \in T$ that satisfy $f_i(X_i(t_i)) = f_j(X_j(t_j))$ for all $i \neq j$, i.e. this form of F corresponds to such intersections.

The following definition contains the conditions that are related to the existence or non-existence of intersections points, meaning the points in the set $M(H,T)$. As stated in the Theorem 2 below, these conditions are equivalent to the conditions provided by Theorem 1, but also they are simpler and more explicit.

For any differentiable function $g : \mathbb{R}^d \to \mathbb{R}^m$, such that $rank\, g'(y) = m$ for all $y \in \mathbb{R}^d$, and for any $i = 1, \ldots, n$ and $x \in G_i$ denote

$$m_i(g,x) = \min\{\sum_{q=1}^{m} p_i(j_q) | 1 \leqslant j_1 < \ldots < j_m \leqslant d : \\ L_{i,j_1} g(x), \ldots, L_{i,j_m} g(x) \text{ are linearly independent}\}$$

Definition 3. We denote as Λ a subset of H, such that

1. if $m = 1$ then $\Lambda = H$;
2. if $m = 2$, $n = 2$ then $\Lambda = H$;
3. if $m \geqslant 4$ then $\Lambda = \emptyset$;
4. if $m = 3$, $n \geqslant 3$ then $\Lambda = \emptyset$;
5. if $m = 2$, $n \geqslant 3$, then $x \in \Lambda$ if and only if
$$\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_n} > n - 2; \tag{2}$$
where $q_i = m_i(f_i, x_i) - 1$, $i = 1, \ldots, n$.
6. if $m = 3$, $n = 2$ then $x \in \Lambda$ if and only if at least one of the following conditions holds
   - there is a choice of i, j, k such that $p_1(i) = p_1(j) = p_2(k) = 1$ and $L_{1,i} f_1(x_1), L_{1,j} f_1(x_1), L_{2,k} f_2(x_2)$ are linearly independent;
   - there is a choice of i, j, k such that $p_1(i) = p_2(j) = p_2(k) = 1$ and $L_{1,i} f_1(x_1), L_{2,j} f_2(x_2), L_{2,k} f_2(x_2)$ are linearly independent;

Theorem 2. *If $x \in NS(F) \cap H$ then $x \in \Theta$ if and only if $x \in \Lambda$.*

3. The relation between the geometry of Carnot groups and intersections

In this section we consider the case

$$F(x) = (f_1(x_1) - f_2(x_2), \ldots, f_{n-1}(x_{n-1}) - f_n(x_n))$$

and prove Theorem 2. First we will simplify the condition Θ using the structure of F. As in the case of Θ the following condition is a reformulation of a special case of the main condition in Theorem 11 from [14], for the specific form of F.

Definition 4. We denote as Θ^* a subset of $G_1 \times \ldots \times G_n$ that $x \in \Theta^*$ if and only if for any $\lambda_i \geqslant 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \lambda_i > 0$, there exist $k_i \in \{0, 1, \ldots, m\}$ and, if $k_i \neq 0$, $t_i : \{1, \ldots, k_i\} \to \{1, \ldots, d\}$, $i = 1, \ldots, n$, satisfying the following conditions:

1. $\sum_{i=1}^{n} \dim K_i = \sum_{i=1}^{n} k_i = m(n-1)$, where $K_i = \{0\}$ if $k_i = 0$ and otherwise

$$K_i = l.sp.\{L_{i,t_i(1)} f_i(x_i), \ldots, L_{i,t_i(k_i)} f_i(x_i)\}, i = 1, \ldots, n,$$

2.

$$\forall w \in \mathbb{R}^d, w \neq 0 \; \exists i = 1, \ldots, n : w \notin K_i$$

3.

$$\sum_{i=1}^{n} \lambda_i \left(2 - \sum_{j=1}^{k_i} p_i(t_i(j)) \right) > 0$$

First we are going to show that Θ and Θ^* coincide for a given F.

Lemma 1. *Suppose that we have a family of vector fields $L_{i,t_i(j)}$, $j = 1, \ldots, k_i$, $i = 1, \ldots, n$, where $t_i : \{1, \ldots, k_i\} \to \{1, \ldots, d\}$. Then the vectors $L^{x_i}_{i,t_i(j)} F(x)$, $j = 1, \ldots, k_i$, $i = 1, \ldots, n$ are linearly independent for $x \in G_1 \times \ldots \times G_n$ if and only if*

- *$L_{i,t_i(j)} f_i(x_i)$, $j = 1, \ldots, k_i$ are linearly independent for all $i = 1, \ldots, n$,*
- $$\forall w \in \mathbb{R}^m, w \neq 0, \exists i \in \{1, \ldots, n\} : w \notin B_i$$
  where
  $$B_i = l.sp.\{L_{i,t_i(j)} f_i(x_i), j = 1, \ldots, k_i\}$$

Proof. It is convenient to think of the vector $L^{x_i}_{i,t_i(j)} F(x)$ as $n-1$ vectors of size m put together, since F itself has similar structure. We will refer to such vector in $L^{x_i}_{i,t_i(j)} F(x)$ that corresponds to $f_l(x_l) - f_{l+1}(x_{l+1})$ in F as block number l in $L^{x_i}_{i,t_i(j)} F(x)$. Note that all blocks of $L^{x_i}_{i,t_i(j)} F(x)$ are equal zero, except maybe two: block number i, which is equal to $L_{i,t_i(j)} f_i(x_i)$, and block number $i-1$, which is equal to $-L_{i,t_i(j)} f_i(x_i)$ (if block number is not between 1 and $n-1$ then the corresponding condition should be simply ignored). This leads us to following conclusion: the vectors $L^{x_i}_{i,t_i(j)} F(x)$, $j = 1, \ldots, k_i$, $i = 1, \ldots, n$ are linearly dependent if and only if there exists $w \in \mathbb{R}^m$ such that it can be represented as a linear combination of $L_{i,t_i(j)} f_i(x_i)$, $j = 1, \ldots, k_i$ for all $i = 1, \ldots, n$ and at least one of these representations has non-zero coefficients. But this is equivalent to the statement of Lemma. $\square$

Corollary 1. *$\Theta = \Theta^*$.*

To prepare the proof of Theorem 2 we need some additional information regarding the structure of $NS(F)$ for the given F.

Lemma 2. *Suppose that differentiable vector fields $A_1, \ldots, A_k$ in $\mathbb{R}^d$ and a twice differentiable function $f : \mathbb{R}^d \to \mathbb{R}^m$, where $m \leqslant d$ and rank $f'(x) = m$ at some point $x \in \mathbb{R}^d$, are such that*

- *$\dim B(y) = 1$ for all y in some neighbourhood of x, where*
  $$B(y) = l.sp.\{A_1 f(y), \ldots, A_k f(y)\}$$
- *$B(y) = B(z)$ as long as $f(y) = f(z)$ for all y, z in some neighbourhood of x.*

Then $[A_i, A_j]f(y) \in B(y)$ *for all* y *in some neighbourhood of* x *and all* i, j.

Proof. Choose i such that $A_i f(x) \neq 0$ and so $A_i f(y) \neq 0$ for all y in some neighbourhood of x. Then for each $j \neq i$ there exists a differentiable function $\lambda_j(y)$, such that $A_j f(y) = \lambda_j(y) A_i f(y)$ in some neighbourhood of x. Denote $\gamma_i(y) = |A_i f(y)|$. Note that, by our assumptions, $\frac{A_i f(y)}{\gamma_i(y)}$ has to be constant as long as $f(y)$ is constant for all y in some neighbourhood of x. We obtain

$$[A_i, A_j] = [\gamma_i \frac{A_i}{\gamma_i}, A_j - \lambda_j A_i + \lambda_j A_i] = \alpha A_i + \gamma_i [\frac{A_i}{\gamma_i}, A_j - \lambda_j A_i]$$

with some differentiable function $\alpha(y)$. Now to show that $[A_i, A_j]f(y) \in B(y)$ it is enough to prove that $[\frac{A_i}{\gamma_i}, A_j - \lambda_j A_i]f(y) = 0$ for all y in some neighbourhood of x.

Since any change of variables (meaning the replacement of y with $\psi(z)$ where ψ is a diffeomorphism on some neighbourhood of x) does not change the linear independence of vector fields and their commutators, we may assume that $f(y) = (y_1, \dots, y_m)$ and $x = 0$. In this case m first coordinates of $\frac{A_i}{\gamma_i}$ are differentiable functions of $y_1, \dots, y_m$ (they do not depend on $y_{m+1}, \dots, y_d$) and m first coordinates of $A_j - \lambda_j A_i$ are equal to zero in some neighbourhood of x. But the vector $[\frac{A_i}{\gamma_i}, A_j - \lambda_j A_i]f(y)$ is equal to m first coordinates of $[\frac{A_i}{\gamma_i}, A_j - \lambda_j A_i](y) = \frac{A_i}{\gamma_i}(A_j - \lambda_j A_i)(y) - (A_j - \lambda_j A_i)\frac{A_i}{\gamma_i}(y)$, which are all equal to zero. To finish the proof we note that for any j_1, j_2

$$[A_{j_1}, A_{j_2}] = \alpha_1 A_i + [A_{j_1} - \lambda_{j_1} A_i, A_{j_2} - \lambda_{j_2} A_i] + \\ + \lambda_{j_2}\gamma_i [A_{j_1} - \lambda_{j_1} A_i, \frac{A_i}{\gamma_i}] + \lambda_{j_1}\gamma_i [\frac{A_i}{\gamma_i}, A_{j_2} - \lambda_{j_2} A_i]$$

and, similarly to what was done before, we can claim that $[A_{j_1} - \lambda_{j_1} A_i, A_{j_2} - \lambda_{j_2} A_i]f(y) = 0$ for all y in some neighbourhood of x. ☐

Theorem 3. *If* $x = (x_1, \dots, x_n) \in NS(F) \cap H$, *then for each* $i = 1, \dots, n$ *there exists* $l(i) \in \{1, \dots, d\}$ *such that* $p_i(l(i)) = 1$ *and* $L_{i,l(i)} f_i(x_i) \neq 0$. *Moreover, if* $m \geqslant 2$, $n \geqslant 2$, *then for each pair* $q \neq r$ *in* $\{1, \dots, n\}$ *we can choose* $l(q)$, $l(r)$ *so that* $L_{q,l(q)} f_q(x_q), L_{r,l(r)} f_r(x_r)$ *are linearly independent and* $p_q(l(q)) = 1$, $p_r(l(r)) = 1$.

Proof. First of all we can see that the condition $x \in NS(F) \cap H$ also provides a similar condition for f_i at x_i: for each i all sets of vectors $L_{i,j_1} f_i(x_i), \dots, L_{i,j_k} f_i(x_i)$ are either linearly independent at x_i or linearly dependent for all y in some neighbourhood of x_i. Note that the condition $f(y) = f(x_i)$ is not required, since the projection of H on G_i is an open set, which follows from the fact that for each $j = 1, \dots, n$ the set $f_j(G_j)$ is open.

