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NONPARAMETRIC ESTIMATION OF LINEAR MULTIPLIER FOR PROCESSES DRIVEN BY A HERMITE PROCESS

We study the problem of nonparametric estimation of the linear multiplier function $\theta(t)$ for processes satisfying stochastic differential equations of the type

$$dX_t = \theta(t)X_t dt + \epsilon\, dZ_t^{q,H}, X_0 = x_0, 0 \leq t \leq T$$

where $\{Z_t^{q,H}, t \geq 0\}$ is a Hermite process with known order q and known self-similarity parameter $H \in (\frac{1}{2}, 1)$. We study the asymptotic behaviour of the estimator of the unknown function $\theta(t)$ as $\epsilon \to 0$.

1. Introduction

Statistical inference for fractional diffusion processes satisfying stochastic differential equations driven by a fractional Brownian motion (fBm) has been studied earlier and a comprehensive survey of various methods is given in Mishura (2008) and in Prakasa Rao (2010). There has been a recent interest to study similar problems for stochastic processes driven by mixed fractional Brownian motion, sub-fractional Brownian motion, α-stable noises, fractional Levy processes and general Gaussian processes. Prakasa Rao (2024) investigated nonparametric estimation of linear multiplier in SDEs driven by general Gaussian processes.

In modeling processes with possible long range dependence, it is possible that no special functional form is available for modeling the trend a priori and it is necessary to estimate the trend function based on the observed process over an interval. This problem of estimation is known as nonparametric function estimation in classical statistical inference (cf. Prakasa Rao (1983)).

Our aim in this paper is to study nonparametric estimation of the trend function, which is a linear multiplier, when the process is governed by a stochastic differential equation driven by a Hermite process following the ideas of density function estimation and regression function estimation in classical statistical inference. Several methods are present for nonparametric function estimation as described in Prakasa Rao (1983). The method of kernels is widely used for the estimation of a density function or a regression function and it is known that the properties of such an estimator do not depend on the choice of the kernel in general but on the choice of the bandwidth. Properties of the estimators of a density function and a regression function, using the method of kernels, are described in Prakasa Rao (1983). Our aim is to propose a kernel type estimator for the trend function and study its properties. We will show that the kernel type estimator is uniformly consistent over a class of trend functions and obtain the asymptotic distribution of the estimator in the presence of small noise. We will also obtain the optimum rate of convergence of the kernel type estimators for the trend function. Results derived in this paper will be useful when there is no information on the functional form of the trend coefficient and the trend has to be estimated from the observed path of the underlying process. For a comprehensive review of parametric and nonparametric inference for

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processes driven by fractional processes such as sub-fractional Brownian motion, mixed fractional Brownian motion and fractional Levy process, see Prakasa Rao (2026).

Hermite processes is another class of non-Gaussian processes which have been proposed as the driving forces for modeling stochastic phenomenon with long range dependence in a non-Gaussian environment (cf. Taqqu (1978); Lawrance and Kottegoda (1977)). Statistical inference for Vasicek-type model driven by Hermite process was studied in Nourdin and Diu Tran (2017) (cf. Diu Tran (2018)). Coupek and Kriz (2025) discuss parametric and nonparametric inference for nonlinear stochastic differential equations driven by Hermite processes. Our aim in this paper is to study nonparametric estimation of the linear multiplier in the trend function when the process is governed by a stochastic differential equation driven by a Hermite process.

2. Hermite Processes

As the definition and properties of Hermite processes are not widely known, we now give a short review of the properties of Hermite processes for completeness following Diu Tran (2018) and Tudor (2013).

Let $W = \{W(h), h \in L^2(R)\}$ be a Brownian field defined on a probability space $(\Omega, \mathcal{F}, P)$, that is, a centered Gaussian family of random variables satisfying

$$E[W(h)W(g)] = < h, g >_{L^2(R)}$$

for any $h, g \in L^2(R)$. For every $q \geq 1$, the q-th Wiener chaos $\mathcal{H}_q$ is defined as the closed linear subspace of $L^2(\Omega)$ generated by the family of random variables $\{H_q(W(h)), h \in L^2(R), ||h||_{L^2(R)} = 1\}$, where H_q is the q-th Hermite polynomial. Recall that $H_1(x) = x, H_2(x) = x^2 - 1, H_3(x) = x^3 - 3x, \ldots$. The mapping $I_q^W(h) = H_q(W(h))$ can be extended to a linear isometry between $L_s^2(R^q)$, the space of symmetric square integrable functions on R^q equipped with the modified norm $\sqrt{q!}||.||_{L^2(R^q)}$, and the q-th Wiener chaos $\mathcal{H}_q$. If $f \in L_s^2(R^q)$, , then the random variable $I_q^W(f)$, defined below, is called the *multiple Wiener integral* of the function f of order q and it is denoted by

$$I_q^W(f) = \int_{R^q} f(\psi_1, \ldots, \psi_q) dW_{\psi_1} \ldots dW_{\psi_q}.$$

For the existence and the properties of such multiple Wiener integrals, see Nualart (2006) and Major (2014).

We now define the Hermite process and discuss its properties following Diu Tran (2018) and Tudor (2013).

