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MONTE CARLO APPROXIMATION FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH INTERACTION

We study a Monte Carlo approximation scheme for stochastic differential equations with interaction. The initial measure is replaced by its empirical approximation constructed from independent samples, and the corresponding equation is discretized in time by the Euler–Maruyama scheme. For initial measures with finite moments of order $q > 2$, we derive a quantitative estimate for the empirical approximation in the Wasserstein distance of order two. Combining this estimate with stability estimates for stochastic flows with interaction, we obtain a convergence bound for the total Euler–Maruyama–Monte Carlo approximation. We also compare the particle-based approximation with the deterministic tensor-grid construction used in earlier work.

1. Introduction

Stochastic differential equations with interaction were introduced by A. A. Dorogovtsev; see [1, 2]. We consider equations of the form

$$
\begin{aligned}
&\mathrm{d}x(u,t) = a\big(x(u,t),\mu_t\big)\,\mathrm{d}t + \int_{\mathbb{R}^d} b\big(x(u,t),\mu_t,p\big)\,W(\mathrm{d}p,\mathrm{d}t),\\
&x(u,0)=u, \qquad u\in\mathbb{R}^d, \quad t\in[0,1], \qquad \mu_t=\mu_0\circ x(\cdot,t)^{-1}.
\end{aligned} \tag{1}
$$

In such equations, the coefficients depend not only on the current position of an individual particle, but also on the current distribution of the whole system. More precisely, if $x(u,t)$ denotes the trajectory of the particle starting from $u \in \mathbb{R}^d$, then the measure argument is given by the image measure

$$\mu_t = \mu_0 \circ x(\cdot,t)^{-1}.$$

Thus, μ_t describes the mass distribution generated by the stochastic flow. The dependence of the coefficients on μ_t reflects the interaction between particles and makes the equation essentially infinite-dimensional.

The numerical approximation of such equations requires, in particular, an approximation of the measure argument. In previous work [4], a deterministic spatial discretization of the initial measure was used. The idea of that approach was to replace the initial measure by a discrete measure supported on a regular grid and then to construct an approximation of the corresponding stochastic flow. Although this method gives convergence estimates, its computational cost grows rapidly with the dimension. Indeed, if the spatial domain is discretized by a regular grid with N nodes in each coordinate direction, then the total number of grid points is of order N^d.

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A natural alternative is to replace the deterministic spatial grid by a Monte Carlo approximation. In this case the initial measure is approximated by the empirical measure

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{U_i}, \tag{2}$$

where $U_1, \dots, U_N$ are independent random variables with common distribution μ_0. This approximation uses only N support points and therefore avoids the direct construction of an N^d-point grid.

The convergence of empirical measures in Wasserstein distances has been studied extensively. Let

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i} \tag{3}$$

be the empirical measure generated by independent random variables $X_1, \dots, X_N$ with common law μ. Fournier and Guillin [5] obtained explicit finite-sample estimates for

$$\mathbb{E}\gamma_p^p(\mu_N, \mu)$$

under moment assumptions. In particular, if the distribution μ has a finite moment of order q, for some $q > p$, then, outside the corresponding critical moment exponents, their estimates imply bounds of the form

$$\mathbb{E}\gamma_p^p(\mu_N, \mu) \le C \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-p}{q}}, & p > \frac{d}{2}, \\ N^{-\frac{1}{2}} \log(1+N) + N^{-\frac{q-p}{q}}, & p = \frac{d}{2}, \\ N^{-\frac{p}{d}} + N^{-\frac{q-p}{q}}, & p < \frac{d}{2}, \end{cases} \tag{4}$$

where the constant C depends on p, d, q and on the q-th moment of μ. Here the critical values are $q = 2p$ when $p \ge \frac{d}{2}$, and $q = \frac{dp}{d-p}$ when $p < \frac{d}{2}$. For the Wasserstein distance of order two, corresponding to $p = 2$, this gives, outside the critical values specified in Theorem 3.1,

$$\mathbb{E}\gamma_2^2(\mu_N, \mu) \le C \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}}, & d < 4, \\ N^{-\frac{1}{2}} \log(1+N) + N^{-\frac{q-2}{q}}, & d = 4, \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}}, & d > 4. \end{cases} \tag{5}$$

Dereich, Scheutzow and Schottstedt [6] studied the approximation of a probability measure by empirical measures interpreted as random quantizations. For the empirical measure μ_N defined by (3), they obtained asymptotic estimates for the averaged Wasserstein quantization error in the range $2p < d$, under the moment condition $q > dp/(d-p)$. In particular,

$$\mathbb{E}\gamma_p^p(\mu, \mu_N) \le C \left(\int_{\mathbb{R}^d} |x|^q \, \mu(\mathrm{d}x) \right)^{\frac{p}{q}} N^{-\frac{p}{d}}. \tag{6}$$

For $p = 2$, this yields, in dimensions $d > 4$,

$$\mathbb{E}\gamma_2^2(\mu, \mu_N) \le C \left(\int_{\mathbb{R}^d} |x|^q \, \mu(\mathrm{d}x) \right)^{\frac{2}{q}} N^{-\frac{2}{d}}. \tag{7}$$

Deterministic empirical approximations were studied by Bencheikh and Jourdain [7] on the real line. For $p \ge 1$, they considered the minimal deterministic Wasserstein error

$$e_N(\mu, p) := \inf_{x_1, \dots, x_N \in \mathbb{R}} \gamma_p \left(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}, \mu \right). \tag{8}$$

Their results characterize the order of convergence of $e_N(\mu,p)$ in terms of the quantile function, the absolutely continuous part of the measure, and the tail behaviour of μ. They also show that for measures with heavier tails the order can be slower and is determined by the tail behaviour of μ.

The estimates described above concern the approximation of a probability measure by an empirical or deterministic finitely supported measure. The problem considered in the present paper is different. We use an empirical approximation of the initial measure as part of a numerical scheme for a stochastic differential equation with interaction. Therefore, it is not sufficient to estimate only the Wasserstein distance between μ_0 and μ_0^N. One has to transfer this error through the stochastic flow and then combine it with the time discretization error.