Fix $i \in \{1, \dots, n\}$ and suppose that $L_{ij} f_i(x_i) = 0$ for all j such that $p_i(j) = 1$. Then, since $x \in NS(F)$, this is also true for all y in some neighbourhood of x_i. Recall that we can obtain each L_{ij} with $p_i(j) = 2$ as a linear combination of commutators of L_{ij} with $p_i(j) = 1$, and similarly we can obtain each L_{ij} with $p_i(j) = k > 2$ as a linear combination of commutators of L_{ij} with smaller $p_i(j)$. But $[L_{i,j_1}, L_{i,j_2}]f_i(z) = 0$ if $L_{i,j_1} f_i(y) = L_{i,j_2} f_i(y) = 0$ for all y in some neighbourhood of z. Applying this argument repeatedly we obtain that $L_{ij} f_i(x_i) = 0$ for all $j = 1, \dots, d$, which is a contradiction. So there has to be at least one j such that $L_{ij} f_i(x_i) \neq 0$ and $p_i(j) = 1$.

Now suppose that $q \neq r$ in $\{1, \dots, n\}$ are fixed and we cannot choose such $l(q)$, $l(r)$ that $L_{q,l(q)} f_q(x_q), L_{r,l(r)} f_r(x_r)$ are linearly independent and $p_q(l(q)) = p_r(l(r)) = 1$. Denote

$$B_{i1} = B_{i1}(y) = l.sp.\{L_{ij} f_i(y) | j = 1, \dots, d, p_i(j) = 1\}, i = 1, \dots, n$$

Since we already proved that $\dim B_{q1}(x_q) \neq 0$, $\dim B_{r1}(x_r) \neq 0$ then to satisfy our assumptions it is necessary that $\dim B_{q1}(x_q) = \dim B_{r1}(x_r) = 1$ and moreover $B_{q1}(x_q) =$

$B_{r1}(x_r)$ also has to be true. Consider a set K of all $L_{ij}^{x_i}F(x)$ with $i \neq q$, $i \neq r$ and all $L_{ij}^{x_i}F(x)$ corresponding to the vector fields L_{ij} in B_{q1}, B_{r1}, in other words

$$K = \{L_{ij}^{x_i}F(x) | i \neq q, i \neq r \text{ or } p_i(j) \leqslant 1\}.$$

Since $x \in NS(F)$ then $w(y) = \dim l.sp.K$ has to be constant for all y in some neighbourhood of x with $F(y) = F(x)$. On the other hand we have $w(x) = m(n-2) + 1$ and to maintain this value as claimed it is necessary that:

- $\dim B_{q1}(y) = 1$ for all y in some neighbourhood of x_q,
- there is a small neighbourhood U of x_q such that $B_{q1}(y) = B_{q1}(z)$ for all $y \in U$, $z \in U$ with $f_q(y) = f_q(z)$.

But it means that we can apply Lemma 2 and obtain that $[L_{q,j_1}, L_{q,j_2}]f_q(y) \in B_{q1}(y)$ for all y in some neighbourhood of x_q and every j_1, j_2 such that $p_q(j_1) = p_q(j_2) = 1$. Therefore $L_{qj}f_q(y) \in B_{q1}(y)$ if $p_i(j) = 2$. Continuing this argument we obtain that $L_{qj}f_q(x_q) \in B_{q1}(x_q)$ for all $j = 1, \ldots, d$, which is a contradiction since $m \geqslant 2$.

□

Before we prove Theorem 2 we also need to consider the following lemma.

Lemma 3. *Let $\alpha_1, \ldots, \alpha_n$ be such that $\alpha_i > 1$, $i = 1, 2 \ldots, n$. Then there exist non-negative numbers $\lambda_1, \ldots, \lambda_n$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$ such that $\alpha_i\lambda_i + \alpha_j\lambda_j \leqslant 1$ for all $i = 1, \ldots, n$, $j = 1, \ldots, n$, $i \neq j$ if and only if $\sum\limits_{i=1}^{n} \frac{1}{\alpha_i} \geqslant 2$.*

Proof. Assume that such $\lambda_1, \ldots, \lambda_n$ exist, but $\sum\limits_{i=1}^{n} \frac{1}{\alpha_i} < 2$. Note that if $\alpha_i\lambda_i \leqslant \frac{1}{2}$ for all $i = 1, \ldots, n$ then we obtain:

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \geqslant \sum_{i=1}^{n} 2\lambda_i = 2$$

So without loss of generality we may assume that $\alpha_1\lambda_1 > \frac{1}{2}$. Then we obtain

$$1 = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \sum_{i=2}^{n} \frac{\lambda_i\alpha_i}{\alpha_i} \leqslant \lambda_1 + \sum_{i=2}^{n} \frac{1 - \lambda_1\alpha_1}{\alpha_i} \leqslant$$
$$\leqslant \lambda_1 + (2 - \frac{1}{\alpha_1})(1 - \lambda_1\alpha_1) = 1 + (1 - \frac{1}{\alpha_1})(1 - 2\lambda_1\alpha_1) < 1$$

But this is a contradiction, meaning that one of the assumptions does not hold.

On the other hand if $\sum\limits_{i=1}^{n} \frac{1}{\alpha_i} \geqslant 2$ we may set $\lambda_i = \frac{1}{\alpha_i}(\sum\limits_{i=1}^{n} \frac{1}{\alpha_i})^{-1}$ and then $\alpha_i\lambda_i + \alpha_j\lambda_j = 2(\sum\limits_{i=1}^{n} \frac{1}{\alpha_i})^{-1} \leqslant 1$ for all $i = 1, \ldots, n$, $j = 1, \ldots, n$, $i \neq j$. □

Proof of Theorem 2. Note that for all $x \in \mathbb{R}^d$, $i = 1, \ldots, n$ we have

$$\dim l.sp.\{L_{i,1}f_i(x_i), \ldots, L_{i,d}f_i(x_i)\} = m$$

due to our assumptions on L_{ij} and f_i. Therefore we can always choose k_i and t_i that satisfies the first two statements in the condition for $x \in \Theta^*$ by setting $k_1 = k_2 = \ldots = k_{n-1} = m$ and $k_n = 0$ and selecting t_i as the set of indices of the corresponding maximal linear independent subset of $\{L_{i,1}f_i(x_i), \ldots, L_{i,d}f_i(x_i)\}$ for all $i = 1, \ldots, n-1$.

Suppose that $m = 1$, then to satisfy the conditions for $x \in \Theta^*$ it is enough to set $k_1 = k_2 = \ldots = k_{n-1} = 1$, $k_n = 0$ and for each $i = 1, \ldots, n-1$ select $t_i(1)$ so that $p_i(t_i(1)) = 1$, which is possible due to Theorem 3. So in this case $\Theta^* \cap NS(F) = \Lambda \cap NS(F) = H \cap NS(F)$.

If $m = 2$, $n = 2$ we set $k_1 = k_2 = 1$, and $t_i(1)$ can be selected so that $p_1(t_1(1)) = p_2(t_2(1)) = 1$ and $L_{1,t_1(1)}f_1(x_1)$, $L_{2,t_2(1)}f_2(x_2)$ are linearly independent, due to Theorem 3, fulfilling the conditions for Θ^* at every point $x \in NS(F) \cap H$.

Suppose that $m = 2$, $n \geqslant 3$ and $x \in \Lambda \cap NS(F)$, so the condition (2) holds at x. It means that there are $s_i : \{1, 2\} \to \{1, \dots, d\}$, $i = 1, \dots, n$ such that $p_{i,s_i(1)} = 1$, $i = 1, \dots, n$ (such choice is possible by Theorem 3), and $L_{i,s_i(1)}f_i(x_i), L_{i,s_i(2)}f_i(x_i)$ are linearly independent for all $i = 1, \dots, n$, and moreover $q_i = p_i(s_i(2))$, $i = 1, \dots, n$ in the condition (2) (since we can choose s_i so that $m_i(f_i, x_i) = p_i(s_i(1)) + p_i(s_i(2)) = 1 + p_i(s_i(2))$). For any $l_1 \neq l_2$ we set $k_{l_1} = k_{l_2} = 1$, $k_i = 2$, $i \neq l_1$, $i \neq l_2$. Then we take $t_i(j) = s_i(j)$ for $j = 1, 2$ and $i \neq l_1$, $i \neq l_2$, and select $t_{l_1}(1)$, $t_{l_2}(1)$, using Theorem 3, so that $p_{l_1}(t_{l_1}(1)) = p_{l_2}(t_{l_2}(1)) = 1$ and $L_{l_1,t_{l_1}(1)}f_{l_1}(x_{l_1})$, $L_{l_2,t_{l_2}(1)}f_{l_2}(x_{l_2})$ are linearly independent. Then the first two parts of the condition for Θ^* are satisfied and at the same time:

$$\sum_{i=1}^{n} \lambda_i \left(2 - \sum_{j=1}^{k_i} p_i(t_i(j)) \right) = \sum_{i=1}^{n} \lambda_i (1 - q_i) + \lambda_{l_1} q_{l_1} + \lambda_{l_2} q_{l_2}$$

where $q_i = p_i(s_i(2))$. So the third part of the condition for Θ^* can be rewritten as

$$\lambda_{l_1} q_{l_1} + \lambda_{l_2} q_{l_2} > \sum_{i=1}^{n} \lambda_i (q_i - 1)$$

or equivalently

$$\tilde{\lambda}_{l_1} \frac{q_{l_1}}{q_{l_1} - 1} + \tilde{\lambda}_{l_2} \frac{q_{l_2}}{q_{l_2} - 1} > 1$$

where

$$\tilde{\lambda}_i = \lambda_i (q_i - 1) (\sum_{i=1}^{n} \lambda_i (q_i - 1))^{-1}$$

Note that if $q_l = 1$ then the index l can be simply ignored, i.e. never selected as l_1 or l_2, Removing such index completely, while also lowering n by 1, does not change the condition (2), and if we show the conditions for $x \in \Theta^*$ in the new situation then these conditions also hold in the initial form. Then, if after such operation we have $n \geqslant 2$, we can apply Lemma 3 with $\lambda_i = \tilde{\lambda}_i$ and $\alpha_i = \frac{q_i}{q_i - 1}$, to prove that the conditions for $x \in \Theta^*$ can be satisfied using one of the above choices (note that the condition (2) in terms of α_i has the following form: $\sum\limits_{i=1}^{n} \frac{1}{\alpha_i} < 2$). It remains to consider the case when $q_i > 1$ for no more than one i. Suppose l is the only value of i with $q_i > 1$. If such l does not exist or if $\lambda_l = 0$ then we select any l with $\lambda_l > 0$. In any case we obtain that $\lambda_i(q_i - 1) = 0$ for all $i \neq l$. Then if we select $k_l = 0$ (and choose the rest of k_i and t_i appropriately) the third part of the conditions for $x \in \Theta^*$ has the following form:

$$\sum_{i=1}^{n} \lambda_i (2 - \sum_{j=1}^{k_i} p_{i,t_i(j)}) = \sum_{i=1}^{n} \lambda_i (1 - q_i) + \lambda_l (1 + q_i) = 2\lambda_l > 0$$

so $x \in \Theta^*$.

Now suppose that $m = 2$, $n \geqslant 3$ and $x \in \Theta^* \cap NS(F)$. For all $i = 1, \dots, n$ choose $s_i(1)$, $s_i(2)$ so that $p_i(s_i(1)) + p_i(s_i(2))$ has the smallest possible value with $L_{i,t_i(s_i(1))}f_i(x_i)$, $L_{i,t_i(s_i(2))}f_i(x_i)$ being linearly independent. Then one of these values, say $p_i(s_i(1))$, is equal to 1, due to Theorem 3. The second of these values, i.e. $p_i(s_i(2))$, we denote as q_i. It is enough to prove that these q_i satisfy the condition (2) to show that $x \in \Lambda$, since we also have that $q_i = m_i(f_i, x_i) - 1$. Note that $p_i(t_i(1)) + p_i(t_i(2)) \geqslant 1 + q_i$ for any t_i with $k_i = 2$ as in the conditions for $x \in \Theta^*$.