Definition 2.1. The *Hermite process* $\{Z_t^{q,H}, t \geq 0\}$ of order $q \geq 1$ and the self-similarity parameter $H \in (\frac{1}{2}, 1)$ is defined as

$$Z_t^{q,H} = c(q,H) \int_{R^q} (\int_0^t \Pi_{j=1}^q (s - \psi_j)_+^{H_0 - \frac{3}{2}} ds) dW_{\psi_1} \ldots dW_{\psi_q} \tag{2.1}$$

where

$$c(q,H) = \sqrt{\frac{H(2H-1)}{q!(\beta(H_0 - \frac{1}{2}, 2 - 2H_0))^q}} \tag{2.2}$$

and

$$H_0 = 1 + \frac{H-1}{q}. \tag{2.3}$$

Note that $H_0 \in (1 - \frac{1}{2q}, 1)$. The integral in (2.1) is a multiple Wiener integral of order q. Observe that $Z_0^{q,H} = 0$ *a.s.*The constant $c(q,H)$ is chosen in (2.1) so that $E[(Z_1^{q,H})^2] = 1$. Hermite process of order $q = 1$ is the fractional Brownian motion. It is the only Hermite process that is Gaussian. The Hermite process of order $q = 2$ is called the *Rosenblatt process.*

For any two processes $X = \{X_t, t \geq 0\}$ and $Y = \{Y_t, t \geq 0\}$, we say that $\{X_t, t \geq 0\} \stackrel{\Delta}{=} \{Y_t, t \geq 0\}$ if the finite dimensional distributions of the process X are the same as the finite dimensional distributions of the process Y.

The following properties are satisfied by the Hermite processes.

Theorem 2.1. *Suppose $Z = \{Z_t^{q,H}, t \geq 0\}$ is the Hermite process of order $q \geq 1$ and the parameter $H \in (\frac{1}{2}, 1)$. Then the process is centered and the following properties hold:*

(i) the process Z is self-similar, that is, for all $a > 0$

$$\{Z_{at}^{q,H}, t \geq 0\} \stackrel{\Delta}{=} \{a^H Z_t^{q,H}, t \geq 0\};$$

(ii) the process Z has stationary increments, that is, for any $h > 0$,

$$\{Z_{t+h}^{q,H} - Z_h^{q,H}, t \geq 0\} \stackrel{\Delta}{=} \{Z_t^{q,H}, t \geq 0\};$$

(iii) the covariance function of the process Z is given by

$$E[Z_t^{q,H} Z_s^{q,H}] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), t \geq 0, s \geq 0;$$

(iv) the process Z has long range dependence property, that is,

$$\sum_{n=0}^{\infty} E[Z_1^{q,H}(Z_{n+1}^{q,H} - Z_n^{q,H})] = \infty;$$

(v) for any $\alpha \in (0, H)$ and any interval $[0, T] \subset R_+$, the process $\{Z_t^{q,H}, 0 \leq t \leq T\}$ admits a version with Holder continuous sample paths of order α; and

(vi) for every $p \geq 1$, there exists a positive constant $C_{p,q,H}$ such that

$$E[|Z_t^{q,H}|^p] \leq C_{p,q,H} t^{pH}, t \geq 0.$$

For proof of this theorem, see Tudor (2013).

Wiener integral with respect to the Hermite process:

The Wiener integral of a non-random function f, with respect to the Hermite process $Z = \{Z_t^{q,H}, t \geq 0\}$, is studied in Maejima and Tudor (2007) and it is denoted by

$$E[\int_R f(u) dZ_u^{q,H}]$$

whenever it exists. It is known that the Wiener integral of the function f with respect to the Hermite process of order q and parameter H exists if

$$\int_R \int_R |f(u)f(v)||u-v|^{2H-2}\, dudv < \infty. \tag{2.4}$$

Let $|\mathcal{H}|$ denote the class of functions f satisfying the inequality (2.4). It can be shown that , for any $f, g \in |\mathcal{H}|$,

$$E[\int_R f(u) dZ_u^{q,H} \int_R g(u) dZ_u^{q,H}] = H(2H-1) \int_R \int_R f(u) g(v) |u-v|^{2H-2} dudv. \tag{2.5}$$

Furthermore, the Wiener integral of f with respect to the process Z admits the representation

$$\int_R f(u) dZ_u^{q,H} = c(q,H) \int_{R^q} (\int_R f(u) \Pi_{j=1}^q (u - \psi_j)_+^{H_0 - \frac{3}{2}} du) dW_{\psi_1} \ldots dW_{\psi_q} \tag{2.6}$$

where $c(q,H)$ and H_0 are as defined by the equations (2.2) and (2.3) respectively.

We now derive a maximal inequality for the Hermite process Z of order q and self-similarity parameter H. Let $Z_t^{q,H,*} = \sup_{0\leq s\leq t}|Z_s^{q,H}|$.

Maximal inequalities for processes such as the fractional Brownian motion and sub-fractional Brownian motion are known and are reviewed in Prakasa Rao (2014, 2017, 2020) following the property of self-similarity for such processes. Maximal inequalities for general Gaussian processes are presented in Li and Shao (2001), Berman (1985), Marcus and Rosen (2006) and Borovkov et al. (2017) among others. As far as we are aware, there are no maximal inequalities available for Hermite processes in the literature.

For any process $X = \{X_t, t \geq 0\}$, define

$$X_t^* = \sup\{|X_s|, 0 \leq s \leq t\}.$$

Since the Hermite process $\{Z_t^{q,H}, t \geq 0\}$ is a self-similar process with self-similarity parameter H, it follows that

$$Z_{at}^{q,H} \triangleq a^H Z_t^{q,H}$$

for every $t \geq 0$ and $a > 0$. This in turn implies that

$$Z_{at}^{q,H,*} \triangleq a^H Z_t^{q,H,*}$$

for every $a > 0$. Hence the following result holds.

Theorem 2.2. *Let $T > 0$ and $\{Z_t^{q,H}, t \geq 0\}$ be the Hermite process with self-similarity parameter H. Suppose that $E[|Z_1^{q,H,*}|^p] < \infty$ for some $p > 0$.. Then*

$$E[(Z_T^{q,H,*})^p] = K(p,q,H)T^{pH}$$

where $K(p,q,H) = E[(Z_1^{q,H,*})^p]$.