Motivated by these considerations, we propose an Euler–Maruyama–Monte Carlo approximation scheme for stochastic differential equations with interaction. The initial measure μ_0 is replaced by the empirical measure μ_0^N defined in (2), and the corresponding stochastic equation is discretized in time by the Euler–Maruyama scheme.

The first step is to estimate the empirical approximation error of the initial measure. Under the assumption

$$\mu_0 \in \mathcal{M}_q(\mathbb{R}^d), \qquad q > 2, \tag{9}$$

and outside the critical values in (24), we use the non-asymptotic empirical Wasserstein bounds of Fournier and Guillin to obtain

$$\mathbb{E}\gamma_2^2(\mu_0,\mu_0^N) \le C\, R_{d,q}(N), \tag{10}$$

where

$$R_{d,q}(N) = \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}}, & d < 4, \\ N^{-\frac{1}{2}}\log(1+N) + N^{-\frac{q-2}{q}}, & d = 4, \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}}, & d > 4. \end{cases} \tag{11}$$

The second step is to transfer the approximation error of the initial measure to the corresponding stochastic flows. Using stability estimates with respect to the initial measure, we obtain a bound for the difference between the solution $x(u,t)$ of the original equation and the solution $x^N(u,t)$ of the equation starting from the empirical measure μ_0^N.

The final step is to add the Euler–Maruyama time discretization. If $x_n^N(u,t)$ denotes the Euler–Maruyama–Monte Carlo approximation, then for every compact set $K \subset \mathbb{R}^d$, every $\theta \in (0,1)$, and

$$\alpha = \frac{\theta}{2\theta + d},$$

we obtain the convergence estimate

$$\mathbb{E}\sup_{u\in K}\sup_{t\in[0,1]} |x(u,t) - x_n^N(u,t)| \le C_K\Big(R_{d,q}(N)^\alpha + n^{-\alpha}\Big), \tag{12}$$

where $C_K > 0$ is independent of N and n.

Thus, compared with the deterministic grid-based approximation, the proposed method replaces the spatial grid by random sampling. The number of support points used for the approximation of the initial measure is N, rather than N^d. Compared with the results of Fournier–Guillin and Dereich–Scheutzow–Schottstedt, the novelty of the present work is not the empirical approximation of the measure alone, but the propagation of this error through an interacting stochastic flow and its combination with a time discretization scheme. Compared with the deterministic empirical approximations studied by Bencheikh–Jourdain, the present method is based on random sampling, works in a multi-dimensional setting, and is designed for stochastic differential equations with interaction.

The paper is organized as follows. Section 2 defines the equation and the numerical scheme. Section 3 recalls the empirical Wasserstein bound. Section 4 proves the moment, regularity, measure-stability and time-discretization estimates. Section 5 establishes the full convergence theorem. Section 6 compares the Monte Carlo construction with the deterministic grid approximation from [4].

2. Equation and approximation scheme

For $p \geq 1$, denote by $\mathcal{M}_p(\mathbb{R}^d)$ the set of all probability measures μ on the Borel σ-field of $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |v|^p \, \mu(\mathrm{d}v) < +\infty.$$

For $r \geq 1$ and $\mu \in \mathcal{M}_r(\mathbb{R}^d)$, set

$$M_r(\mu) := \int_{\mathbb{R}^d} |z|^r \, \mu(\mathrm{d}z).$$

For $\mu, \nu \in \mathcal{M}_2(\mathbb{R}^d)$, we use the quadratic Wasserstein distance

$$\gamma_2^2(\mu,\nu) := \inf_{\varkappa \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \, \varkappa(\mathrm{d}x, \mathrm{d}y), \tag{13}$$

where $\Pi(\mu,\nu)$ is the set of all couplings of μ and ν.

Let W be a Wiener sheet on $\mathbb{R}^d \times [0,1]$. We consider

$$\begin{aligned} \mathrm{d}x(u,t) &= a\big(x(u,t), \mu_t\big)\, \mathrm{d}t + \int_{\mathbb{R}^d} b\big(x(u,t), \mu_t, p\big)\, W(\mathrm{d}p, \mathrm{d}t), \\ x(u,0) &= u, \qquad \mu_t = \mu_0 \circ x(\cdot,t)^{-1}. \end{aligned} \tag{14}$$

We assume that

$$a : \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \to \mathbb{R}^d$$

and

$$b : \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$$

are measurable, and that $p \mapsto b(x,\mu,p)$ belongs to $L_2(\mathbb{R}^d;\mathbb{R}^d)$ for every (x,μ).

Assumption 2.1. *There exists $L > 0$ such that, for all $x, y \in \mathbb{R}^d$ and $\mu, \nu \in \mathcal{M}_2(\mathbb{R}^d)$,*

$$|a(x,\mu) - a(y,\nu)| \leq L\big(|x-y| + \gamma_2(\mu,\nu)\big), \tag{15}$$

$$\int_{\mathbb{R}^d} |b(x,\mu,p) - b(y,\nu,p)|^2 \, \mathrm{d}p \leq L^2\big(|x-y|^2 + \gamma_2^2(\mu,\nu)\big), \tag{16}$$

Under Assumption 2.1, existence and pathwise uniqueness follow by the standard fixed-point argument for stochastic equations with interaction; see [1, 2].