Suppose that the condition (2) with q_i defined as above does not hold. There are two types of possible selections of k_i, t_i that may satisfy the conditions for $x \in \Theta^*$ for given λ_i. The first type has $k_i = 2$ for all i except $i = l$ and $k_l = 0$. The second type has $k_i = 2$ for all i except $i = l_1$, $i = l_2$ and $k_{l_1} = k_{l_2} = 1$. Select $\lambda_i = \frac{1}{2q_i}$ (the reason for such selection can be found in the proof of Lemma 3). If the conditions for $x \in \Theta^*$ with such λ can be satisfied using the first type of selection of k_i, t_i, then

$$0 < \sum_{i=1}^{n} 1_{i\neq l}\lambda_i(2 - p_i(t_i(1)) - p_i(t_i(2))) + 2\lambda_l \leqslant \sum_{i=1}^{n} 1_{i\neq l}\lambda_i(1 - q_i) + 2\lambda_l =$$
$$= \sum_{i=1}^{n} \frac{1-q_i}{2q_i} + \frac{1+q_l}{2q_l} \leqslant \frac{n-2}{2} - \frac{n}{2} + \frac{1+q_l}{2q_l} = \frac{1-q_l}{2q_l} \leqslant 0$$

which is a contradiction. If the conditions for $x \in \Theta^*$ with such λ can be satisfied using the second type of selection of k_i, t_i, then

$$0 < \sum_{i=1}^{n} 1_{i\neq l_1}1_{i\neq l_2}\lambda_i(2-p_i(t_i(1))-p_i(t_i(2)))+(2-p_{l_1}(t_{l_1}(1)))\lambda_{l_1}+(2-p_{l_2}(t_{l_2}(1)))\lambda_{l_2} \leqslant$$
$$\leqslant \sum_{i=1}^{n} 1_{i\neq l_1}1_{i\neq l_2}\lambda_i(1-q_i) + \lambda_{l_1} + \lambda_{l_2} = \sum_{i=1}^{n} \frac{1-q_i}{2q_i} + 1 \leqslant \frac{n-2}{2} - \frac{n}{2} + 1 = 0$$

which is also a contradiction. We conclude that the condition (2) has to hold with such q_i and so $x \in \Lambda$.

Suppose that $m = 3$, $n = 2$ and $x \in \Lambda \cap NS(F)$. Assume that the first set of the conditions for $x \in \Lambda$ in such case holds. In this case we select $k_1 = 2$, $k_2 = 1$, $t_1(1) = s_1(1)$, $t_1(2) = s_1(2)$, $t_2(1) = s_2(1)$. Then all statements in the conditions for $x \in \Theta^*$ are satisfied, if $\lambda_2 > 0$. However if $\lambda_2 = 0$, then $\lambda_1 > 0$ and we may select $k(1) = 0$, $k(2) = 3$, so the conditions for $x \in \Theta^*$ are satisfied. The second set of conditions for $x \in \Lambda$ in the case $m = 3$, $n = 2$ also provides that $x \in \Theta^*$ in a similar fashion.

Suppose that $m = 3$, $n = 2$ and $x \in \Theta^* \cap NS(F)$. Select $\lambda_i = 1$, $i = 1, \ldots, n$ and suppose that the conditions for $x \in \Theta^*$ are satisfied with such λ_i and with $k_1 = 2$, $k_2 = 1$. But then $4 - p_1(t_1(1)) - p_1(t_1(2)) - p_2(t_2(1)) > 0$ and therefore $p_1(t_1(1)) = p_1(t_1(2)) = p_2(t_2(1)) = 1$, which means that the first set of conditions for $x \in \Lambda$ in the case $m = 3$, $n = 2$ is satisfied (the linear independence also follows from the conditions for $x \in \Theta^*$). Similarly if the conditions for $x \in \Theta^*$ are satisfied with $k_1 = 1$, $k_2 = 2$ then the second set of conditions for $x \in \Lambda$ in the case $m = 3$, $n = 2$ is satisfied. Now suppose that the conditions for Θ^* are satisfied with $k_1 = 3$, $k_2 = 0$ and some t_1, t_2. But then $4 - p_1(t_1(1)) - p_1(t_1(2)) - p_1(t_1(3)) > 0$ and therefore $p_1(t_1(1)) = p_1(t_1(2)) = p_1(t_1(3)) = 1$. According to Theorem 3 we may select $s_2(1)$ so that $p_2(s_2(1)) = 1$ and $L_{2,s_2(1)}f_2(x_2) \neq 0$, but then we also may choose i, j among $t_1(1)$, $t_1(2)$, $t_1(3)$ to satisfy the first set of conditions for $x \in \Lambda$ in the case $m = 3$, $n = 2$. The last remaining case is when the conditions for Θ^* are satisfied with $k_1 = 0$, $k_2 = 3$, but it is similar to one we just considered, so we have that $x \in \Lambda$ in all cases.

Finally suppose that $m = 3$, $n \geqslant 3$ or $m \geqslant 4$ and $x \in \Theta^*$. Select $\lambda_i = 1$, $i = 1, \ldots, n$ then the third part of the conditions for $x \in \Theta^*$ with such λ_i provides the following inequality:

$$0 < \sum_{i=1}^{n} \lambda_i(2 - \sum_{j=1}^{k_i} p_{i,t_i(j)}) \leqslant 2n - m(n-1)$$

or $m < \frac{2n}{n-1}$. But if $m = 3$, $n \geqslant 3$ or $m \geqslant 4$ then this inequality is false, so in fact the conditions for $x \in \Theta^*$ cannot hold in such case, and $\Theta^* = \emptyset = \Lambda$.

□

4. The estimate for the surface measure of a ball in Carnot group

In this section we consider a Carnot group $G = (\mathbb{R}^{m_1}, \bullet)$ and suppose that $f : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ is an infinitely differentiable function such that $rank\, f'(x) = m_2 \leqslant m_1$ for all $x \in \mathbb{R}^{m_1}$. Additionally instead of using standard homogeneous degrees of vector fields in Carnot group (as it was done in [14] for a similar upper bound), we use arbitrary positive numbers, under condition that the corresponding dilations are still preserved by the group operation. In this way we are able to include into our consideration a special case, when G is a product of Carnot groups, but the homogeneous degrees in different groups have different "weights". More precisely we fix $\gamma : \{1, 2, \dots, m_1\} \to [0, +\infty)$ and assume that

$$\delta_\lambda(x \bullet y) = \delta_\lambda(x) \bullet \delta_\lambda(y), \delta_\lambda(x)_i = \lambda^{\gamma(i)} \cdot x_i, i = 1, \dots, m_1 \tag{3}$$

For example if all $\gamma(i)$ are equal to the corresponding homogeneous degree of the coordinate i multiplied by the same non-negative constant, then this condition is satisfied by the definition of Carnot group. Note that $\gamma(i) = 0$ is possible, but we assume that at least one $\gamma(i)$ is not zero, meaning $\sum\limits_{i=1}^{m_1} \gamma(i) > 0$.

We also fix a set of left-invariant vector fields $L_1, \dots, L_{m_1}$ on G, defined as in (1) and denote

$$\begin{aligned} m_f(x) = \min\{\sum_{q=1}^{m_2} \gamma(i_q) | 1 \leqslant i_1 < \ldots < i_{m_2} \leqslant m_1 : \\ L_{i_1} f(x), \dots, L_{i_{m_2}} f(x) \text{ are linearly independent}\} \end{aligned}$$

A similar definition was used above for $m_i(g, x)$, but with homogeneous degrees used in place of $\gamma(i)$ (the definition of $m_f(x)$ in [14] is the same as the definition of $m_i(g, x)$, while the definition above is its generalization).

We define a family of measures $\mu(u, dy)$ on the surfaces

$$W_u = \{x \in \mathbb{R}^{m_1} | f(x) = u\}, u \in \mathbb{R}^{m_2}$$

as it was done in the Proposition 1 in [14], where they were defined using a variant of the standard construction for the integration on a manifold (for a general theory of such integration see for example [17]). Additionally it was shown that for any $x \in \mathbb{R}^{m_1}$ we can choose such $\delta > 0$ and an infinitely differentiable injective function $\psi : B(0, \delta) \to \mathbb{R}^{m_1}$, where $B(0, \delta)$ is an open ball in $\mathbb{R}^{m_2} \times \mathbb{R}^{m_1 - m_2}$, that $\psi(0, 0) = x$, $rank\, \psi'(0, 0) = m_1$ and additionally for all $u \in \mathbb{R}^{m_2}$ and all $a \in \mathbb{R}^{m_1 - m_2}$ with $|(u, a)| < \delta$ we have $f(\psi(u, a)) = u + f(x)$. Then, according to Corollary 1 in [14], for all continuous functions g with support inside $\psi(B(0, \delta))$ and for all $u \in \mathbb{R}^{m_2}$ with $|u| < \delta$ the following equality holds:

$$\int\limits_{\mathbb{R}^{m_1}} g(y) \mu(u + f(x), dy) = \int\limits_{(u,a) \in B(0,\delta)} g(\psi(u, a)) \cdot |\det(\psi'(u, a))| da \tag{4}$$

Theorem 4. *Let $x \in \mathbb{R}^{m_1}$ and suppose that $m_f(y) = m_f(x)$ for all y, such that $f(x) = f(y)$, in some neighbourhood U of x. Then there exist constants $C > 0$ and $\delta > 0$ such that for all $\varepsilon \in (0, 1)$ and all $y \in B(x, \delta)$ with $f(y) = f(x)$ we have $\mu(f(y), D(y, \varepsilon)) \geqslant C \varepsilon^{Q_\gamma - m_f(x)}$, where*

$$D(y, \varepsilon) = \{z \in G : |(y^{-1} \bullet z)_i| \leqslant \varepsilon^{\gamma(i)}, i = 1, \dots, m_1\}$$

and $Q_\gamma = \sum\limits_{k=1}^{m_1} \gamma(k)$.

To prepare the proof of this Theorem we first present several useful lemmas.

In the following lemma we take a standard argument, involving the Banach fixed point theorem, often used to prove the inverse function theorem, and extend it, using additional bounds on the second derivative, to show that the choice of the neighbourhoods in the inverse function theorem can be made uniformly.

For any real-valued matrix A we denote as $\|A\|$ the norm of A as a linear operator between Euclidean spaces. For any twice continuously differentiable function $g : \mathbb{R}^d \to \mathbb{R}^d$ we denote as $g''(x)(u, v) = \frac{\partial}{\partial u}\frac{\partial}{\partial v}g(x)$ the second directional derivative at x of g in the directions of $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$. This allows us to write:

$$(g'(x) - g'(y))u = \frac{\partial}{\partial u}g(x) - \frac{\partial}{\partial u}g(y) = \int_0^1 g''(tx + (1-t)y)(u, x-y)dt$$

Additionally we denote $\|g''(x)\| = \sup_{|u|=|v|=1} |g''(x)(u, v)|$ and note that for all $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$ we have $|g''(x)(u, v)| \leqslant \|g''(x)\||u||v|$.