Proof. The result is a consequence of the fact $Z_T^{q,H,*} \triangleq T^H\ Z_1^{q,H,*}$ by self-similarity. $\square$

3. Preliminaries

Let $Z = \{Z_t^{q,H}, t \geq 0\}$ be a Hermite process with known parameters q, H such that $H \in (\frac{1}{2}, 1)$. Consider the problem of estimating the unknown function $\theta(t), 0 \leq t \leq T$ (linear multiplier) from the observations $\{X_t, 0 \leq t \leq T\}$ of process satisfying the stochastic differential equation

$$dX_t = \theta(t)X_t dt + \epsilon\, dZ_t^{q,,H}, X_0 = x_0, 0 \leq t \leq T \tag{3.1}$$

and study the properties of the estimator as $\epsilon \to 0$. Consider the differential equation in the limiting system of (3.1), that is, for $\epsilon = 0$, given by

$$\frac{dx_t}{dt} = \theta(t)x_t, x_0, 0 \leq t \leq T. \tag{3.2}$$

Observe that

$$x_t = x_0 \exp\{\int_0^t \theta(s)ds).$$

We assume that the following condition holds:

(A_1) The trend coefficient $\theta(t)$, over the interval $[0, T]$, is bounded by a constant L.

Lemma 3.1. *Let the condition (A_1) hold and $\{X_t, 0 \leq t \leq T\}$ and $\{x_t, 0 \leq t \leq T\}$ be the solutions of the equations (3.1) and (3.2) respectively. Then, with probability one,*

$$|X_t - x_t| < e^{Lt}\epsilon \sup_{0\leq s\leq t}|Z_s^{q,H}| \tag{3.3}$$

and

$$\sup_{0\leq t\leq T} E(X_t - x_t)^2 \leq e^{2LT}\epsilon^2 T^{2H}. \tag{3.4}$$

Proof. **Proof of (a):** Let $u_t = |X_t - x_t|$. Then by (A_1); we have,

$$
\begin{aligned}
u_t &\leq \int_0^t |\theta(v)(X_v - x_v)|dv + \epsilon |Z_t^{q,H}| \\
&\leq L\int_0^t u_v dv + \epsilon \sup_{0\leq s\leq t} |Z_s^{q,H}|.
\end{aligned} \tag{3.5}
$$

Applying the Gronwall's lemma (cf. Lemma 1.12, Kutoyants (1994), p. 26), it follows that

$$u_t \leq \epsilon \sup_{0\leq s\leq t} |Z_s^{q,H}| e^{Lt}. \tag{3.6}$$

Proof of (b): From the equation (3.3), we have

$$
\begin{aligned}
E(X_t - x_t)^2 &\leq e^{2Lt}\epsilon^2 E(\sup_{0\leq s\leq t} |Z_s^{q,H}|)^2 \\
&= e^{2Lt}\epsilon^2 t^{2H}.
\end{aligned} \tag{3.7}
$$

Hence

$$\sup_{0\leq t\leq T} E(X_t - x_t)^2 \leq e^{2LT}\epsilon^2 T^{2H}. \tag{3.8}$$

□

4. Main Results

Let $\Theta_0(L)$ denote the class of all functions $\theta(.)$ with the same bound L. Let $\Theta_k(L)$ denote the class of all functions $\theta(.)$ which are uniformly bounded by the same constant L and which are k-times differentiable with respect to t satisfying the condition

$$|\theta^{(k)}(x) - \theta^{(k)}(y)| \leq L_1|x-y|, x, y \in R$$

for some constant $L_1 > 0$. Here $g^{(k)}(x)$ denotes the k-th derivative of $g(.)$ at x for $k \geq 0$. If $k = 0$, we interpret the function $g^{(0)}(x)$ as $g(x)$.

Let $G(u)$ be a bounded function with compact support $[A, B]$ with $A < 0 < B$ satisfying the condition

(A_2) $\int_A^B G(u)du = 1.$

It is obvious that the following conditions are satisfied by the function $G(.)$:

: (i) $\int_{-\infty}^{\infty} |G(u)|^2 du < \infty$;

: (ii) $\int_{-\infty}^{\infty} |u^{k+1} G(u)|^2 du < \infty$.

We define a kernel type estimator $\hat{\theta}_t$ of the function $\theta(t)$ by the relation

$$\widehat{\theta}_t X_t = \frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) dX_\tau \tag{4.1}$$

where the normalizing function $\varphi_\epsilon \to 0$ as $\epsilon \to 0$. Let $E_\theta(.)$ denote the expectation when the function $\theta(.)$ is the linear multiplier.

Theorem 4.1. *Suppose that the linear multiplier $\theta(.) \in \Theta_0(L)$ and the function $\varphi_\epsilon \to 0$ and $\epsilon^2\varphi_\epsilon^{2H-2} \to 0$ as $\epsilon \to 0$. Suppose the conditions $(A_1) - (A_2)$ hold. Then, for any $0 < a \leq b < T$, the estimator $\hat{\theta}_t$ is uniformly consistent, that is,*

$$\lim_{\epsilon\to 0} \sup_{\theta(.)\in\Theta_0(L)} \sup_{a\leq t\leq b} E_\theta(|\hat{\theta}_t X_t - \theta(t)x_t|^2) = 0. \tag{4.2}$$

In addition to the conditions (A_1) and (A_2), suppose the following condition holds: $(A_3)\int_{-\infty}^{\infty} u^j G(u)du = 0$ for $j = 1, 2, ...k$.