It follows from Assumption 2.1 that there exists a constant $C > 0$ such that, for every $x \in \mathbb{R}^d$ and $\mu \in \mathcal{M}_2(\mathbb{R}^d)$,

$$|a(x,\mu)|^2 + \int_{\mathbb{R}^d} |b(x,\mu,p)|^2 \, dp \leq C\big(1 + |x|^2 + M_2(\mu)\big). \tag{17}$$

2.1. **Monte Carlo approximation of the initial measure.** Let $U_1, \ldots, U_N$ be i.i.d. with law μ_0, independent of W, and define

$$\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{U_i}. \tag{18}$$

Conditionally on $U_1, \ldots, U_N$, let x^N solve

$$\begin{aligned} &\mathrm{d}x^N(u,t) = a\big(x^N(u,t), \mu_t^N\big)\,\mathrm{d}t + \int_{\mathbb{R}^d} b\big(x^N(u,t), \mu_t^N, p\big)\, W(\mathrm{d}p, \mathrm{d}t), \\ &x^N(u,0) = u, \end{aligned} \tag{19}$$

where

$$\mu_t^N = \mu_0^N \circ x^N(\cdot,t)^{-1} = \frac{1}{N} \sum_{i=1}^N \delta_{x^N(U_i,t)}. \tag{20}$$

2.2. **Euler–Maruyama discretization.** Let $n \in \mathbb{N}$, $h = \frac{1}{n}$, $t_k = kh$, and

$$\kappa_n(s) = t_k, \qquad s \in [t_k, t_{k+1}).$$

The continuous-time Euler–Maruyama approximation is

$$\begin{aligned} x_n^N(u,t) = u &+ \int_0^t a\big(x_n^N(u,\kappa_n(s)), \mu_{\kappa_n(s)}^{N,n}\big)\,\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} b\big(x_n^N(u,\kappa_n(s)), \mu_{\kappa_n(s)}^{N,n}, p\big) W(\mathrm{d}p, \mathrm{d}s), \end{aligned} \tag{21}$$

where

$$\mu_t^{N,n} = \mu_0^N \circ x_n^N(\cdot,t)^{-1} = \frac{1}{N} \sum_{i=1}^N \delta_{x_n^N(U_i,t)}. \tag{22}$$

At the grid points,

$$\begin{aligned} x_{k+1}^{N,n}(u) = x_k^{N,n}(u) &+ h\, a\big(x_k^{N,n}(u), \mu_k^{N,n}\big) \\ &+ \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} b\big(x_k^{N,n}(u), \mu_k^{N,n}, p\big) W(\mathrm{d}p, \mathrm{d}s), \end{aligned} \tag{23}$$

with $x_0^{N,n}(u) = u$ and

$$\mu_k^{N,n} = \frac{1}{N} \sum_{i=1}^N \delta_{x_k^{N,n}(U_i)}.$$

The continuous-time interpolation and the analytical structure of the Euler–Maruyama scheme follow the framework developed in [4, Section 2]. The difference is that the deterministic grid approximation of the initial measure is replaced here by the random empirical measure (18).

3. Empirical approximation of the initial measure

The following is the specialization of Theorem 1 in [5] to the quadratic transportation cost.

Theorem 3.1 (Fournier–Guillin)**.** *Let $q > 2$, let $\mu \in \mathcal{M}_q(\mathbb{R}^d)$, and let*

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \qquad X_1, \ldots, X_N \text{ i.i.d. with law } \mu.$$

Assume

$$q \neq 4 \quad \text{if } d \leq 4, \qquad q \neq \frac{2d}{d-2} \quad \text{if } d > 4. \tag{24}$$

Then there exists $C = C(d, q, M_q(\mu))$ *such that*

$$\mathbb{E}\gamma_2^2(\mu, \mu^N) \leq C R_{d,q}(N), \tag{25}$$

where

$$R_{d,q}(N) = \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}}, & d < 4, \\ N^{-\frac{1}{2}} \log(1+N) + N^{-\frac{q-2}{q}}, & d = 4, \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}}, & d > 4. \end{cases} \tag{26}$$

Remark 3.2. *The exceptional values in* (24) *are the values at which the two powers in the bound coincide. If* $\mu \in \mathcal{M}_q$ *for a critical exponent* q*, then* $\mu \in \mathcal{M}_{q'}$ *for every* $q' \in (2, q)$*. Hence Theorem 3.1 may be applied with any noncritical* $q' < q$*, producing a rate arbitrarily close to the critical one. The critical logarithmic refinements are not needed for the arguments below.*

Remark 3.3. *In the deterministic construction of* [4]*, if* μ_0 *is supported on a bounded set and* $\widetilde{\mu}_0^{N_{\mathrm{grid}}}$ *is the corresponding grid projection, then*

$$\gamma_2^2(\mu_0, \widetilde{\mu}_0^{N_{\mathrm{grid}}}) \leq C N_{\mathrm{grid}}^{-2}.$$

The Monte Carlo construction replaces this deterministic error by (25) *and uses exactly* N *sampled support points.*

4. A priori, stability, and discretization estimates

Throughout this section, pointwise estimates are formulated for arbitrary initial points in $\mathbb{R}^d$. A compact set $K \subset \mathbb{R}^d$ is introduced only when an estimate contains a supremum over the initial spatial variable. No relation between K and $\operatorname{supp} \mu_0$ is assumed, in particular, the initial measure may have unbounded support.

4.1. **Second-moment estimates.**

Lemma 4.1. *Assume* $\mu_0 \in \mathcal{M}_2(\mathbb{R}^d)$*. There exists* $C > 0$*, independent of* N*,* n*, and* u*, such that, for every* $u \in \mathbb{R}^d$*,*

$$\begin{aligned} &\mathbb{E} \sup_{t \in [0,1]} |x(u,t)|^2 + \sup_{N \geq 1} \mathbb{E} \sup_{t \in [0,1]} |x^N(u,t)|^2 \\ &\qquad + \sup_{N,n \geq 1} \mathbb{E} \sup_{t \in [0,1]} |x_n^N(u,t)|^2 \leq C\big(1 + |u|^2 + M_2(\mu_0)\big). \end{aligned} \tag{27}$$

Consequently, for every compact set $K \subset \mathbb{R}^d$*, the left-hand side of* (27)*, with an additional supremum over* $u \in K$*, is bounded by* $C_K(1 + M_2(\mu_0))$*.*

Moreover,

$$\sup_{N \geq 1} \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \sup_{t \in [0,1]} |x^N(U_i,t)|^2\right] + \sup_{N,n \geq 1} \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \sup_{t \in [0,1]} |x_n^N(U_i,t)|^2\right] \leq C\big(1 + M_2(\mu_0)\big). \tag{28}$$

Proof. The argument is analogous to the moment estimate in [4, Lemma 2.1]. We give the details because the present setting involves a random empirical initial measure and does not require compact support.