Lemma 4. *Suppose that $a > 0$, $b > 0$. For all twice continuously differentiable functions $g : \mathbb{R}^d \to \mathbb{R}^d$, satisfying $\|(g'(0))^{-1}\| \leqslant a$ and $\|g''(x)\| \leqslant b$, $x \in B(0, \frac{1}{ab})$, we have:*

- *g is an injection on the set $\{x : |x| \leqslant \frac{1}{3ab}\}$;*
- *$\|g'(x)^{-1}\| \leqslant 2a$ for all $|x| \leqslant \frac{1}{3ab}$;*
- *the set $\{y : |y - g(0)| \leqslant \frac{1}{6a^2b}\}$ is in the image of $\{x : |x| < \frac{1}{3ab}\}$ under g.*

Proof. Consider

$$F_y(x) = x + (g'(0))^{-1}(y - g(x))$$

and note that, for $x \in B(0, \frac{1}{ab})$,

$$|F_y(x)| \leqslant |(g'(0))^{-1}(g(x)-g(0)-g'(0)x)|+|(g'(0))^{-1}(y-g(0))| \leqslant \frac{ab}{2}|x|^2+|(g'(0))^{-1}(y-g(0))|$$

and moreover if $y = g(x_0)$, $x_0 \in B(0, \frac{1}{ab})$,

$$|F_y(x)| \leqslant \frac{ab}{2}|x|^2 + |(g'(0))^{-1}(g(x_0) - g'(0)x_0 - g(0))| + |x_0| \leqslant \frac{ab}{2}(|x|^2 + |x_0|^2) + |x_0|$$

So if we set $c(a, b) = \frac{1}{3ab}$ and $y = g(x_0)$ for some x_0 with $|x_0| \leqslant c(a, b)$, then for all x such that $|x| \leqslant 2c(a, b)$:

$$|F_y(x)| \leqslant \frac{ab}{2}(4c(a, b)^2 + c(a, b)^2) + c(a, b) \leqslant \frac{5}{6}c(a, b) + c(a, b) < 2c(a, b)$$

which means that F_y maps the ball $B(0, 2c(a, b))$ into itself. Additionally, under the same assumptions, we obtain

$$\|F_y'(x)\| = \|I - (g'(0))^{-1}g'(x)\| \leqslant ab|x| \leqslant \frac{2}{3}$$

Therefore by the Banach fixed point theorem the equation $F_y(x) = x$ has exactly one solution inside $B(0, 2c(a, b))$ and the same is true about the equation $g(x) = g(x_0)$, which means that g is an injection on $B(0, c(a, b))$.

Now assume that there is a point y, such that $|y - g(0)| \leqslant \frac{1}{6a^2b}$, but $g(x) \neq y$ for $|x| < \frac{1}{3ab}$. Note that the set $g(\{x : |x| < \frac{1}{3ab}\})$ is open, since

$$|g'(x)u| \geqslant |g'(0)u| - |\int_0^1 g''(sx)(x, u)ds| \geqslant |u|(\frac{1}{a} - b|x|) \geqslant \frac{1}{2a}|u|, |x| \leqslant \frac{1}{3ab}$$

so that $\|g'(x)^{-1}\| \leqslant 2a$, $|x| \leqslant \frac{1}{3ab}$ and we can apply inverse function theorem for g at every point of this set. It means that the set

$$(\mathbb{R}^d \backslash g(\{x : |x| < \frac{1}{3ab}\}) \cap \{y : |y - g(0)| \leqslant \frac{1}{6a^2b}\}$$

is compact and nonempty. Consequently, by standard arguments, there exists $y_0 \in \mathbb{R}^d$, such that $|y_0 - g(0)| \leqslant \frac{1}{6a^2b}$ and $g(x) \neq y_0$ for $|x| < \frac{1}{3ab}$, and for all y with $|y - g(0)| < |y_0 - g(0)|$ there exists x with $g(x) = y$ and $|x| < \frac{1}{3ab}$. But then there is x_0 such that $g(x_0) = y_0$ and $|x_0| = \frac{1}{3ab}$, and we have a contradiction since

$$|g(x) - g(0)| = |g'(0)x + \int_0^1 (1-s)g''(sx)(x,x)ds| \geqslant \frac{|x|}{a} - \frac{b|x|^2}{2} \geqslant \frac{5|x|}{6a}, |x| \leqslant \frac{1}{3ab}$$

and therefore $|y_0 - g(0)| = |g(x_0) - g(0)| \geqslant \frac{5}{18a^2b} > \frac{1}{6a^2b}$. □

Lemma 5. *Suppose that A is a $d \times d$ non-degenerate real matrix, $m \in \{1, 2, \ldots, d-1\}$ and $i_1, \ldots, i_m$, $j_1, \ldots, j_{d-m}$ is a permutation of $1, 2, \ldots, d$. Denote $B_{kl} = A_{k,i_l}$, $k = 1, \ldots, m$, $l = 1, \ldots, m$ and $C_{kl} = A^{-1}_{j_k,l}$, $k = 1, \ldots, d-m$, $l = m+1, \ldots, d$ so that B and C are $m \times m$ and $(d-m) \times (d-m)$ matrices correspondingly, where $m \in \{1, \ldots, d-1\}$. Then $|\det B| = |\det A \cdot \det C|$ and therefore B is non-degenerate if and only if C is non-degenerate.*

Proof. This is well-known. □

Denote as L the matrix defined by $L_{ji}(y) = (L_i(y))_j$, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_1$ (note that $L(0) = I$).

Lemma 6. *For all $x \in G$ and $\lambda > 0$ the following equality holds:*

$$L(\delta_\lambda(x)) = T(\lambda)L(x)T(\lambda^{-1}),$$

where $T(\lambda)$ is $m_1 \times m_1$ matrix defined as follows: $(T(\lambda))_{ii} = \lambda^{\gamma(i)}$, $i = 1, \ldots, m_1$; $(T(\lambda))_{ij} = 0$, $i \neq j$.

Proof. By the definition of L we have that for all $x \in G$ and $\lambda > 0$, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_1$:

$$(L_i(\delta_\lambda(x)))_j = \frac{\partial}{\partial y_i}(\delta_\lambda(x) \bullet y)_j|_{y=0} = \lambda^{-\gamma(i)} \frac{\partial}{\partial y_i}(\delta_\lambda(x) \bullet \delta_\lambda(y))_j|_{y=0} =$$
$$= \lambda^{-\gamma(i)} \frac{\partial}{\partial y_i} \delta_\lambda(x \bullet y)_j|_{y=0} = \lambda^{\gamma(j)-\gamma(i)}(L_i(x))_j$$

and the statement of the lemma follows. □

Proof of Theorem 4. Recall that we can choose such $\delta_1 > 0$ and an infinitely differentiable function $\psi : B(0, \delta_1) \to \mathbb{R}^{m_1}$, that $\psi(0,0) = x$, *rank* $\psi'(0,0) = m_1$ and for all $u \in \mathbb{R}^{m_2}$ and all $a \in \mathbb{R}^{m_1 - m_2}$ with $|(u,a)| < \delta_1$ we have $f(\psi(u,a)) = u + f(x)$. Note that the matrix of derivatives ψ' has the following property:

$$(f'(\psi(u,a)) \cdot \psi'(u,a))_{ij} = ((f(\psi(u,a)))')_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}, i = 1, \ldots, m_2; j = 1, \ldots, m_1;$$

so the first m_2 rows of $(\psi')^{-1}(u,a) = (\psi^{-1})'(\psi(u,a))$ coincide with $f'(\psi(u,a))$. We can choose $\delta > 0$ so that $B(x,\delta) \subset U$, $\psi^{-1}(B(x,\delta)) \subset B(0,\delta_1)$, $m_f(x) = m_f(y)$ for $y \in B(x,\delta)$ with $f(x) = f(y)$, and, due to (4), so that for any continuous $g \geqslant 0$ and $u \in \mathbb{R}^{m_2}$:

$$\int_{B(x,\delta)} g(z)\mu(u + f(x), dz) \geqslant C \int_{|\psi(u,a)-x|<\delta} g(\psi(u,a))da$$

where a positive constant C depends only on the choice of ψ and δ. Since $D(x,\varepsilon)$ is compact, we can represent $1_{D(x,\varepsilon)}$ as a pointwise limit of a sequence of uniformly bounded continuous functions, so that the integrals above, with these functions in place of g, converge to the same integrals with $1_{D(x,\varepsilon)}$ in place of g. Therefore to prove our statement it is enough to estimate from below the following expression for all $\varepsilon \in (0,1)$ and all y in some neighbourhood of x, such that $f(y) = f(x)$:

$$\lambda(\{a : \psi(0,a) \in B(x,\delta), |(y^{-1} \bullet \psi(0,a))_i| \leqslant \varepsilon^{\gamma(i)}, i = 1,2,\ldots,m_1\})$$

For a moment let us fix $y \in B(x,\delta)$ and set $u = f(y) - f(x)$. We have for all $i = 1,2,\ldots,m_1 - m_2, j = 1,2,\ldots,m_1$

$$\frac{\partial}{\partial a_i}(y^{-1} \bullet \psi(u,a))_j = \sum_{k=1}^{m_1} \frac{\partial \psi_k}{\partial a_i}(u,a)\frac{\partial}{\partial z_k}(y^{-1} \bullet z)_j|_{z=\psi(u,a)},$$

Define a matrix $R(y,z)$ as follows: $R_{jk}(y,z) = \frac{\partial}{\partial z_k}(y^{-1} \bullet z)_j$. According to the definition of L_i (see (1)) we have that $\sum_{k=1}^{m_1} R_{jk}(y,z)(L_i(z))_k = (L_i(y^{-1}\bullet z))_j$, and therefore $R(y,z) = L(y^{-1} \bullet z)L^{-1}(z)$. Consider $g(a) = y^{-1} \bullet \psi(u,a)$, then we can rewrite the above relation as follows

$$\frac{\partial}{\partial a_i} g_j(a) = \sum_{k=1}^{m_1} R_{jk}(y,\psi(u,a))(\psi')_{k,i+m_2}(u,a), i = 1,2,\ldots,m_1 - m_2, j = 1,2,\ldots,m_1,$$

so that $g'(a)$ is a submatrix of $R(y,\psi(u,a))\psi'(u,a)$ that consists of the last $m_1 - m_2$ columns.

In the following we are only interested in the case $u = 0$, so that $f(y) = f(x)$ and consider all such values of a for which $\psi(0,a)$ is well-defined and inside $B(x,\delta)$. The first m_2 rows of

$$(R(y,\psi(0,a))\psi'(0,a))^{-1} = (\psi^{-1})'(\psi(0,a)) \cdot L(\psi(0,a)) \cdot L^{-1}(y^{-1} \bullet \psi(0,a))$$

coincide with $f'(\psi(0,a)) \cdot L(\psi(0,a)) \cdot L^{-1}(y^{-1} \bullet \psi(0,a))$. Moreover each column $i = 1,\ldots,m_1$ of the matrix $f'(\psi(0,a))\cdot L(\psi(0,a))$ is in fact the vector $L_i f(\psi(0,a))$. Therefore by our assumptions (and after making δ smaller, if necessary), we can select the columns $j_1,\ldots,j_{m_2}$ of $f'(y) \cdot L(y)$, such that they are linear independent and $m_f(y) = \sum_{k=1}^{m_2} \gamma(j_k)$ for all $y \in B(x,\delta)$ with $f(y) = f(x)$.