Theorem 4.2. *Suppose that the function $\theta(.) \in \Theta_{k+1}(L)$ and the conditions $(A_1) - (A_3)$ hold. Further suppose that $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$. Then,*

$$\limsup_{\epsilon\to 0} \sup_{\theta(.)\in\Theta_{k+1}(L)} \sup_{a\le t\le b} E_\theta(|\hat{\theta}_t X_t - \theta(t)x_t|^2)\epsilon^{-\min(2,\frac{2(k+1)}{k+2-H})} < \infty. \tag{4.3}$$

Theorem 4.3. *Suppose that the function $\theta(.) \in \Theta_{k+1}(L)$ for some $k > 1$ and the conditions $(A_1) - (A_3)$ hold. Further suppose that $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$. Let $J(t) = \theta(t)x_t$. Then, as $\epsilon \to 0$, the asymptotic distribution of*

$$\epsilon^{\frac{-(k+1)}{k-HK+2}}(\hat{\theta}_t X_t - J(t) - \frac{J^{(k+1)}(t)}{(k+1)!}\int_{-\infty}^{\infty} G(u)u^{k+1}\ du)$$

has mean zero and variance

$$\sigma_H^2 = H(2H-1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u)G(v)|u-v|^{2H-2}dudv$$

where

$$\frac{\partial^2 R_{H,K}(t,s)}{\partial t\partial s} = \alpha_{H,K}(t^{2H}+s^{2H})^{K-2}(ts)^{2H-1} + \beta_{H,K}|t-s|^{2HK-2}.$$

5. Proofs of Theorems

Proof of Theorem 4.1. From the inequality

$$(a+b+c)^2 \le 3(a^2+b^2+c^2), a, b, c \in R,$$

it follows that

$$\begin{aligned}
E_\theta[|\hat{\theta}(t)X_t - \theta(t)x_t|^2] &= E_\theta[|\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)(\theta(\tau)X_\tau - \theta(\tau)x_\tau)\,d\tau \\
&\quad + \frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)\theta(\tau)x_\tau d\tau - \theta(t)x_t \\
&\quad + \frac{\epsilon}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right) dZ_\tau^{q,H}|^2] \\
&\le 3E_\theta[|\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)(\theta(\tau)X_\tau - \theta(\tau)x_\tau)d\tau|^2] \\
&\quad + 3E_\theta[|\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)\theta(\tau)x_\tau d\tau - \theta(t)x_t|^2] \\
&\quad + 3\frac{\epsilon^2}{\varphi_\epsilon^2}E_\theta[|\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right) dZ_\tau^{q,H}|^2] \\
&= I_1 + I_2 + I_3 \text{ (say)}.
\end{aligned} \tag{5.1}$$

By the boundedness condition on the function $\theta(.)$, the inequality (3.3) in Lemma 3.1 and the condition (A_2), and applying the Hölder inequality, it follows that

(5.2)

$$I_1 = 3E_\theta\left|\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)(\theta(\tau)X_\tau - \theta(\tau)x_\tau)d\tau\right|^2$$

$$
\begin{aligned}
&= 3E_\theta \left| \int_{-\infty}^{\infty} G(u)\left(\theta(t+\varphi_\epsilon u) X_{t+\varphi_\epsilon u} - \theta(t+\varphi_\epsilon u) x_{t+\varphi_\epsilon u}\right) du \right|^2 \\
&\leq 3(B-A) \int_{-\infty}^{\infty} |G(u)|^2 L^2 E \left| X_{t+\varphi_\epsilon u} - x_{t+\varphi_\epsilon u} \right|^2 \; du \text{ (by using the condition } (A_1)) \\
&\leq 3(B-A) \int_{-\infty}^{\infty} |G(u)|^2 \;\; L^2 \sup_{0 \leq t+\varphi_\epsilon u \leq T} E_\theta \left| X_{t+\varphi_\epsilon u} - x_{t+\varphi_\epsilon u} \right|^2 \; du \\
&\leq 3(B-A) L^2 e^{2LT} \epsilon^2 T^{2H} \int_{-\infty}^{\infty} |G(u)|^2 du \;\; \text{(by using (3.4))}
\end{aligned}
$$

which tends to zero as $\epsilon \to 0$. For the term I_2, by the boundedness condition on the function $\theta(.)$, the condition (A_2) and the Hölder inequality, it follows that

(5.3)
$$
\begin{aligned}
I_2 &= 3E_\theta \left| \frac{1}{\varphi_\epsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) \theta(\tau) x_\tau d\tau - \theta(t) x_t \right|^2 \\
&= 3 \left| \int_{-\infty}^{\infty} G(u) \left(\theta(t+\varphi_\epsilon u) x_{t+\varphi_\epsilon u} - \theta(t) x_t \right) \; du \right|^2 \\
&\leq 3(B-A) L^2 \varphi_\epsilon^2 \int_{-\infty}^{\infty} |uG(u)|^2 du \;\; \text{(by } (A_2)).
\end{aligned}
$$

The last term tends to zero as $\varphi_\epsilon \to 0$. We will now get an upper bound on the term I_3. Note that

(5.4)
$$
\begin{aligned}
I_3 &= 3\frac{\epsilon^2}{\varphi_\epsilon^2} E_\theta \left| \int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) dZ_\tau^{q,H} \right|^2 \\
&= 3\frac{\epsilon^2}{\varphi_\epsilon^2} \int_0^T \int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) G\left(\frac{\tau' - t}{\varphi_\epsilon}\right) |\tau - \tau'|^{2H-2} d\tau d\tau' \\
&= C\frac{\epsilon^2}{\varphi_\epsilon^2} \varphi_\epsilon^{2H} H(2H-1) \int_R \int_R G(u) G(v) |u-v|^{2H-2} du dv \\
&\leq C_1 \epsilon^2 \varphi_\epsilon^{2H-2}
\end{aligned}
$$

for some positive constant C_1. The inequality in third line of the eqution (5.4) given above follows by the change of variable technique using the transformation of the vector $(\frac{\tau-t}{\varphi_\epsilon}, \frac{\tau'-t}{\varphi_\epsilon})$ to the vector (u,v) and invoking the computation of covariance of two wiener integrals with respect to the Hermite process given in the equation (2.5). Theorem 4.1 is now proved by using the equations (5.1) to (5.4). □