We first consider the exact flow. By the integral equation, the Cauchy–Schwarz inequality, the Burkholder–Davis–Gundy inequality, and (17), for $t \le 1$,

$$\begin{aligned}\mathbb{E}\sup_{r\le t}|x(u,r)|^2 &\le C|u|^2 + C\int_0^t \mathbb{E}\left[1+|x(u,s)|^2+M_2(\mu_s)\right]\,\mathrm{d}s\\ &\le C(1+|u|^2)+C\int_0^t\left(\mathbb{E}\sup_{r\le s}|x(u,r)|^2+A(s)\right)\mathrm{d}s,\end{aligned}$$

where

$$A(t) := \int_{\mathbb{R}^d}\mathbb{E}\sup_{r\le t}|x(v,r)|^2\,\mu_0(\mathrm{d}v).$$

Since $\mu_s = \mu_0\circ x(\cdot,s)^{-1}$,

$$M_2(\mu_s)=\int_{\mathbb{R}^d}|x(v,s)|^2\,\mu_0(\mathrm{d}v).$$

Integrating the preceding estimate with respect to $\mu_0(\mathrm{d}u)$ gives

$$A(t)\le C\big(1+M_2(\mu_0)\big)+C\int_0^t A(s)\,\mathrm{d}s.$$

Gronwall's lemma yields

$$\sup_{t\le 1}A(t)\le C\big(1+M_2(\mu_0)\big).$$

Substitution into the estimate for a fixed u proves the first term in (27).

For x^N, condition on $\mathcal{G}_N := \sigma(U_1,\dots,U_N)$ and define

$$A_N(t) := \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\sup_{r\le t}|x^N(U_i,r)|^2\,\middle|\,\mathcal{G}_N\right].$$

The same argument gives

$$A_N(t)\le C\big(1+M_2(\mu_0^N)\big)+C\int_0^t A_N(s)\,\mathrm{d}s.$$

Thus

$$A_N(1)\le C\big(1+M_2(\mu_0^N)\big).$$

Taking expectation and using

$$\mathbb{E}M_2(\mu_0^N)=M_2(\mu_0)$$

proves the first term of (28). The estimate for a fixed $u\in\mathbb{R}^d$ follows from the same conditional inequality.

For x_n^N, one applies the same calculation to (21). The coefficients are evaluated at $\kappa_n(s)$, but the right-hand side is bounded by the running supremum up to time s. The resulting discrete-continuous Gronwall estimate is uniform in n. This proves the remaining bounds. □

4.2. **Spatial regularity.**

Lemma 4.2. *For every $r\ge 2$ there exists $C_r>0$ such that, for all $u,v\in\mathbb{R}^d$,*

$$\mathbb{E}\sup_{t\in[0,1]}|x(u,t)-x(v,t)|^r\le C_r|u-v|^r, \tag{29}$$

$$\sup_{N\ge1}\mathbb{E}\sup_{t\in[0,1]}|x^N(u,t)-x^N(v,t)|^r\le C_r|u-v|^r, \tag{30}$$

$$\sup_{N,n\ge1}\mathbb{E}\sup_{t\in[0,1]}|x_n^N(u,t)-x_n^N(v,t)|^r\le C_r|u-v|^r. \tag{31}$$

Proof. The estimate for the Euler–Maruyama field follows by the same argument as in [4, Lemma 2.2]. Indeed, for two trajectories belonging to the same flow, the measure argument is identical and therefore cancels when the equations are subtracted. The Lipschitz assumptions, the Burkholder–Davis–Gundy inequality, and Gronwall's lemma then yield the required estimate. The proofs for x and x^N are identical. □

Corollary 4.3. *Let $\theta \in (0,1)$. The fields*

$$u \mapsto x(u,\cdot), \qquad u \mapsto x^N(u,\cdot), \qquad u \mapsto x_n^N(u,\cdot)$$

admit $C([0,1];\mathbb{R}^d)$-valued modifications that are locally θ-Hölder continuous on $\mathbb{R}^d$. More precisely, for every compact set $K \subset \mathbb{R}^d$, there exist nonnegative random variables H_K, H_K^N, and $H_K^{N,n}$ such that

$$\begin{aligned}\sup_{t\le 1} |x(u,t) - x(v,t)| &\le H_K |u-v|^\theta,\\ \sup_{t\le 1} |x^N(u,t) - x^N(v,t)| &\le H_K^N |u-v|^\theta,\\ \sup_{t\le 1} |x_n^N(u,t) - x_n^N(v,t)| &\le H_K^{N,n} |u-v|^\theta,\end{aligned}$$

for all $u,v \in K$, and, for every sufficiently large $r \ge 2$,

$$\mathbb{E}H_K^r + \sup_{N\ge 1} \mathbb{E}(H_K^N)^r + \sup_{N,n\ge 1} \mathbb{E}(H_K^{N,n})^r \le C_{K,\theta,r}. \tag{32}$$

Proof. Let

$$B = C([0,1];\mathbb{R}^d)$$

with the supremum norm. Choose

$$r > \max\left\{2, \frac{d}{1-\theta}\right\},$$

so that $r > d$ and

$$\theta < 1 - \frac{d}{r}.$$

By Lemma 4.2,

$$\mathbb{E}\|X(u) - X(v)\|_B^r \le C_r |u-v|^r, \qquad u,v \in \mathbb{R}^d,$$

uniformly for the three families of random fields.