Denote as $D(a)$ the selection of the rows $i_1,\ldots,i_{m_1-m_2}$ and the last $m_1 - m_2$ columns from $L^{-1}(\psi(0,a))\psi'(0,a)$, where the set of indices $i_1,\ldots,i_{m_1-m_2}$ is such that its union with $j_1,\ldots,j_{m_2}$ is equal to $\{1,\ldots,m_1\}$. Then by Lemma 5 applied to

$$A = L^{-1}(\psi(0,a))\psi'(0,a),$$

we obtain that for all a, such that $\psi(0,a) \in B(x,\delta)$, $D(a)$ is non-degenerate and also $\sum_{k=1}^{m_1-m_2} \gamma(i_k)$ is maximal possible over all possible selections of non-degenerate $(m_1-m_2)\times(m_1-m_2)$ submatrices from $L^{-1}(\psi(0,a))\psi'(0,a)$ when always choosing the last m_1-m_2 columns and arbitrary $m_1 - m_2$ rows with row indices in place of i_k.

Denote $\varphi(a) = (g_{i_1}(a),\ldots,g_{i_{m_1-m_2}}(a))$ and denote as $B(a)$ the matrix that contains the last $m_2 - m_1$ columns of $L^{-1}(\psi(0,a))\psi'(0,a)$. Then

$$g'(a) = L(y^{-1} \bullet \psi(0,a))B(a)$$

and $\varphi'(a)$ can be obtained as a submatrix of $g'(a)$ by selecting the rows $i_1,\ldots,i_{m_1-m_2}$, just as $D(a)$ can be obtained as a submatrix of $B(a)$ by selecting the rows $i_1,\ldots,i_{m_1-m_2}$. Since the rows $i_1,\ldots,i_{m_1-m_2}$ of B are linearly independent we can write $B_{jl}(a) =$

$\sum\limits_{k=1}^{m_2-m_1} \lambda_{jk}(a)B_{i_k,l}(a)$ where $\lambda_{jk}(a)$ are some real numbers. We claim that $\lambda_{jk}(a) = 0$ if $\gamma(j) > \gamma(i_k)$. Indeed, otherwise we can replace i_k with q in the selection and make $\sum\limits_{k=1}^{m_1-m_2} \gamma(i_k)$ larger, which contradicts the choice of $i_1, \dots, i_{m_2-m_1}$. Note that $\lambda_{jk}(a)$ are in fact the elements of $m_1 \times (m_1 - m_2)$ matrix $B(a)D^{-1}(a)$.

Now suppose that $y = \psi(0, a_0) \in B(x, \delta)$ so all functions of a, defined above, may also depend on a_0. Take $\theta > 0$ and select $\delta_2 = \delta_2(\theta)$ so that $|y^{-1} \bullet \psi(0, a)| < \theta$ as long as $|a - a_0| < \delta_2$. Since $L(0) = I$ there exists such constant C_1 that $\|I - L(z)\| < C_1\theta$ if $|z| < \theta$. Consider

$$g'(a) - B(a) = (L(y^{-1} \bullet \psi(0, a)) - I)B(a)$$

and

$$\varphi'(a) - D(a) = C(a)B(a)$$

where $C(a)$ is a submatrix of $L(y^{-1} \bullet \psi(0, a)) - I$ that contains the rows $i_1, \dots, i_{m_1-m_2}$. But $\|C(a)B(a)\| \leqslant C_2\theta$, if $|a - a_0| < \delta_2$, so by choosing small enough θ we can guarantee that $\varphi'(a)$ is non-degenerate and moreover that $\|(\varphi')^{-1}(a)\|$ is bounded uniformly w.r.t. a and a_0 as long as $\psi(0, a_0) \in B(x, \delta)$, $\psi(0, a) \in B(x, \delta)$ and $|a - a_0| < \delta_2$. Additionally it is easy to see that both $\|\varphi'(a)\|$ and $\|\varphi''(a)\|$ are also bounded uniformly w.r.t. a and a_0 under the same conditions, since L, L^{-1} and their first and second derivatives are bounded on a bounded set.

Therefore we can introduce a new set of variables $b = \varphi(a)$ in some neighbourhood of a_0. More precisely, using the inverse function theorem and Lemma 4 we can choose $\delta_3 > 0$, that does not depend on a_0, such that $\varphi^{-1}(B(\varphi(a_0), \delta_3)) \subset B(a_0, \delta_2)$, φ is injective on $\varphi^{-1}(B(0, \delta_3))$ ($\varphi(a_0) = 0$), and additionally $\|(\varphi^{-1})'(b)\|$, $\|\varphi'(\varphi^{-1}(b))\|$ and $\|\varphi''(\varphi^{-1}(b))\|$ are all bounded if $b \in B(0, \delta_3)$ uniformly w.r.t. a_0 with $\psi(0, a_0) \in B(x, \delta)$. It means that the following inequality holds uniformly w.r.t. a_0 with $\psi(0, a_0) \in B(x, \frac{\delta}{2})$ (if we choose small enough $\delta_3 > 0$):

$$\lambda(\{\psi(0, a) \in B(x, \delta) : |g_i(a)| \leqslant \varepsilon^{\gamma(i)}, i = 1, \dots, m_1\}) \geqslant \\ \geqslant C_3\lambda(\{b \in B(0, \delta_3) : |g_i(\varphi^{-1}(b))| \leqslant \varepsilon^{\gamma(i)}, i = 1, \dots, m_1\})$$

To find the lower bound of the last expression we consider, for $j = 1, \dots, m_1$, $k = 1, \dots, m_1 - m_2$:

$$\frac{\partial}{\partial b_k} g_j(\varphi^{-1}(b)) = \sum_{l=1}^{m_1-m_2} g'_{jl}(\varphi^{-1}(b)) \frac{\partial(\varphi^{-1})_l}{\partial b_k}(b) = (g'(\varphi^{-1}(b)) \cdot (\varphi'(\varphi^{-1}(b)))^{-1})_{jl}$$

These functions are bounded if $b \in B(0, \delta_3)$ uniformly w.r.t. a_0 with $\psi(0, a_0) \in B(x, \delta)$, but we have to find a stronger estimate, also taking into account the conditions $|b_{i_k}| \leqslant \varepsilon^{\gamma(i_k)}$, $k = 1, \dots, m_1 - m_2$.

Using Lemma 6 we can rewrite the corresponding matrix as follows:

$$g'(a) \cdot (\varphi'(a))^{-1} = L(y^{-1} \bullet \psi(0, a))B(a)(\varphi'(a))^{-1} = \\ = L(y^{-1} \bullet \psi(0, a))B(a)D^{-1}(a)(\varphi'(a)D^{-1}(a))^{-1} = \\ = T(\varepsilon)L(d_\varepsilon)T(\varepsilon^{-1})B(a)D^{-1}(a)(\varphi'(a)D^{-1}(a))^{-1}$$

where $d_\varepsilon = d_\varepsilon(a) = \delta_{\frac{1}{\varepsilon}}(y^{-1} \bullet \psi(0, a))$. Note that the conditions $|g_i(a)| \leqslant \varepsilon^{\gamma(i)}, i = 1, \dots, m_1$ can be rewritten as $|(d_\varepsilon)_i| \leqslant 1, i = 1, \dots, m_1$ and so they imply that d_ε is bounded and therefore $\|L(d_\varepsilon)\|$, $\|L^{-1}(d_\varepsilon)\|$ are bounded.

Recall that the elements of $B(a)D^{-1}(a)$ with indices j, k, such that $\gamma(j) > \gamma(i_k)$, are equal to zero. This means that the elements of the matrix $T(\varepsilon^{-1})B(a)D^{-1}(a)\tilde{T}(\varepsilon)$ and their derivatives by a are bounded uniformly w.r.t. a and $\varepsilon \in (0, 1)$ (and they do not depend on a_0), where $\tilde{T}_{kk}(\varepsilon) = \varepsilon^{\gamma(i_k)}$, $k = 1, \ldots, m_1 - m_2$, $\tilde{T}_{ij}(\varepsilon) = 0$, $i \neq j$ ($\tilde{T}$ is a $(m_1 - m_2) \times (m_1 - m_2)$ matrix). Denote as $E(a)$ the submatrix that consists of the rows $i_1, \ldots, i_{m_1 - m_2}$ of $L(d_\varepsilon)T(\varepsilon^{-1})B(a)D^{-1}(a)\tilde{T}(\varepsilon)$. We can also check that $E(a) = \tilde{T}(\varepsilon^{-1})\varphi'(a)D^{-1}(a)\tilde{T}(\varepsilon)$. We can choose $\beta \in (0,1)$ such that $\|E^{-1}(a)\|$ and $\|E(a)\|$ are bounded uniformly w.r.t. a, a_0 and $\varepsilon \in (0,1)$ as long as $|d_\varepsilon| < \beta$, since $E(a_0) = I$ and we can make sure that elements of both $L(d_\varepsilon)$ and $T(\varepsilon^{-1})B(a)D^{-1}(a)\tilde{T}(\varepsilon)$ are in some small neighbourhood of their values at $a = a_0$, uniformly w.r.t. a_0. Moreover the following functions are also uniformly bounded in the same way:

$$\varepsilon^{\gamma(i_k)-\gamma(j)} \frac{\partial}{\partial b_k} g_j(\varphi^{-1}(b)) = (T(\varepsilon^{-1}) g'(\varphi^{-1}(b)) \cdot (\varphi')^{-1}(\varphi^{-1}(b)) \tilde{T}(\varepsilon))_{jk}$$

because

$$\begin{aligned} T(\varepsilon^{-1})g'(a)(\varphi'(a))^{-1}\tilde{T}(\varepsilon) = \\ &= L(d_\varepsilon)T(\varepsilon^{-1})B(a)D^{-1}(a)(\tilde{T}(\varepsilon^{-1})\varphi'(a)D^{-1}(a))^{-1} = \\ &\qquad = L(d_\varepsilon)T(\varepsilon^{-1})B(a)D^{-1}(a)\tilde{T}(\varepsilon)E^{-1}(a). \end{aligned}$$

We conclude that there exist $\beta > 0$ and $C_4 > 0$ such that for all $b \in B(0, \delta_3)$ and $\varepsilon \in (0,1)$, satisfying $|d_\varepsilon(\varphi^{-1}(b))| < \beta$, the following inequality holds uniformly w.r.t. a_0:

$$\begin{aligned} |g_j(\varphi^{-1}(b))| &= |g_j(\varphi^{-1}(b)) - g_j(\varphi^{-1}(\varphi(a_0)))| \leqslant \\ &\leqslant \sum_{k=1}^{m_1-m_2} |\frac{\partial}{\partial b_k} g_j(\varphi^{-1}(b))||b_k| \leqslant C_4 (\sum_{k=1}^{m_1-m_2} 1_{\{\gamma(i_k) \geqslant \gamma(j)\}} |b_k| + \\ &\qquad + \sum_{k=1}^{m_1-m_2} 1_{\{\gamma(i_k) < \gamma(j)\}} \varepsilon^{\gamma(j)-\gamma(i_k)} |b_k|) \end{aligned}$$

Therefore if we assume that $|(d_\varepsilon(\varphi^{-1}(b)))_{i_k}| < \beta_1$, $k = 1, \ldots, m_1 - m_2$ or, equivalently, $|b_k| < \varepsilon^{\gamma(i_k)}\beta_1$, $k = 1, \ldots, m_1 - m_2$, with small enough $\beta_1 \in (0,1)$, that does not depend on a_0, then we can provide that $|d_\varepsilon(\varphi^{-1}(b))| < \frac{\beta}{2}$. Indeed if it is not true, then there exists such $\tilde{b} \in B(0, \delta_3)$ with $|d_\varepsilon(\varphi^{-1}(\tilde{b}))| < \beta$, $|\tilde{b}_k| < \varepsilon^{\gamma(i_k)}\beta_1$, $k = 1, \ldots, m_1 - m_2$ and such $j \in \{1, \ldots, m_1\}$ that $g_j(\varphi^{-1}(\tilde{b})) = \frac{1}{2\sqrt{m_1}}\beta\varepsilon^{\gamma(j)}$ (we can select such $\tilde{b}$ on the segment between 0 and b using continuity). If $j = i_k$ for some $k = 1, \ldots, m_1 - m_2$ then we obtain a contradiction under the condition $\beta_1 < \frac{\beta}{2\sqrt{m_1}}$. But otherwise, setting $b = \tilde{b}$ in the above inequality, we obtain $|g_j(\varphi^{-1}(b))| \leqslant C_4\beta_1\varepsilon^{\gamma(j)}(m_1 - m_2)$, which gives us a contradiction under the condition $\beta_1 < \frac{\beta}{2m_1^{\frac{3}{2}}C_4}$.