Proof of Theorem 4.2. Let $J(t) = \theta(t) x_t$. By the Taylor's formula, for any $x \in R$,

$$
J(y) = J(x) + \sum_{j=1}^{k} J^{(j)}(x) \frac{(y-x)^j}{j!} + [J^{(k)}(z) - J^{(k)}(x)] \frac{(y-x)^k}{k!}
$$

for some z such that $|z - x| \leq |y - x|$. Using this expansion, the equation (3.2) and the condition (A_3) in the expression for I_2 defined in the proof of Theorem 4.1, it follows that

$$\begin{aligned} I_2 &= 3\left[\int_{-\infty}^{\infty} G(u)\left(J(t+\varphi_\epsilon u)-J(t)\right)\ du\right]^2 \\ &= 3[\sum_{j=1}^{k} J^{(j)}(t)(\int_{-\infty}^{\infty} G(u)u^j du)\varphi_\epsilon^j (j!)^{-1} \\ &\quad +(\int_{-\infty}^{\infty} G(u)u^k (J^{(k)}(z_u)-J^{(k)}(t))du\ \varphi_\epsilon^k (k!)^{-1}]^2 \end{aligned}$$

for some z_u such that $|z_u - t| \le C|\varphi_\epsilon u|$. Hence

$$\begin{aligned} I_2 &\le 3L^2\left[\int_{-\infty}^{\infty} |G(u)u^{k+1}|\varphi_\epsilon^{k+1}(k!)^{-1} du\right]^2 \\ &\le 3L^2(B-A)(k!)^{-2}\varphi_\epsilon^{2(k+1)}\int_{-\infty}^{\infty} G^2(u)u^{2(k+1)}\ du \\ &\le C_2\varphi_\epsilon^{2(k+1)} \end{aligned} \tag{5.5}$$

for some positive constant C_2. Combining the equations (5.2)- (5.5), we get that there exists a positive constant C_3 such that

$$\sup_{a\le t\le b} E_\theta|\hat{\theta}_t X_t - \theta(t)x_t|^2 \le C_3(\epsilon^2 + \varphi_\epsilon^{2(k+1)} + \epsilon^2\varphi_\epsilon^{2H-2}).$$

Choosing $\varphi_\epsilon = \epsilon^{\frac{1}{k+2-H}}$, we get that

$$\limsup_{\epsilon\to 0}\ \sup_{\theta(.)\in\Theta_{k+1}(L)}\ \sup_{a\le t\le b} E_\theta|\theta(t)X_t - \theta(t)x_t|^2\epsilon^{-\min(2,\frac{2(k+1)}{k+2-H})} < \infty.$$

This completes the proof of Theorem 4.2. □

Proof of Theorem 4.3. Let $\alpha = \frac{k+1}{k+H-2}$. Note that $0<\alpha<1$ since $0<H<1$. From (3.1), we obtain that

$$\begin{aligned} &\epsilon^{-\alpha}(\hat{\theta}(t)X_t - \theta(t)x_t) \\ &= \epsilon^{-\alpha}[\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)(\theta(\tau)X_\tau - \theta(\tau)x_\tau)\ d\tau \\ &\quad +\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)\theta(\tau)x_\tau d\tau - \theta(t)x_t + \frac{\epsilon}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right) dZ_\tau^{q,H}] \\ &= \epsilon^{-\alpha}[\int_{-\infty}^{\infty} G(u)(\theta(t+\varphi_\epsilon u)X_{t+\varphi_\epsilon u} - \theta(t+\varphi_\epsilon u)x_{t+\varphi_\epsilon u})\ du \\ &\quad +\int_{-\infty}^{\infty} G(u)(\theta(t+\varphi_\epsilon u)x_{t+\varphi_\epsilon u} - \theta(t)x_t)\ du \\ &\quad +\frac{\epsilon}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right) dZ_\tau^{q,H}]. \\ &= R_1 + R_2 + R_3 \quad \text{(say)}. \end{aligned} \tag{5.6}$$

By the boundedness condition on the function $\theta(.)$ and part (a) of Lemma 3.1, it follows that

$$R_1 \le \epsilon^{-\alpha}|\int_{-\infty}^{\infty} G(u)(\theta(t+\varphi_\epsilon u)X_{t+\varphi_\epsilon u} - \theta(t+\varphi_\epsilon u)x_{t+\varphi_\epsilon u})du| \tag{5.7}$$

$$\leq \epsilon^{-\alpha}\epsilon L \int_{-\infty}^{\infty} |G(u)|X_{t+\varphi_\epsilon u} - x_{t+\varphi_\epsilon u}|du$$
$$\leq Le^{LT}\epsilon^{1-\alpha}\int_{-\infty}^{\infty}|G(u)| \sup_{0\leq t+\varphi_\epsilon u\leq T}|Z^{q,H}_{t+\varphi_\epsilon u}|du.$$

Applying the Markov's inequality, it follows that, for any $\eta > 0$,

$$\begin{aligned}(5.8)\quad P(|R_1| > \eta) &\leq \epsilon^{1-\alpha}\eta^{-1}Le^{LT}\int_{-\infty}^{\infty}|G(u)|E_\theta(\sup_{0\leq t+\varphi_\epsilon u\leq T}|Z^{q,H}_{t+\varphi_\epsilon u}|)du\\
&\leq \epsilon^{1-\alpha}\eta^{-1}Le^{LT}\int_{-\infty}^{\infty}|G(u)||E_\theta[(\sup_{0\leq t+\varphi_\epsilon u\leq T}(Z^{q,H}_{t+\varphi_\epsilon u})^2]|^{1/2}du\\
&\leq \epsilon^{1-\alpha}\eta^{-1}Le^{LT}CT^H\int_{-\infty}^{\infty}|G(u)|du\end{aligned}$$

from the maximal inequality for a Hermite process proved in Theorem 2.2 for some constant $C > 0$, and the last term tends to zero as $\epsilon \to 0$. Let $J_t = \theta(t)x_t$. By the Taylor's formula, for any $t \in [0, T]$,