The Kolmogorov–Totoki continuity criterion, applied to the B-valued random fields, yields a modification that is θ-Hölder continuous on every compact subset of $\mathbb{R}^d$, together with the stated moment estimates for the corresponding Hölder seminorms. See [4, Theorem 2.1 and the discussion following Lemma 2.2]; see also [3, Lemmas 1.8.1–1.8.2]. Applying the result on the increasing sequence of compact balls $\overline{B}(0,m)$, $m \in \mathbb{N}$, and using modification argument gives locally θ-Hölder continuous modifications on $\mathbb{R}^d$. □

4.3. **A compactification lemma.**

Lemma 4.4. *Let B be a separable Banach space and let $X, Y : K \to B$ be random fields. Suppose that, for some $\theta \in (0,1)$,*

$$\|X(u) - X(v)\|_B \le H_X |u-v|^\theta, \qquad \|Y(u) - Y(v)\|_B \le H_Y |u-v|^\theta,$$

where $\mathbb{E}H_X + \mathbb{E}H_Y \le C_H$. Assume also that

$$\sup_{u\in K} \mathbb{E}\|X(u) - Y(u)\|_B^2 \le A \tag{33}$$

for some $A \in [0, A_0]$, where $A_0 > 0$ is fixed. Then

$$\mathbb{E}\sup_{u\in K} \|X(u) - Y(u)\|_B \le C_{K,\theta,C_H,A_0} A^\alpha, \qquad \alpha = \frac{\theta}{2\theta + d}. \tag{34}$$

Proof. For $\delta \in (0,1]$, choose a δ-net $\{u_1, \dots, u_m\}$ of K with

$$m \le C_K \delta^{-d}.$$

For every $u \in K$, choose u_j with $|u - u_j| \le \delta$. Then

$$\|X(u) - Y(u)\|_B \le \|X(u_j) - Y(u_j)\|_B + (H_X + H_Y)\delta^\theta.$$

Consequently,

$$\begin{aligned}\mathbb{E} \sup_{u \in K} \|X(u) - Y(u)\|_B &\le \mathbb{E} \max_{1 \le j \le m} \|X(u_j) - Y(u_j)\|_B + C_H \delta^\theta \\ &\le \left(\sum_{j=1}^m \mathbb{E}\|X(u_j) - Y(u_j)\|_B^2 \right)^{1/2} + C_H \delta^\theta \\ &\le C_K \delta^{-d/2} A^{1/2} + C_H \delta^\theta.\end{aligned}$$

If $A = 0$, the conclusion follows by approximation, or directly from (33) and continuity. Assume therefore that $A > 0$ and choose

$$\delta = \min\{1, A^{1/(2\theta + d)}\}.$$

For $A \le 1$ this gives the power $A^{\theta/(2\theta+d)}$. Since A is bounded above by A_0, the case $A > 1$ is absorbed into the constant. This proves (34). □

4.4. **Stability with respect to the initial measure.** A related stability estimate for deterministic approximations of the initial measure was obtained in [4, Lemma 4.1]. We give here a direct second-moment coupling proof adapted to the empirical initial measure.

Let $\mu_0^1, \mu_0^2 \in \mathcal{M}_2(\mathbb{R}^d)$ be deterministic, and let x^1, x^2 be the corresponding solutions driven by the same Wiener sheet. Set

$$\mu_t^i = \mu_0^i \circ x^i(\cdot, t)^{-1}, \qquad i = 1, 2.$$

Lemma 4.5. *There exists $C > 0$ such that, for every $u \in \mathbb{R}^d$,*

$$\mathbb{E} \sup_{t \in [0,1]} |x^1(u,t) - x^2(u,t)|^2 \le C \gamma_2^2(\mu_0^1, \mu_0^2). \tag{35}$$

Proof. Fix a coupling $\varkappa \in \Pi(\mu_0^1, \mu_0^2)$. For $v, w \in \mathbb{R}^d$, set

$$D_{v,w}(t) := \mathbb{E} \sup_{s \le t} |x^1(v,s) - x^2(w,s)|^2.$$

Subtracting the two equations, applying the Cauchy–Schwarz and Burkholder–Davis–Gundy inequalities, and using (15)–(16), we obtain

$$D_{v,w}(t) \le C|v - w|^2 + C \int_0^t \left(D_{v,w}(s) + \mathbb{E}\gamma_2^2(\mu_s^1, \mu_s^2) \right) \mathrm{d}s. \tag{36}$$

The image measure

$$\varkappa_s := \varkappa \circ \left(x^1(\cdot, s), x^2(\cdot, s) \right)^{-1}$$

is a coupling of μ_s^1 and μ_s^2. Therefore

$$\gamma_2^2(\mu_s^1, \mu_s^2) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x^1(v,s) - x^2(w,s)|^2 \, \varkappa(\mathrm{d}v, \mathrm{d}w). \tag{37}$$

Define

$$H(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} D_{v,w}(t) \, \varkappa(\mathrm{d}v, \mathrm{d}w).$$

Integrating (36) and using (37) yields

$$H(t) \le C \int |v - w|^2 \, \varkappa(\mathrm{d}v, \mathrm{d}w) + C \int_0^t H(s) \, \mathrm{d}s.$$

Hence, by Gronwall's lemma,

$$\sup_{t\le 1} H(t) \le C \int |v-w|^2\, \varkappa(\mathrm{d}v,\mathrm{d}w). \tag{38}$$

For the two trajectories starting from the same point u, the initial difference vanishes. Repeating the estimate leading to (36) gives

$$\mathbb{E}\sup_{s\le t} |x^1(u,s)-x^2(u,s)|^2 \le C\int_0^t \left(\mathbb{E}\sup_{r\le s}|x^1(u,r)-x^2(u,r)|^2 + H(s)\right)\mathrm{d}s.$$

Using (38) and Gronwall's lemma,

$$\mathbb{E}\sup_{t\le 1} |x^1(u,t)-x^2(u,t)|^2 \le C\int |v-w|^2\,\varkappa(\mathrm{d}v,\mathrm{d}w).$$