We obtain

$$\begin{aligned} \lambda(\{b \in B(0,\delta_3) : |g_i(\varphi^{-1}(b))| \leqslant \varepsilon^{\gamma(i)}, i = 1, \ldots, m_1\}) \geqslant \\ &\geqslant \lambda(\{b \in B(0,\delta_3) : |g_{i_k}(\varphi^{-1}(b))| < \beta_1\varepsilon^{\gamma(i_k)}, k = 1, \ldots, m_1 - m_2, \\ &\qquad |g_j(\varphi^{-1}(b))| < \frac{\beta}{2}\varepsilon^{\gamma(j)}, j \neq i_k, k = 1, \ldots, m_1 - m_2\}) = \\ = \lambda(\{b \in B(0,\delta_3) : |b_k| < \beta_1\varepsilon^{\gamma(i_k)}, k = 1, \ldots, m_1 - m_2\}) \geqslant C_5 \varepsilon^{\sum_{k=1}^{m_1-m_2} \gamma(i_k)} = \\ &\qquad = C_5\varepsilon^{Q_\gamma - m_f(x)} \end{aligned}$$

uniformly w.r.t. a_0, which concludes the proof of the Theorem.

□

5. The joint points of trajectories inside a set

The purpose of this section is to fill the gap left in [16] where in Theorem 1 the case $\delta = 0$ was omitted. The following Theorem allows us to fill that gap and consequently the condition $x \in \Theta$ becomes necessary and sufficient for $M(H \cap V, (0,t]^n)$ to be nonempty with positive probability for all open neighbourhoods V of x, as long as $x \in NS(F) \cap H$.

Theorem 5. *Suppose that $d \geqslant 2$. For all $t > 0$, all $X_i(0) \in \mathbb{R}^d$, $i = 1, \dots, n$ and all Borel sets $V \subset G_1 \times \dots \times G_n$ we have*

$$P(M(V, [0,t]^n \backslash \{0\}) \textit{ is a nonempty finite set}) = 0.$$

Proof. Denote as $X^{n-1}(T) = \{(X_1(t_1), \dots, X_{n-1}(t_{n-1}))|(t_1, \dots, t_{n-1}) \in T\}$ the set of all points on the joint trajectories of $X_1, \dots, X_{n-1}$ that correspond to the time parameters in $T \subset [0, +\infty)^{n-1}$. Due to our assumptions on X_i the set $X^{n-1}([0,t]^{n-1})$ is compact a.s. for all $t > 0$.

Fix $t > 0$ and a Borel set $V \subset G_1 \times \dots \times G_n$. Consider a set

$$M^{n-1}(V, [0,t]^{n-1}) = p_{G_n}(X^{n-1}([0,t]^{n-1}) \times G_n \cap V),$$

where p_{G_n} is the canonical projection of $G_1 \times \dots \times G_n$ on G_n. Note that $x \in M^{n-1}(V, [0,t]^{n-1})$ if and only if $X^{n-1}([0,t]^{n-1}) \times \{x\} \cap V \neq \emptyset$. It follows from the definition that $M^{n-1}(V, [0,t]^{n-1})$ is a $\mathfrak{B}(\mathbb{R}^d)$-analytic set a.s.

Then

$$\begin{aligned} M(V, [0,t]^{n-1} \times (0,t]) &= \bigcup_{t_n \in (0,t]} X^{n-1}([0,t]^{n-1}) \times \{X_n(t_n)\} \cap V = \\ &= \bigcup_{t_n \in (0,t] : X_n(t_n) \in M^{n-1}(V, [0,t]^{n-1})} X^{n-1}([0,t]^{n-1}) \times \{X_n(t_n)\} \cap V \end{aligned}$$

We conclude from this equality that if $M(V, [0,t]^{n-1} \times (0,t])$ is a finite set then the number of points on the trajectory of $X_n(t_n), t_n \in (0,t]$ that belong to the set $M^{n-1}(V, [0,t]^{n-1})$ has to be finite. Moreover if $M(V, [0,t]^{n-1} \times (0,t])$ is nonempty then $X_n(t_n), t_n \in [0,t]$ visits $M^{n-1}(V, [0,t]^{n-1})$. Consider the conditional probability with fixed $X_1, \dots, X_{n-1}$. Since X_i are independent, this does not change the distribution of X_n and $M^{n-1}(V, [0,t]^{n-1})$ can be considered as a non-random $\mathfrak{B}(\mathbb{R}^d)$-analytic set (with probability 1 w.r.t. the distribution of $X_1, \dots, X_{n-1}$).

Consider a subset Ω_1 of the probability space that corresponds to the event that $X_n(t_n), t_n \in (0,t]$ visits $M^{n-1}(V, [0,t]^{n-1})$. It is known that Ω_1 is measurable, moreover the moment of the first such visit τ is a stopping time, and also the probability of the event $\tau = t$ is 0. It is also known that $X_n(\tau)$ is a regular point for $M^{n-1}(V, [0,t]^{n-1})$ a.s. All of this can be proved for Brownian motions on Carnot groups, in the same way as it was proved for standard Brownian motion in [12] (some of these facts can be derived from the potential theory of Markov processes developed in [2], but it is important that our processes are essentially hypoelliptic diffusions, meaning that they have everywhere positive infinitely differentiable density).

Since X_n satisfies strong Markov property we may consider a process with the same distribution as X_n, but started at the moment τ at $X_n(\tau)$ and its trajectories will have the same distribution as that of the process X_n. But since $X_n(\tau)$ is a regular point for $M^{n-1}(V, [0,t]^{n-1})$ a.s. then for any $\delta > 0$ a.s. on Ω_1 there is a point y on the trajectory of $X_n(t), t \in (\tau, \tau + \delta)$ such that $y \in M^{n-1}(V, [0,t]^{n-1})$. Consequently a.s. on Ω_1 there are infinite number of different times $s_1, s_2, \dots,$ on the interval $(0, t - \tau)$ such that $X_n(\tau + s_k) \in M^{n-1}(V, [0,t]^{n-1}))$.

So if the process $X_n(t_n), t_n \in (0,t]$ visits $M^{n-1}(V, [0,t]^{n-1})$, then $X_n(t_n), t_n \in (0,t]$ visits this set infinite number of times with the exception of the event of probability 0. Since $d \geqslant 2$ the probability of returning to a fixed point for X_n is 0. Therefore with probability 1 on Ω_1 the point $X_n(\tau)$ is not visited by $X_n(s), \tau < s \leqslant t$. Consider

$\tau_k = \inf\{s > \tau + \frac{1}{k} | X_n(s) \in M^{n-1}(V, [0,t]^{n-1})\}$, which is the first moment of hitting $M^{n-1}(V, [0,t]^{n-1})$ after $\tau + \frac{1}{k}$ (we set $\tau_k = +\infty$ if such moment does not exist). With probability 1 on Ω_1 there exists such k that $\tau_k < t$. It means that with probability 1 on Ω_1 there are at least 2 different points on the trajectory of $X_n(t_n), t_n \in (0,t]$ in $M^{n-1}(V, [0,t]^{n-1})$. Repeating this argument, starting at each τ_k instead of τ, we obtain that one additional point of $X_n(t_n), t_n \in (0,t]$ are in $M^{n-1}(V, [0,t]^{n-1})$ with probability 1 on Ω_1. But then we can claim that an infinite number of such points also exists with probability 1 on Ω_1.

So the event, that $M(V, [0,t]^{n-1} \times (0,t])$ is a nonempty finite set, is a subset of the set of probability 0. And since the probability space is complete then

$$P(M(V, [0,t]^{n-1} \times (0,t]) \text{ is a nonempty finite set}) = 0.$$

Repeating this proof for each X_k instead of X_n, we obtain that

$$P(M(V, [0,t]^n \backslash \{0\}) \text{ is a nonempty finite set}) = 0,$$

as claimed.

□

6. Proof of Theorem 1

Proof of Theorem 1. Suppose that the set $U_0 \cap V \cap \Theta$ is empty. It is enough to show that $M(U_0 \cap V, (0,t]^n)$ is empty with probability 1 for arbitrary $t > 0$. We are going to apply the main result of [16], and to provide the lower bound (6) in [16], we will use Theorem 4 with $G = G_1 \times \ldots \times G_n$ and $f = F$.

To start we check the conditions of Theorem 4. There are non-negative numbers $\lambda_1, \ldots, \lambda_n$, for which the condition for $x \in \Theta$ fails at the fixed point $x \in U_0$ selected in Theorem 4. Then we can choose γ for Theorem 4 to be equal $\lambda_i p_i(j)$ for the corresponding L_{ij}. We can see that such choice of γ obviously satisfies the required scaling condition, and $m_f(x) = m_f(y)$ for all y in some neighbourhood of x with $f(y) = f(x)$, since $x \in NS(f)$ and the value of $m_f(y)$ is uniquely determined by the ranks of the sets of vector fields considered in the definition of $NS(f)$. Therefore all the conditions of Theorem 4 are satisfied.

But then we can claim the conclusion of Theorem 4 gives us the condition (6) in Theorem 1 of [16] with measure $\mu(0, \cdot)$ as measure ν, $r_i = \lambda_i$ and $\delta \leqslant 0$ (note that the notation for Carnot groups G_i in [16] is similar, in particular $Q(i)$ in [16] is the same as Q_i). To prove this statement we take an optimal choice of i_k in the definition of $m_f(x)$ in Theorem 4 and the corresponding $t_i(j)$ defined as in the condition for $x \in \Theta$ (the correspondence is the same as the correspondence between L_{i_k} and $L_{i,t_i(j)}$). Note that $t_i(j)$ satisfies the first condition in the definition for $x \in \Theta$ and we can estimate

$$\delta = (Q_\gamma - \sum_{q=1}^{m_2} \gamma(i_q)) - \sum_{i=1}^{n} \lambda_i (Q_i - 2) = \sum_{i=1}^{n} \lambda_i (2 - \sum_{j=1}^{k_i} p_i(t_i(j)) \leqslant 0$$

where the last inequality has to hold because $x \notin \Theta$. It remains to note that $D(y, \varepsilon)$ in Theorem 4 is equivalent to $B(y_1, \varepsilon^{r_1}) \times \ldots \times B(y_n, \varepsilon^{r_n})$ in [16] in the following sense: the latter set belong to $D(y, a_1\varepsilon)$ and contains $D(y, a_2\varepsilon)$ for some positive a_1, a_2. This statement is a consequence of the known equivalence of homogeneous norms on G (see Proposition 5.1.4 in [3]), since both sets are in fact the balls with radius ε w.r.t some homogeneous norms.