$$J_t = J_{t_0} + \sum_{j=1}^{k+1} J^{(j)}_{t_0}\frac{(t-t_0)^j}{j!} + [J^{(k+1)}_{t_0+\gamma(t-t_0)} - J^{(k+1)}_{t_0}]\frac{(t-t_0)^{k+1}}{(k+1)!}$$

where $0 < \gamma < 1$ and $t_0 \in (0, T)$. Applying the Condition (A_3) and the Taylor's expansion, it follows that

$$\begin{aligned}(5.9)\qquad R_2 &= \epsilon^{-\alpha}[\sum_{j=1}^{k+1} J^{(j)}_t(\int_{-\infty}^{\infty} G(u)u^j\ du)\varphi^j_\epsilon(j!)^{-1}\\
&\quad +\frac{\varphi^{k+1}_\epsilon}{(k+1)!}\int_{-\infty}^{\infty} G(u)u^{k+1}(J^{(k+1)}_{t+\gamma\varphi_\epsilon u} - J^{(k+1)}_t)\ du]\\
&= \epsilon^{-\alpha}\frac{J^{(k+1)}_t}{(k+1)!}\int_{-\infty}^{\infty} G(u)u^{k+1}\ du\\
&\quad +\varphi^{k+1}_\epsilon\epsilon^{-\alpha}\frac{1}{(k+1)!}\int_{-\infty}^{\infty} G(u)u^{k+1}(J^{(k+1)}_{t+\gamma\varphi_\epsilon u} - J^{(k+1)}_t)\ du.\end{aligned}$$

.

Observing that $\theta(t) \in \Theta_{k+1}(L)$, we obtain that

$$\begin{aligned}(5.10)\qquad &\frac{1}{(k+1)!}\int_{-\infty}^{\infty} G(u)u^{k+1}(J^{(k+1)}_{t+\gamma\varphi_\epsilon u} - J^{(k+1)}_t)du\\
&\leq \frac{1}{(k+1)!}\int_{-\infty}^{\infty}|G(u)u^{k+1}(J^{(k+1)}_{t+\gamma\varphi_\epsilon u} - J^{(k+1)}_t)|du\\
&\leq \frac{L\varphi_\epsilon}{(k+1)!}\int_{-\infty}^{\infty}|G(u)u^{k+2}|du.\end{aligned}$$

Combining the equations given above, it follows that

$$\begin{aligned}(5.11)\qquad &\epsilon^{-\alpha}(\hat{\theta}_t X_t - J(t) - \frac{J^{(k+1)}_t}{(k+1)!}\int_{-\infty}^{\infty} G(u)u^{k+1}\ du)\\
&= O_p(\epsilon^{1-\alpha}) + O_p(\epsilon^{-\alpha}\varphi^{k+2}_\epsilon) + \epsilon^{1-\alpha}\varphi^{-1}_\epsilon\int_0^T G(\frac{\tau-t}{\varphi_\epsilon})dZ^{q,H}_\tau.\end{aligned}$$

Let

$$\eta_\epsilon(t) = \epsilon^{\frac{1-H}{k-H+2}}\varphi_\epsilon^{-1}\int_0^T G(\frac{\tau-t}{\varphi_\epsilon})dZ_\tau^{q,H}. \tag{5.12}$$

Note that $E[\eta_\epsilon(t)] = 0$, and

$$\begin{aligned} E([\eta_\epsilon(t)]^2) &= (\epsilon^{\frac{1-H}{k-H+2}}\varphi_\epsilon^{-1})^2 E([\int_0^T G(\frac{\tau-t}{\varphi_\epsilon})dZ_\tau^{q,H}]^2) \\ &= (\epsilon^{\frac{1-H}{k-H+2}}\varphi_\epsilon^{-1})^2[\varphi_\epsilon^{2H}H(2H-1)\int_R\int_R G(u)G(v)|u-v|^{2H-2}dudv]. \end{aligned}$$

Choosing $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$, we get that

$$E([\eta_\epsilon(t)]^2) = H(2H-1)\int_R\int_R G(u)G(v)|u-v|^{2H-2}dudv.$$

From the choice of φ_ϵ and α, it follows that

$$\epsilon^{1-\alpha}\varphi_\epsilon^{-1} = {\varphi_\epsilon}^H,$$

and,

(5.13)

$$\begin{aligned} &Var[\varphi_\epsilon^{-H}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)dZ_\tau^{q,H}] \\ &= \varphi_\epsilon^{-2H}H(2H-1)\int_0^T\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)G\left(\frac{\tau'-t}{\varphi_\epsilon}\right)|\tau-\tau'|^{2H-2}d\tau d\tau' \end{aligned}$$

and the last term tends to

$$H(2H-1)\int_R\int_R G(u)G(v)|u-v|^{2H-2}dudv = \sigma_H^2$$

as $\epsilon \to 0$. Applying the Slutsky's theorem and the equations derived above, it can be checked that the random variable

$$\epsilon^{-\alpha}(\hat{\theta}_t X_t - J_t - \frac{J_t^{(k+1)}}{(k+1)!}\int_{-\infty}^{\infty} G(u)u^{k+1}\,du)$$

has a limiting distribution as $\epsilon \to 0$ as that of the family of random variables

$$\varphi_\epsilon^{-H}\int_{-\infty}^{\infty} G\left(\frac{\tau-t}{\varphi_\epsilon}\right)dZ_\tau^{q,H}$$

as $\epsilon \to 0$ which has mean zero and variance σ_H^2. This completes the proof of Theorem 4.3. □