Taking the infimum over $\varkappa\in\Pi(\mu_0^1,\mu_0^2)$ proves (35). □

Theorem 4.6. *Let $K\subset\mathbb{R}^d$ be compact and let $\theta\in(0,1)$. Set*

$$\alpha = \frac{\theta}{2\theta+d}.$$

Then

$$\mathbb{E}\sup_{u\in K}\sup_{t\in[0,1]} |x(u,t)-x^N(u,t)| \le C_K\left(\mathbb{E}\gamma_2^2(\mu_0,\mu_0^N)\right)^{\alpha}. \tag{39}$$

Proof. Conditionally on $\mathcal{G}_N=\sigma(U_1,\dots,U_N)$, the measure μ_0^N is deterministic and the conditional law of W is unchanged. Lemma 4.5 therefore gives

$$\mathbb{E}\left[\sup_{t\le 1}|x(u,t)-x^N(u,t)|^2 \,\middle|\, \mathcal{G}_N\right] \le C\gamma_2^2(\mu_0,\mu_0^N).$$

After taking expectation,

$$\sup_{u\in K}\mathbb{E}\sup_{t\le 1}|x(u,t)-x^N(u,t)|^2 \le C\mathbb{E}\gamma_2^2(\mu_0,\mu_0^N). \tag{40}$$

Corollary 4.3 provides spatial θ-Hölder constants with uniformly bounded first moments. Apply Lemma 4.4 with

$$B=C([0,1];\mathbb{R}^d),\qquad X(u)=x(u,\cdot),\qquad Y(u)=x^N(u,\cdot),$$

and $A=C\mathbb{E}\gamma_2^2(\mu_0,\mu_0^N)$. Moreover, using the independent coupling of μ_0 and μ_0^N,

$$\mathbb{E}\gamma_2^2(\mu_0,\mu_0^N)\le 2M_2(\mu_0)+2\mathbb{E}M_2(\mu_0^N)=4M_2(\mu_0),$$

so the upper bound required in Lemma 4.4 is uniform in N. This proves (39). □

4.5. **Euler–Maruyama time discretization.** The following argument is related to the time-discretization part of [4, Corollary 4.2]. In the present setting the estimates must be uniform in the number of empirical particles N, so we include the details.

Lemma 4.7. *There exists $C>0$, independent of N, n, u, and s, such that, for all $N,n\ge 1$, all $u\in\mathbb{R}^d$, and all $s\in[0,1]$,*

$$\mathbb{E}|x_n^N(u,s)-x_n^N(u,\kappa_n(s))|^2 \le C\big(1+|u|^2+M_2(\mu_0)\big)n^{-1}. \tag{41}$$

Moreover,

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N |x_n^N(U_i,s)-x_n^N(U_i,\kappa_n(s))|^2\right] \le C\big(1+M_2(\mu_0)\big)n^{-1}. \tag{42}$$

Proof. Let $s \in [t_k, t_{k+1})$. From (21),

$$x_n^N(u,s) - x_n^N(u,t_k) = (s-t_k)a\big(x_k^{N,n}(u), \mu_k^{N,n}\big) + \int_{t_k}^s \int_{\mathbb{R}^d} b\big(x_k^{N,n}(u), \mu_k^{N,n}, p\big) W(\mathrm{d}p, \mathrm{d}r).$$

By the Itô isometry and (17),

$$\begin{aligned}\mathbb{E}|x_n^N(u,s) - x_n^N(u,t_k)|^2 &\le Ch^2 \mathbb{E}|a(x_k^{N,n}(u), \mu_k^{N,n})|^2 \\ &\quad + Ch\mathbb{E}\int_{\mathbb{R}^d} |b(x_k^{N,n}(u), \mu_k^{N,n}, p)|^2 \,\mathrm{d}p \\ &\le Ch\mathbb{E}\left[1 + |x_k^{N,n}(u)|^2 + M_2(\mu_k^{N,n})\right].\end{aligned}$$

Lemma 4.1 bounds the right-hand side by $C(1+|u|^2+M_2(\mu_0))h$, proving (41). Conditioning on $\mathcal{G}_N$, averaging the same estimate over the deterministic values $u = U_i$, and then taking expectation, (28) proves (42). ☐

Lemma 4.8. *There exists $C > 0$, independent of N, n, and u, such that, for every $u \in \mathbb{R}^d$,*

$$\sup_{N\ge1} \mathbb{E} \sup_{t\in[0,1]} |x^N(u,t) - x_n^N(u,t)|^2 \le C\big(1+|u|^2+M_2(\mu_0)\big)n^{-1}. \tag{43}$$

Consequently, for every compact set $K \subset \mathbb{R}^d$,

$$\sup_{N\ge1} \sup_{u\in K} \mathbb{E} \sup_{t\in[0,1]} |x^N(u,t) - x_n^N(u,t)|^2 \le C_K n^{-1}.$$

Proof. Write

$$e_u(t) := \mathbb{E}\sup_{r\le t} |x^N(u,r) - x_n^N(u,r)|^2.$$

Subtracting (21) from (19), applying the Cauchy–Schwarz and Burkholder–Davis–Gundy inequalities, and using (15)–(16), we obtain

$$\begin{aligned}e_u(t) &\le C\int_0^t e_u(s)\,\mathrm{d}s + C\int_0^t \mathbb{E}|x_n^N(u,s) - x_n^N(u,\kappa_n(s))|^2\,\mathrm{d}s \\ &\quad + C\int_0^t \mathbb{E}\gamma_2^2\big(\mu_s^N, \mu_{\kappa_n(s)}^{N,n}\big)\,\mathrm{d}s.\end{aligned} \tag{44}$$

The coupling which pairs particles having the same initial sample point gives

$$\begin{aligned}\gamma_2^2\big(\mu_s^N, \mu_{\kappa_n(s)}^{N,n}\big) &\le \frac{1}{N}\sum_{i=1}^N |x^N(U_i,s) - x_n^N(U_i,\kappa_n(s))|^2 \\ &\le \frac{2}{N}\sum_{i=1}^N |x^N(U_i,s) - x_n^N(U_i,s)|^2 \\ &\quad + \frac{2}{N}\sum_{i=1}^N |x_n^N(U_i,s) - x_n^N(U_i,\kappa_n(s))|^2.\end{aligned} \tag{45}$$