We proved that for each $x \in U_0$ the condition (6) in Theorem 1 of [16] holds with $\delta \leqslant 0$ for all $y \in U_0$ in some neighbourhood V_x of x. Therefore we can apply Theorem 1 of [16] with $V = V_x$. If $\delta < 0$ then the set $A = M(U_0 \cap V_x, (0,t]^n)$ is empty as claimed by Theorem 1 in [16], and in the case $\delta = 0$ it can be seen from the proof of this

Theorem that the set A is at most finite with probability 1. But due to Theorem 5 the set $M(U_0 \cap V_x, (0,t]^n)$ is in fact always empty with probability 1 and since the choice of x was arbitrary and $U_0 \cap V$ can be covered by a countable number of sets from the family $\{V_x, x \in U_0 \cap V\}$, the same statement is also true for arbitrary V.

Suppose that $U_0 \cap V \cap \Theta$ is nonempty. To prove that $M(U_0 \cap V, (0,t]^n)$ is infinite with positive probability for any fixed $t > 0$ we use Theorem 10 and Theorem 11 from [14] in the case when the processes, that were considered in [14], are Brownian motions on Carnot groups. We can select M in [14] to be a small neighbourhood of $x \in U_0 \cap V \cap \Theta$, such that $M \subset \Theta \cap V$ and $M \cap H \subset U_0$. The set A in Theorem 10 from [14] can be chosen as $[s,t]^n$ with any $s \in (0,t)$ and the function ψ can be any continuous non-negative function with support inside M, which is positive at some $x \in M \cap H$. Such choice eliminates the singularities related to starting points, so we may set $\lambda_{2j-1} = 0$ in the main condition of Theorem 11 from [14], because the corresponding multipliers are not present in the integral (15) from [14]. Then the main condition of Theorem 11 from [14] is satisfied with $k_j = 2 - Q_j$, since it is essentially the same condition as the condition for $x \in \Theta$. To see this we need to link the selection of linear independent rows in Theorem 11 from [14] with the linear independence of vectors $L_{ij}^{x_i} F(x)$. But the matrix $R'_{\sigma,x} \cdot T$, considered in Theorem 11 from [14], taken without its odd numbered row blocks of the size $d \times (2nd - 2m)$ (those correspond to λ_{2j-1} and therefore irrelevant in our case), is the same, for the case of Brownian motions on Carnot groups, as the matrix $g'(a)$ in the proof of Theorem 4 above. And as was shown there, Lemma 5 provides the required connection.

The result that we obtained using Theorem 11 from [14] provides the main condition in the Theorem 10 from [14], because of the upper bounds for q_{2n} using density estimates for X_i. And it is easy to check, using other known properties of the densities for X_i, that all other conditions in the Theorem 10 from [14] are also satisfied (for example it is known that q_{2n} in Theorem 10 from [14] is always positive for our choice of processes). Then the conclusion of Theorem 10 from [14] is that there exists a non-zero limit in L_2 as $\varepsilon \to 0+$ of the corresponding approximations:

$$\gamma_\varepsilon = \int_s^t \int_s^t \dots \int_s^t \psi(X_1(s_1), \dots, X_n(s_n)) f_{m(n-1),\varepsilon}(F(X_1(s_1), \dots, X_n(s_n))) ds_1 \dots ds_n$$

where $f_{m(n-1),\varepsilon}$ is an approximation of δ_0 defined in [14].

But if there are no points on the trajectory of $(X_1(t_1), \dots, X_n(t_n))$, $t_1 \in [s,t], \dots, t_n \in [s,t]$ that belong to $M \cap H$, then γ_ε is equal to zero for small enough ε. So if this happens with probability 1 then the a.s. limit of γ_ε is equal to zero. This proves that $M(U_0 \cap V, [s,t]^n)$ has to be nonempty with positive probability. Then by Theorem 5 we conclude that $M(U_0 \cap V, (0,t]^n)$ is infinite with positive probability.

□

7. Self-intersections of Brownian motions on Carnot groups

In this section we consider the case of multiple self-intersections for a single process X, or more precisely the condition for the $(X(t_1), \dots, X(t_n))$, $t_1 \geqslant 0, \dots, t_n \geqslant 0$ to visit a given relatively open subset of the set H. The self-intersections of the process $f(X(t))$, $t > 0$ with some infinitely differentiable function f is a special case of this construction.

Assume that there is only one d-dimensional Carnot group G, and we have a special case of our definitions when $G_i = G$. Then we may keep all the notation related to group structure and function F (in particular $H = \{x \in G^n | F(x) = 0\}$) as well as the definitions of $NS(F) \subset G^n$ and of the set $\Theta \subset G^n$.

Suppose that $X(t)$, $t \geqslant 0$ is a Brownian motion on a Carnot group G, defined as in [14, 16]. Denote $M_s(V,T) = \{(x_1,\ldots,x_n) \in V | \exists (t_1,\ldots,t_n) \in T, t_1 < t_2 < \ldots < t_n : X(t_i) = x_i, i = 1,\ldots,n\}$.

Theorem 6. *Suppose that $V \subset G^n$ is an open set and denote $U_0 = NS(F) \cap H$. Then if $U_0 \cap \Theta \cap V$ is nonempty set then $M_s(U_0 \cap V, (0,t]^n)$ is infinite with positive probability for all $t > 0$. Otherwise, if $U_0 \cap \Theta \cap V$ is an empty set, then $M_s(U_0 \cap V, (0,+\infty)^n)$ is empty with probability 1.*

Proof. Suppose that $U_0 \cap \Theta \cap V$ is nonempty set. We will show that $M_s(U_0 \cap V, [t,2t] \times [3t,4t] \times \ldots [(2n-1)t, 2nt])$ is nonempty with positive probability for any $t > 0$. To do that we apply Theorem 1 and Theorem 2 from [14].

Note that we can take M from [14] to be a small neighbourhood of $x \in U_0 \cap V \cap \Theta$, such that $M \subset \Theta \cap V$ and $M \cap H \subset U_0$. Moreover ψ can be any continuous non-negative function with support inside M, which is positive at some $x \in M \cap H$ (we do not need additional multipliers for ψ, that were used in Theorem 3 from [14]). Then we can use Theorem 1 and Theorem 2 from [14] in the same way as we used Theorem 10 and Theorem 11 from [14] in the proof of Theorem 1 above. Because of our choice of the set $A = [t,2t] \times \ldots [(2n-1)t, 2nt]$ there is no singularity in the integral (5) from [14] related to the starting point x so we may set $\lambda_1 = 0$. Also many of the permutations σ in Theorem 1 from [14] can be skipped, since the corresponding integral is zero. And for those permutations that are left we can also set $\lambda_{2j-1} = 0$ since the corresponding multipliers can be omitted in the integral (5) from [14]. The latter means that all the conditions of Theorem 1 from [14] can be checked in the same way as it was done in the proof of Theorem 1 for Theorem 10 from [14].

We obtain that

$$\gamma_\varepsilon = \int_t^{2t} \int_{3t}^{4t} \ldots \int_{(2n-1)t}^{2nt} \psi(X(s_1),\ldots,X(s_n)) f_{m(n-1),\varepsilon}(F(X(s_1),\ldots,X(s_n))) ds_1 \ldots ds_n$$

converges in L_2 to a non-zero limit as $\varepsilon \to 0+$. But that means that $M_s(U_0 \cap V, [t,2t] \times [3t,4t] \times \ldots [(2n-1)t, 2nt])$ is nonempty with positive probability, because otherwise such limit should be zero a.s.

To show that the number of points in $M_s(U_0 \cap V, [t,2t] \times [3t,4t] \times \ldots [(2n-1)t, 2nt])$ is infinite with positive probability it is enough to repeat the argument from the proof of Theorem 5. More precisely we can consider the distribution of $X([(2n-1)t, 2nt]) = \{X(s), s \in [(2n-1)t, 2nt]\}$, under the condition that $X([0,(2n-1)t]) = \{X(s), s \in [0,(2n-1)t]\}$ is fixed, and prove that if $X([(2n-1)t, 2nt])$ intersects the set

$$M_s^{n-1} = p_{n,G}(X([t,2t]) \times \ldots \times X([(2n-3)t,(2n-2)t]) \times G \cap U_0 \cap V),$$

where $p_{n,G}$ is the canonical projection of G^n on the last coordinate, then it does so at the infinite number of different points with positive probability. We obtain that $M_s(U_0 \cap V, (0,t]^n)$ is infinite with positive probability for all $t > 0$.

Suppose that $U_0 \cap \Theta \cap V$ is an empty set. It is enough to show that $M_s(U_0 \cap V, [s_1,t_1] \times \ldots \times [s_n,t_n])$ is empty with probability 1 for all $0 < s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n$, because $M_s(U_0 \cap V, (0,+\infty)^n)$ is the union of the countable number of such sets.

Let $P(x)$ be the probability for the distribution of X, started at $x = X(0)$. Denote as $X_i(t), t \geqslant 0$, $i = 1,\ldots,n$ the family of independent copies of X. We will denote the corresponding probabilities as $P_i(x)$, $i = 1,\ldots,n$ (they are the same as P but correspond to different variables). Then $M(U_0 \cap V, [0,t_1 - s_1] \times \ldots \times [0, t_n - s_n])$ is empty with probability 1 if and only if

$$P_1(x_1)P_2(x_2)\ldots P_n(x_n) I(X_1,\ldots,X_n) = 0$$

for all $x_i = X_i(0)$, where

$$I(X_1,\ldots,X_n) = 1_{\{\exists r_1\in[0,t_1-s_1],\ldots,r_n\in[0,t_n-s_n]:(X_1(r_1),\ldots,X_n(r_n))\in U_0\cap V\}}.$$

Note that it is known that $I(\cdot)$ and the result of any number of consequent integrations with $P_i(x_i)$ of $I(X_1,\ldots,X_n)$ is a jointly measurable function of all free variables, including x_i. By Theorem 1 the statement above holds under our assumptions.

On the other hand, by the Markov property for X, $M_s(U_0\cap V,[s_1,t_1]\times\ldots\times[s_n,t_n])$ is empty with probability 1 if and only if

$$P(x)P_1(X(s_1))P_2(X_1(s_2-s_1))\ldots P_n(X_{n-1}(s_n-s_{n-1}))I(X_1,\ldots,X_n)=0$$

for all $x = X(0)$. If we denote as $p(t,x,y)$ the transition density of X then this statement can be rewritten as (also using Fubini theorem):

$$\int\limits_{G^n} P_1(y_1)P_2(y_2)\ldots P_n(y_n)p(s_1,x,y_1)p(s_2-t_1,X_1(t_1-s_1),y_2)\ldots$$
$$\ldots p(s_n-t_{n-1},X_{n-1}(t_{n-1}-s_{n-1}),y_n)I(X_1,\ldots,X_n)dy_1\ldots dy_n=0$$

But this equality holds under our assumptions, since all multipliers with p in the integral above are bounded.