6. Alternate Estimator for the Multiplier $\theta(.)$

Let $\Theta_\rho(L_\gamma)$ be a class of functions $\theta(t)$ uniformly bounded by a constant L and k-times continuously differentiable for some integer $k \geq 1$ with the k-th derivative satisfying the Hölder condition of the order $\gamma \in (0,1)$,

$$|\theta^{(k)}(t) - \theta^{(k)}(s)| \leq L_\gamma|t-s|^\gamma, \rho = k+\gamma$$

and observe that $\rho > H$ since $\rho > 1$ and $H \in (\frac{1}{2}, 1)$. Suppose the process $\{X_t, 0 \leq t \leq T\}$ satisfies the stochastic differential equation given by the equation (3.1) where the linear multiplier is an unknown function in the class $\Theta_\rho(L_\gamma)$ and further suppose that $x_0 > 0$ and is *known*. From the Lemma 3.1, it follows that

$$|X_t - x_t| \leq \epsilon e^{Lt}\sup_{0\leq s\leq T}|Z_s^{q,H}|.$$

Let

$$A_t = \{\omega : \inf_{0\leq s\leq t} X_s(\omega) \geq \frac{1}{2}x_0 e^{-Lt}\}$$

and let $A = A_T$. Following the technique suggested in Kutoyants (1994), p. 156, we define another process Y with the differential

$$dY_t = \theta(t)I(A_t)dt + \epsilon 2x_0^{-1}e^{LT}I(A_t)\, dZ_t^{q,H}, 0 \leq t \leq T.$$

We will now construct an alternate estimator of the linear multiplier $\theta(.)$ based on the process Y over the interval $[0, T]$. Define the estimator

$$\tilde{\theta}(t) = I(A)\frac{1}{\varphi_\epsilon}\int_0^T G(\frac{t-s}{\varphi_\epsilon})dY_s$$

where the kernel function $G(.)$ satisfies the conditions $(A_1) - (A_3)$. Observe that

$$\begin{aligned}
E|\tilde{\theta}(t)-\theta(t)|^2 &= E_\theta|I(A)\frac{1}{\varphi_\epsilon}\int_0^T G(\frac{t-s}{\varphi_\epsilon})(\theta(s)-\theta(t))ds \\
&\quad +I(A^c)\theta(t) + I(A)\frac{\epsilon}{\varphi_\epsilon}\int_0^T G(\frac{t-s}{\varphi_\epsilon})2x_0^{-1}e^{LT}dZ_s^{q,H}|^2 \\
&\leq 3E_\theta|I(A)\int_R G(u)[\theta(t+u\varphi_\epsilon)-\theta(t)]du|^2 + 3|\theta(t)|^2[P(A^c)]^2 \\
&\quad +3\frac{\epsilon^2}{\varphi_\epsilon^2}|E[I(A)\int_0^T G(\frac{t-s}{\varphi_\epsilon})2x_0^{-1}e^{LT}dZ_s^{q,H}]|^2 \\
&= D_1 + D_2 + D_3.\ \text{(say)}.
\end{aligned}$$

Applying the Taylor's theorem and using the fact that the function $\theta(t) \in \Theta_\rho(L_\gamma)$, it follows that

$$D_1 \leq C_1\frac{1}{(k+1)!}\varphi_\epsilon^{2\rho}\int_R |G^2(u)u^{2\rho}|du.$$

Note that, by Lemma 3.1,

$$\begin{aligned}
P(A^c) &= P(\inf_{0\leq t\leq T} X_t < \frac{1}{2}x_0e^{-LT}) \\
&\leq P(\inf_{0\leq t\leq T}|X_t - x_t| + \inf_{0\leq t\leq T} x_t < \frac{1}{2}x_0e^{-LT}) \\
&\leq P(\inf_{0\leq t\leq T}|X_t - x_t| < -\frac{1}{2}x_0e^{-LT}) \\
&\leq P(\sup_{0\leq t\leq T}|X_t - x_t| > \frac{1}{2}x_0e^{-LT}) \\
&\leq P(\epsilon e^{LT}\sup_{0\leq t\leq T}|Z_t^{q,H}| > \frac{1}{2}x_0e^{-LT}) \\
&= P(\sup_{0\leq t\leq T}|Z_t^{q,H}| > \frac{x_0}{2\epsilon}e^{-2LT}) \\
&\leq (\frac{x_0}{2\epsilon}e^{-2LT})^{-2}E[\sup_{0\leq t\leq T}|Z_t^{q,H}|^2] \\
&\leq (\frac{x_0}{2\epsilon}e^{-2LT})^{-2}C_2T^{2H}
\end{aligned}$$

by Theorem 2.2 for some positive constant C_2. The upper bound obtained above and the fact that $|\theta(s)| \leq L, 0 \leq s \leq T$ leads an upper bound for the term D_2. We have used the inequality

$$x_t = x_0 \exp(\int_0^t \theta(s)ds) \geq x_0 e^{-Lt}$$

in the computations given above. Applying Theorem 2.1, it follows that

$$\begin{aligned}
& E[|I(A)\int_0^T G(\frac{t-s}{\varphi_\epsilon})2x_0^{-1}e^{LT}dZ_s^{q,H}|^2] \\
\leq\ & CE[|\int_0^T G(\frac{t-s}{\varphi_\epsilon})dZ_s^{q,H}|^2] \\
=\ & C\,Var[\int_0^T G(\frac{t-s}{\varphi_\epsilon})dZ_s^{q,H}] \\
=\ & C\varphi_\epsilon^{2H}H(2H-1)\int_R\int_R G(u)G(v)|u-v|^{2H-2}dudv
\end{aligned}$$

for some positive constant C which leads to an upper bound on the term D_3. Combining the above estimates, it follows that

$$E|\tilde{\theta}(t)-\theta(t)|^2 \leq C_1\varphi_\epsilon^{2\rho} + C_2\epsilon^4 + C_3\epsilon^2\varphi_\epsilon^{2H}$$

for some positive constants $C_i, i = 1,2,3$. Choosing $\varphi_\epsilon = \epsilon^{\frac{1}{\rho-H}}$, we obtain that

$$E|\tilde{\theta}(t)-\theta(t)|^2 \leq C_4(\epsilon^{\frac{2\rho}{\rho-H}} + \epsilon^4)$$

for some positive constants C_4. Hence we obtain the following result implying the uniform consistency of the estimator $\tilde{\theta}(t)$ as an estimator of $\theta(t)$ as $\epsilon \to 0$.