Set

$$E_N(t) := \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \sup_{r\le t} |x^N(U_i,r) - x_n^N(U_i,r)|^2\right].$$

Condition on $\mathcal{G}_N$. Apply the inequality leading to (44) to the deterministic values $u = U_i$, average over i, and then take expectation. Using (45) together with Lemma 4.7, we obtain

$$E_N(t) \le C\int_0^t E_N(s)\,\mathrm{d}s + Cn^{-1}.$$

Hence

$$\sup_{t\le 1} E_N(t) \le Cn^{-1}. \tag{46}$$

Returning to (44), using (45), (46), and (41), we find, for every $u \in \mathbb{R}^d$,

$$e_u(t) \le C \int_0^t e_u(s)\,\mathrm{d}s + C\big(1 + |u|^2 + M_2(\mu_0)\big)n^{-1}.$$

Gronwall's lemma proves (43); the compact-set estimate then follows by taking the supremum over $u \in K$. □

Theorem 4.9. *Let $K \subset \mathbb{R}^d$ be compact, let $\theta \in (0,1)$, and let*

$$\alpha = \frac{\theta}{2\theta + d}.$$

Then

$$\sup_{N\ge 1} \mathbb{E} \sup_{u\in K} \sup_{t\in[0,1]} |x^N(u,t) - x_n^N(u,t)| \le C_K n^{-\alpha}. \tag{47}$$

Proof. Apply Lemma 4.4 with

$$B = C([0,1];\mathbb{R}^d), \qquad X(u) = x^N(u,\cdot), \qquad Y(u) = x_n^N(u,\cdot).$$

Corollary 4.3 gives spatial Hölder constants with first moments bounded uniformly in N, n, while Lemma 4.8 provides the pointwise hypothesis of Lemma 4.4 with $A = C_K n^{-1}$. Therefore

$$\mathbb{E} \sup_{u\in K} \sup_{t\le 1} |x^N(u,t) - x_n^N(u,t)| \le C_K (n^{-1})^{\alpha}.$$

□

5. Full Euler–Maruyama–Monte Carlo approximation

Theorem 5.1. *Let $K \subset \mathbb{R}^d$ be compact. Assume Assumption 2.1 and let $\mu_0 \in \mathcal{M}_q(\mathbb{R}^d)$ for some $q > 2$ satisfying (24). Fix $\theta \in (0,1)$ and set*

$$\alpha = \frac{\theta}{2\theta + d}.$$

Then there exists $C_K > 0$, independent of N and n, such that

$$\mathbb{E} \sup_{u\in K} \sup_{t\in[0,1]} |x(u,t) - x_n^N(u,t)| \le C_K \left(R_{d,q}(N)^{\alpha} + n^{-\alpha}\right), \tag{48}$$

where $R_{d,q}$ is defined by (26).

Proof. By the triangle inequality,

$$\begin{aligned}\mathbb{E} \sup_{u\in K} \sup_{t\le 1} |x(u,t) - x_n^N(u,t)| &\le \mathbb{E} \sup_{u\in K} \sup_{t\le 1} |x(u,t) - x^N(u,t)| \\ &\quad + \mathbb{E} \sup_{u\in K} \sup_{t\le 1} |x^N(u,t) - x_n^N(u,t)|.\end{aligned}$$

Theorem 4.6 and Theorem 3.1 imply

$$\mathbb{E} \sup_{u\in K} \sup_{t\le 1} |x(u,t) - x^N(u,t)| \le C_K R_{d,q}(N)^{\alpha}.$$

Theorem 4.9 gives

$$\mathbb{E} \sup_{u\in K} \sup_{t\le 1} |x^N(u,t) - x_n^N(u,t)| \le C_K n^{-\alpha}.$$

Combining the two estimates proves (48). □

Corollary 5.2. *For every*

$$\beta \in \left(0, \frac{1}{d+2}\right)$$

there exists $C_{K,\beta} > 0$ *such that*

$$\mathbb{E} \sup_{u \in K} \sup_{t \le 1} |x(u,t) - x_n^N(u,t)| \le C_{K,\beta} \left(R_{d,q}(N)^\beta + n^{-\beta} \right).$$

Proof. Given $\beta \in (0, 1/(d+2))$, choose

$$\theta = \frac{\beta d}{1 - 2\beta}.$$

Then $\theta \in (0,1)$ and

$$\frac{\theta}{2\theta + d} = \beta.$$

The conclusion follows from Theorem 5.1. □

6. Comparison with the deterministic grid approximation

In this section, the Monte Carlo approximation is compared with the deterministic grid construction considered in [4]. The comparison is first made only for the approximation of the initial measure. A possible additional discretization of the initial spatial variable is discussed separately.

Let $K \subset \mathbb{R}^d$ be a fixed compact set on which the approximation error is measured, and define

$$K^+ := \left\{ x \in \mathbb{R}^d : \operatorname{dist}(x, K) \le 1 \right\}.$$

The set K is not required to contain the support of the initial measure.

For the deterministic construction, assume additionally that μ_0 has compact support. Choose a bounded rectangle

$$D = \prod_{j=1}^{d} [a_j, b_j]$$

such that

$$K^+ \cup \operatorname{supp} \mu_0 \subset D.$$

For $N_{\text{grid}} \in \mathbb{N}$, divide each coordinate interval $[a_j, b_j]$ into N_{grid} subintervals of equal length. This produces

$$M = N_{\text{grid}}^d$$

grid cells. Let $\widetilde{\mu}_0^{N_{\text{grid}}}$ denote the grid projection of μ_0, obtained by assigning the mass of each cell to a fixed representative point of that cell.