□

8. Examples

Let us discuss some examples that show applications of the results of this paper as well as their limitations.

Let $d\geqslant 3$ and consider a Carnot group G such that the corresponding set of vector fields is defined as follows: $L_1=\frac{\partial}{\partial x_1}+\sum\limits_{i=3}^{d}x_{i-1}\frac{\partial}{\partial x_i}$, $L_i=\frac{\partial}{\partial x_i}$, $i=2,\ldots,d$. Note that $[L_i,L_1]=L_{i+1}$, $i=2,\ldots,d-1$. It is known that such group exists (see [15] where the explicit form of the group action on $(G,\bullet)$ was shown). Moreover $p_1=p_2=1$, $p_i=i-1$, $i=3,\ldots,d$ and $Q=1+\frac{d(d-1)}{2}$. The corresponding Brownian motion on G can be defined as follows

$$X(t)=x\bullet\left(W_1(t),W_2(t),\int\limits_0^t W_2(s)\circ dW_1(s),\int\limits_0^t\int\limits_0^{s_2}W_2(s_1)\circ dW_1(s_1)\circ dW_1(s_2),\ldots\right)$$

for any $t>0$, where W_1 and W_2 are independent standard Brownian motions with $W_1(0)=W_2(0)=0$. In the following we fix all G_i, $i=1,\ldots,n$ to be equal to such group G and consequently each X_i has the same distribution as X.

Let us first consider an example of a function F, which has non-trivial structure of the set $NS(F)$ of "non-singular" points.

Example 1. Take $n=2$, $m=1$ and $F:G^2\to\mathbb{R}$ defined as $F(x)=g((x_1)_2)-(x_2)_2$, $x=(x_1,x_2)\in G^2$, where $g:\mathbb{R}\to\mathbb{R}$ is an arbitrary infinitely differentiable function. Then $NS(F)$ contains all points in G^2 except those for which $(x_1)_2\in A=\{t\in\mathbb{R}|g'(t)=0,\exists s_n\to t,n\to+\infty:g'(s_n)\neq 0\}$.

The case $m=1$ in Theorem 1 actually does not require us to consider $NS(F)$, but it may be possible to find similar examples if $m\geqslant 2$.

Consider $m=2$, $n\geqslant 3$ and $f_i(x)=(x_1,x_{k_i})$, $k_i\geqslant 3$, $i=1,\ldots,n$. Take

$$F(x)=(f_1(x_1)-f_2(x_2),\ldots,f_{n-1}(x_{n-1})-f_n(x_n)).$$

It is not hard to check that $G^n\backslash NS(F)=\bigcup\limits_{i\neq j}\{(x_1,\ldots,x_n)\in G^n|(x_i)_{k_i-1}=(x_j)_{k_j-1}\}$ so all the "singular" points inside a linear subspace H are those, which belong to one of the several linear subspaces of the dimension equal to the dimension of H minus 1, obtained by adding an additional equality $(x_i)_{k_i-1} = (x_j)_{k_j-1}$ for some $i \neq j$.

We could also investigate whether the process $(X_1(t_1), X_2(t_2), \dots, X_n(t_n))$ visits the set $H \backslash NS(F)$ using the same approach, by considering several linear functions F_1, such that $H \backslash NS(F)$ is a union of all the sets $\{x \in G^n | F_1(x) = 0\}$. Each such function may be obtained by adding an extra coordinate $(x_i)_{k_i-1} - (x_j)_{k_j-1}$ for some fixed $i \neq j$. Note that if the condition $x \in \Theta$ holds for F_1, then it also holds for F, so if we knew that $H \cap NS(F) \cap \Theta \cap V$ was empty then the corresponding set for F_1 is guaranteed to be empty as well. At the same time if $H \cap NS(F) \cap \Theta \cap V$ is not empty then there is no need to consider the points in $H \backslash NS(F)$, because we already can prove that $M(H, (0,t]^n)$ is nonempty with positive probability. Unfortunately the structure of $NS(F_1)$ is more complicated, since it depends on the choice of k_i and we have $\frac{n(n-1)}{2}$ functions F_1 to consider, so we are not going to describe it explicitly. And generally speaking we do not know how to guarantee that this procedure can be continued for $G^n \backslash NS(F_1)$. Therefore we cannot claim that $(X_1(t_1), X_2(t_2), \dots, X_n(t_n))$ does not visit the set $H \backslash NS(F)$ with probability 1 even when it is known that it does not visit the set $H \cap NS(F)$ with probability 1.

The condition that $x \in \Lambda$ for the given f_i, $i = 1, \dots, n$ is the same in each point and equivalent to the condition (2) with $q_i = p_{k_i} = k_i - 1$. Since $k_i \geqslant 3$ can be selected arbitrarily then p_{k_i} can be any integer greater than 1, and consequently the condition (2) can contain any possible integer values $q_i \geqslant 2$. Note that it is also possible to have $q_i = 1$, for example if we set $f_i(x) = (x_1, x_2)$, but in this case is the structure of $NS(F)$ is different, so it has to be considered separately.

We can apply Theorem 1 and Theorem 2 to obtain the following example.

Example 2. Consider n independent two-dimensional processes Y_i, $i = 1, \dots, n$, such that

$$Y_i(t) = (W_{i1}(t), \int_0^t \int_0^{s_{k_i-2}} \dots \int_0^{s_2} W_{i2}(s_1) \circ dW_{i1}(s_1) \circ dW_{i1}(s_2) \dots \circ dW_{i1}(s_{k_i-2})), i = 1, \dots, n,$$

where W_{i1}, W_{i2}, $i = 1, \dots, n$ are independent standard Brownian motions with $W_{i1}(0) = W_{i2}(0) = 0$. If

$$\frac{1}{k_1 - 1} + \dots + \frac{1}{k_n - 1} > n - 2$$

then for any $t > 0$ with positive probability there are $t_1 \in (0,t)$, $\dots$, $t_n \in (0,t)$ such that $Y_1(t_1) = Y_2(t_2) = \dots = Y_n(t_n)$. If

$$\frac{1}{k_1 - 1} + \dots + \frac{1}{k_n - 1} \leqslant n - 2$$

then with probability 1 there are no $t_1 \in (0,t), \dots, t_n \in (0,t)$ such that $Y_1(t_1) = Y_2(t_2) = \dots = Y_n(t_n)$ and at the same time $Z_i(t_i) \neq Z_j(t_j)$ for all $i \neq j$, where

$$Z_i(t) = \int_0^t \int_0^{s_{k_i-3}} \dots \int_0^{s_2} W_{i2}(s_1) \circ dW_{i1}(s_1) \circ dW_{i1}(s_2) \dots \circ dW_{i1}(s_{k_i-3})$$

if $k_i > 3$ and $Z_i(t) = W_{i2}(t)$ if $k_i = 3$.

According to the Theorem 6 we have the same result if all Y_i are replaced with one process Y with the same distribution and we forbid equalities $t_i = t_j$ for all $i \neq j$. In this case it is interesting to see what happens if all f_i are the same, or in other words if $k_i = k_j = k$ for all i, j.

Example 3. Consider the process

$$Y(t) = (W_1(t), \int_0^t \int_0^{s_{k-2}} \dots \int_0^{s_2} W_2(s_1) \circ dW_1(s_1) \circ dW_1(s_2) \dots \circ dW_1(s_{k-2}))$$

where W_1, W_2 are independent standard Brownian motions with $W_1(0) = W_2(0) = 0$. If $k = 3$ and $n \leqslant 3$ then for any $t > 0$ and any open set V with positive probability there are $0 < t_1 < \dots < t_n < t$ such that $Y(t_1) = Y(t_2) = \dots = Y(t_n) \in V$. If $k \geqslant 4$, $n \geqslant 3$ or if $k = 3$, $n \geqslant 4$ then with probability 1 there are no $0 < t_1 < \dots < t_n < t$ such that $Y(t_1) = Y(t_2) = \dots = Y(t_n)$ and at the same time $Z(t_i) \neq Z(t_j)$ for all $i \neq j$, where

$$Z(t) = \int_0^t \int_0^{s_{k-3}} \dots \int_0^{s_2} W_2(s_1) \circ dW_1(s_1) \circ dW_1(s_2) \dots \circ dW_1(s_{k-3})$$

if $k > 3$ and $Z(t) = W_2(t)$ if $k = 3$.

Consider $m = 3$, $n = 2$ and $f_i(x) = (x_1, x_2, x_{k_i})$, $i = 1, 2$ with $k_i \geqslant 3$. By the definition of Λ, we have $x = (x_1, x_2) \in \Lambda$ if and only if the dimension of the linear span of $L_{11}f_1(x_1), L_{12}f_1(x_1), L_{11}f_2(x_2), L_{12}f_2(x_2)$ is equal to 3. But in our case it means that $(x_1)_{k_1-1} \neq (x_2)_{k_2-1}$. The only case when it is not possible for $x \in H$ is when $k_1 = k_2 = 3$, since $(x_1)_2 = (x_2)_2$ for $x \in H$. But the case $k_1 = k_2 = 3$ is the well-known case of intersections of two independent 3-dimensional hypoelliptic processes with the same distribution, and it was proved in [4] that such intersections do not exist with probability 1. So in the following we assume that $k_1 > 3$ or $k_2 > 3$. Then it is also not hard to check that $G^2 \backslash NS(F) = \{x \in G^2 | (x_1)_{k_1-1} = (x_2)_{k_2-1}\}$ and by Theorem 1 and Theorem 2 we obtain the following.

Example 4. Consider 2 independent 3-dimensional processes Y_i, $i = 1, 2$, such that

$$Y_i(t) = (W_{i1}(t), W_{i2}(t), \int_0^t \int_0^{s_{k_i-2}} \dots \int_0^{s_2} W_{i2}(s_1) \circ dW_{i1}(s_1) \circ dW_{i1}(s_2) \dots \circ dW_{i1}(s_{k_i-2})), i = 1, 2,$$

where W_{i1}, W_{i2}, $i = 1, 2$ are independent standard Brownian motions with $W_{i1}(0) = W_{i2}(0) = 0$. If $k_1 > 3$ or $k_2 > 3$ then for any $t > 0$ and any open set V with positive probability there are $t_1 \in (0, t)$, $t_2 \in (0, t)$ such that $Y_1(t_1) = Y_2(t_2) \in V$.

Additionally by Theorem 6 we obtain the same result for self-intersections.

Example 5. Consider the process

$$Y(t) = (W_1(t), W_2(t), \int_0^t \int_0^{s_{k-2}} \dots \int_0^{s_2} W_2(s_1) \circ dW_1(s_1) \circ dW_1(s_2) \dots \circ dW_1(s_{k-2}))$$

where W_1, W_2 are independent standard Brownian motions with $W_1(0) = W_2(0) = 0$. If $k > 3$ then for any $t > 0$ and any open set V with positive probability there are $0 < t_1 < t_2 < t$ such that $Y(t_1) = Y(t_2) \in V$.

If $k = 3$ then we again have the case of 3-dimensional hypoelliptic process considered in [4] and it was proved there that the corresponding self-intersections do not exist with probability 1.

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