Theorem 6.1. *Let $\theta \in \Theta_\rho(L)$ where $\rho > H$. Let $\varphi_\epsilon = \epsilon^{1/(\rho-H)}$. Suppose the conditions $(A_1)-(A_3)$ hold. Then, for any interval $[a,b] \subset [0,T]$,*

$$\limsup_{\epsilon\to 0} \sup_{\theta(.)\in\Theta_\rho(L)} \sup_{a\leq t\leq b} E|\tilde{\theta}(t)-\theta(t)|^2\epsilon^{-\min(4,\frac{2\rho}{\rho-H})} < \infty.$$

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References

1. S. M. Berman, *An asymptotic bound for the tail of the distribution of the maximum of a Gaussian process*, Ann. Inst. H. Poincare Probab. Statist. **21** (1985), 47-57.
2. K. Borovkov, Y. Mishura, A. Novikov and M. Zhitlukhin, *Bounds for expected maxima of Gaussian processes and their discrete approximations*, Stochastics **89** (2017), 21-37. https://doi.org/10.1080/17442508.2015.1126282
3. P. Coupek, and P. Kriz, *Inference for SDEs driven by Hermite processes* (2025), arXiv: 2506.16916. https://arxiv.org/abs/2506.16916
4. T. T. Diu Tran, *Contributions to the asymptotic study of Hermite driven processes*, Ph.D. Thesis, University of Luxembourg, 2018.
5. Y. A. Kutoyants, *Identification of dynamical systems with small noise*, Dordrecht, Kluwer, 1994. https://doi.org/10.1007/978-94-011-1020-4
6. A. J. Lawrance and N. T. Kottegoda, *Stochastic modeling of river flow time series*, J. Roy. Stat. Soc., Ser.A **140** (1977), 1-47. https://doi.org/10.2307/2344516
7. W. V. Li and Q. M. Shao, *Gaussian processes: Inequalities, small ball probabilities and applications*, Stochastic Processes: Theory and Methods, Handbook of Statistics **19**, Elseveir, Amsterdam, 2001, pp. 533-597. https://doi.org/10.1016/S0169-7161(01)19019-X

8. M. Maejima and C. Tudor, *Wiener integrals with respect to the Hermite process and a non-central limit theorem*, Stoch. Anal. Appl. **25** (2007), no. 5, 1043-1056. https://doi.org/10.1080/07362990701540519
9. P. Major, *Multiple Wiener-Ito integrals, with applications to limit theorems*, Second Edition, Lecture Notes in Mathematics **849**, Springer, 2014. https://doi.org/10.1007/978-3-319-02642-8
10. M. B. Marcus, and J. Rosen, *Markov processes, Gaussian processes and local times*, Cambridge Studies in Advanced Mathematics, Vol. 100, Cambridge University Press, Cambridge, 2006. https://doi.org/10.1017/CBO9780511617997
11. M. N. Mishra and B. L. S. Prakasa Rao, *Nonparametric estimation of linear multiplier for fractional diffusion processes*, Stoch. Anal. Appl. **29** (2011), no. 4, 706-712. https://doi.org/10.1080/07362994.2011.581109
12. Y. Mishura, *Stochastic calculus for fractional Brownian motion and related processes*, Berlin, Springer, 2008. https://doi.org/10.1007/978-3-540-75873-0
13. I. Nourdin and T. T. Diu Tran, *Statistical inference for Vasicek-type model driven by Hermite process* (2017), arxiv: 1712.05915. https://arxiv.org/abs/1712.05915
14. D. Nualart, *The Malliavin calculus and related topics*, 2nd Edition, Springer, New York, 2006. https://doi.org/10.1007/3-540-28329-3
15. B. L. S. Prakasa Rao, *Nonparametric functional estimation*, Academic Press, New York, 1983. https://doi.org/10.1016/C2013-0-11326-8
16. B. L. S. Prakasa Rao, *Minimum L_1-norm estimation for fractional Ornstein-Uhlenbeck type process*, Theor. Prob. and Math. Statist. **71** (2005), 181-189. https://doi.org/10.1090/S0094-9000-05-00657-5
17. B. L. S. Prakasa Rao, *Statistical inference for fractional diffusion processes*, Wiley, London, 2010. https://doi.org/10.1002/9780470667125
18. B. L. S. Prakasa Rao, *Maximal inequalities for fractional Brownian motion: An overview*, Stoch. Anal. Appl. **32** (2014), no. 3, 450-479. https://doi.org/10.1080/07362994.2014.886264
19. B. L. S. Prakasa Rao, *On some maximal and integral inequalities for sub-fractional Brownian motion*, Stoch. Anal. Appl. **35** (2017), no. 2, 279-287. https://doi.org/10.1080/07362994.2016.1241182
20. B. L. S. Prakasa Rao, *More on maximal inequalities for sub-fractional Brownian motion*, Stoch. Anal. Appl. **38** (2020), no. 2, 238-247. https://doi.org/10.1080/07362994.2019.1686395
21. B. L. S. Prakasa Rao, *Nonparametric estimation of linear multiplier in stochastic differential equations driven by general Gaussian processes*, J. Nonparametric Stat. **36** (2024), no. 4, 981-993. https://doi.org/10.1080/10485252.2023.2288842
22. B. L. S. Prakasa Rao, *Advances in statistical inference for processes driven by fractional processes: inference for fractional processes*, World Scientific, Singapore, 2026.
23. M. S. Taqqu, *A representation for self-similar processes*, Stoch. Process Appl. **7** (1978), no. 1, 55-64. https://doi.org/10.1016/0304-4149(78)90037-6
24. C. Tudor, *Analysis of variations for self-similar processes*, Springer, Switzerland, 2013. https://doi.org/10.1007/978-3-319-00936-0

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