Since the diameter of every grid cell is bounded by $C_D N_{\text{grid}}^{-1}$, the coupling induced by the grid projection gives

$$\gamma_2^2 \left(\mu_0, \widetilde{\mu}_0^{N_{\text{grid}}} \right) \le C_D N_{\text{grid}}^{-2}. \tag{49}$$

Let $\widetilde{x}^{N_{\text{grid}}}(u,t)$ be the stochastic flow corresponding to the initial measure $\widetilde{\mu}_0^{N_{\text{grid}}}$. Fix $\theta \in (0,1)$ and set

$$\alpha = \frac{\theta}{2\theta + d}.$$

The measure-stability estimate and the compactification argument from Section 4 imply

$$\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} \left| x(u,t) - \widetilde{x}^{N_{\text{grid}}}(u,t) \right| \le C_K N_{\text{grid}}^{-2\alpha}. \tag{50}$$

Let $\widetilde{x}_n^{N_{\text{grid}}}$ denote the Euler–Maruyama approximation of $\widetilde{x}^{N_{\text{grid}}}$. The proof of Theorem 4.9 applies in the same way to a deterministic discrete measure with non-equal weights. Consequently,

$$\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} \left| \widetilde{x}^{N_{\text{grid}}}(u,t) - \widetilde{x}_n^{N_{\text{grid}}}(u,t) \right| \le C_K n^{-\alpha}, \tag{51}$$

where the constant is independent of N_{grid} and n.

Combining (50) and (51), we obtain

$$\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} \left| x(u,t) - \widetilde{x}_n^{N_{\text{grid}}}(u,t) \right| \le C_K \left(N_{\text{grid}}^{-2\alpha} + n^{-\alpha} \right). \tag{52}$$

Since

$$M = N_{\text{grid}}^d, \qquad N_{\text{grid}} = M^{1/d},$$

estimate (52) can be rewritten in terms of the total number M of grid cells as

$$\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} \left| x(u,t) - \widetilde{x}_n^{N_{\text{grid}}}(u,t) \right| \le C_K \left(M^{-\frac{2\alpha}{d}} + n^{-\alpha} \right). \tag{53}$$

For the Monte Carlo approximation, the number of sampled atoms is itself the approximation parameter. Setting $N = M$ in Theorem 5.1 gives

$$\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} \left| x(u,t) - x_n^M(u,t) \right| \le C_K \left(R_{d,q}(M)^\alpha + n^{-\alpha} \right), \tag{54}$$

where

$$R_{d,q}(M) = \begin{cases} M^{-\frac{1}{2}} + M^{-\frac{q-2}{q}}, & d < 4, \\ M^{-\frac{1}{2}} \log(1+M) + M^{-\frac{q-2}{q}}, & d = 4, \\ M^{-\frac{2}{d}} + M^{-\frac{q-2}{q}}, & d > 4. \end{cases}$$

Under the compact-support assumption imposed for the deterministic construction, the measure μ_0 belongs to $\mathcal{M}_q(\mathbb{R}^d)$ for every $q > 0$. Therefore, q may be chosen sufficiently large so that the tail term $M^{-(q-2)/q}$ does not determine the leading order.

More precisely, one may choose

$$q > 4 \qquad \text{when } d \le 4,$$

and

$$q > \frac{2d}{d-2} \qquad \text{when } d > 4.$$

Hence, under assumptions for which both methods are applicable,

$$R_{d,q}(M)^\alpha \le C \begin{cases} M^{-\frac{\alpha}{2}}, & d < 4, \\ M^{-\frac{\alpha}{2}} \left(\log(1+M) \right)^\alpha, & d = 4, \\ M^{-\frac{2\alpha}{d}}, & d > 4. \end{cases} \tag{55}$$

Comparison of (53) and (55) gives the following conclusions for the derived upper bounds.

If $d < 4$, the deterministic tensor-grid estimate has a faster polynomial rate in M. If $d = 4$, the two estimates have the same polynomial exponent, while the Monte Carlo estimate contains an additional logarithmic factor. If $d > 4$, both methods have the same polynomial order

$$M^{-2\alpha/d}.$$

Thus, under the common compact-support assumptions, the estimates obtained above do not show an asymptotic rate advantage of the Monte Carlo approximation over the deterministic tensor grid. The advantage of the Monte Carlo construction is instead structural. It does not require the construction of a product grid, permits an arbitrary number of sampled atoms, and remains applicable to measures with unbounded support under the finite-moment assumptions of Theorem 5.1. It is particularly convenient when independent samples from μ_0 are available, whereas the masses of individual grid cells are difficult to compute.

If the initial spatial variable is also projected onto the grid, let $\pi_{N_{\text{grid}}}(u)$ denote the representative point of the cell containing u. For all sufficiently large N_{grid},

$$\pi_{N_{\text{grid}}}(u) \in K^+, \qquad \left|u - \pi_{N_{\text{grid}}}(u)\right| \le C_D N_{\text{grid}}^{-1}, \qquad u \in K.$$

The spatial Hölder estimate from Corollary 4.3 then yields

$$\mathbb{E} \sup_{u\in K} \sup_{t\in[0,1]} \left|\widetilde{x}_n^{N_{\text{grid}}}(u,t) - \widetilde{x}_n^{N_{\text{grid}}}\left(\pi_{N_{\text{grid}}}(u),t\right)\right| \le C_K N_{\text{grid}}^{-\theta}. \tag{56}$$

Consequently, the fully spatially discretized grid approximation satisfies

$$\begin{aligned} &\mathbb{E} \sup_{u\in K} \sup_{t\in[0,1]} \left|x(u,t) - \widetilde{x}_n^{N_{\text{grid}}}\left(\pi_{N_{\text{grid}}}(u),t\right)\right| \\ &\quad\le C_K \left(N_{\text{grid}}^{-2\alpha} + N_{\text{grid}}^{-\theta} + n^{-\alpha}\right) \\ &\quad= C_K \left(M^{-2\alpha/d} + M^{-\theta/d} + n^{-\alpha}\right). \end{aligned} \tag{57}$$

The additional term $M^{-\theta/d}$ is caused by the discretization of the initial spatial variable and is not part of the error associated with the approximation of the initial measure itself.

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