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JACOBIAN BOUNDS, FUNCTIONAL INEQUALITIES, AND DISPERSION FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH INTERACTION

We study evolutions of measures under flows of solutions to stochastic differential equations with interaction. Under suitable regularity assumptions on the coefficients, we prove moment bounds for the compact-set Jacobian norm. These bounds yield pathwise transfer of logarithmic Sobolev, concentration, and Talagrand transport inequalities from the initial measure to the random pushforward measure at time t. Finally, for a kernel model satisfying a pairwise dissipativity condition, we prove exponential decay of the expected dispersion around the random center of mass.

1. Introduction

We consider stochastic differential equations with interaction (SDEWI), introduced by Andrey A. Dorogovtsev in [1]. In these equations, the coefficients depend on the measure transported by the entire stochastic flow. More precisely, we study systems of the form

$$\begin{cases} \mathrm{d}X_t(u) &= a\big(X_t(u),\mu_t\big)\,\mathrm{d}t + \displaystyle\sum_{\ell=1}^{\infty}\sigma_\ell\big(X_t(u)\big)\,\mathrm{d}B_t^\ell, \\ X_0(u) &= u, \\ \mu_t &= \mu_0\circ X_t^{-1}. \end{cases} \tag{1}$$

Here $\mu_0\in\mathcal{P}_2(\mathbb{R}^d)$ is a deterministic initial probability measure. For $p\geq 1$, we denote by $\mathcal{P}_p(\mathbb{R}^d)$ the space of Borel probability measures on $\mathbb{R}^d$ with finite p-th moment. For $\mu,\nu\in\mathcal{P}_p(\mathbb{R}^d)$, define

$$W_p(\mu,\nu) := \left(\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\,\pi(\mathrm{d}x,\mathrm{d}y)\right)^{1/p}.$$

Here $\Pi(\mu,\nu)$ denotes the set of couplings of μ and ν. The drift is $a:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$, and the state-dependent diffusion coefficient is $\sigma=(\sigma_\ell)_{\ell\geq1}:\mathbb{R}^d\to\ell^2(\mathbb{R}^d)$. Let $(\Omega,(\mathcal{F}_t)_{t\geq0},\mathbb{P})$ be a filtered probability space satisfying the usual conditions and carrying a sequence $B=(B^\ell)_{\ell\geq1}$ of independent one-dimensional Brownian motions.

Assumption 1 (Well-posedness). *The coefficients $a:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$ and $\sigma=(\sigma_\ell)_{\ell\geq1}:\mathbb{R}^d\to\ell^2(\mathbb{R}^d)$ are continuous, where*

$$\ell^2(\mathbb{R}^d) := \left\{ z=(z_\ell)_{\ell\geq1}\in(\mathbb{R}^d)^{\mathbb{N}} : \sum_{\ell=1}^{\infty}|z_\ell|^2<\infty \right\}.$$

There exist constants $L_a, L_\sigma<\infty$ such that, for all $x,y\in\mathbb{R}^d$ and $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$,

$$|a(x,\mu)-a(y,\nu)| \leq L_a\big(|x-y|+W_2(\mu,\nu)\big),$$

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and

$$\left(\sum_{\ell=1}^{\infty} |\sigma_\ell(x) - \sigma_\ell(y)|^2\right)^{1/2} \le L_\sigma |x-y|.$$

Under Assumption 1, the SDEWI (1) admits a unique global solution; see [2, Theorem 2.3.1].

Unlike a classical stochastic differential equation, the drift depends on μ_t, the pushforward of μ_0 by the entire stochastic flow. This dependence represents the interaction between particles. For each realization $\omega \in \Omega$, the random map

$$T_t(\omega, \cdot) : u \mapsto X_t(u, \omega)$$

acts on the initial measure by pushforward,

$$\mu_t(\omega) = \mu_0 \circ T_t(\omega)^{-1},$$

so the evolution can be viewed as a random transport of measures generated by the interacting system (1). Throughout, D denotes differentiation with respect to spatial variables for vector-valued maps, while ∇ and ∇^2 denote the gradient and Hessian of scalar-valued functions, respectively. Whenever the flow is differentiable with respect to the starting point, we write

$$J_t(u) := DT_t(u).$$

Throughout the paper, $|\cdot|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on finite-dimensional vector spaces. For matrices $M, N \in \mathbb{R}^{d\times d}$, we write

$$\langle M, N \rangle := \sum_{i,j} M_{ij} N_{ij}, \qquad \|M\| := \left(\sum_{i,j} M_{ij}^2\right)^{1/2}.$$

The main purpose of this paper is to relate quantitative properties of the random measure μ_t to estimates on the flow T_t and its Jacobian. The Jacobian $J_t(u)$ measures the local stretching of the flow near the point $X_t(u)$. Uniform compact-set Jacobian bounds allow us to transfer functional inequalities from μ_0 to μ_t.

The paper is organized around two related problems.

First, in Section 2, we study the maximal Jacobian norm $\mathcal{K}_t(K)$ on the compact set K

$$\mathcal{K}_t(K) := \sup_{u\in K} \|J_t(u)\|.$$

This random constant controls the Lipschitz constant of the flow on K. We first prove the compact-set Jacobian moment estimate in Theorem 2.2. We then use this estimate in Propositions 2.1 to 2.3 to transfer logarithmic Sobolev, concentration, and Talagrand transport inequalities from the initial measure μ_0 to the random pushforward measure μ_t.

Second, in Section 3, we study a specific kernel model. We impose a pairwise dissipativity condition and prove exponential decay of the dispersion

$$\mathcal{V}_t := \int_{\mathbb{R}^d} |T_t(u) - \bar{X}_t|^2 \,\mu_0(\mathrm{d}u) = W_2^2(\mu_t, \delta_{\bar{X}_t}),$$

where $\bar{X}_t := \int_{\mathbb{R}^d} X_t(u)\,\mu_0(\mathrm{d}u)$ is the random center of mass.

In summary, we prove compact-set moment bounds for the maximal Jacobian norm, use them to transfer functional inequalities, and establish exponential decay of dispersion under a pairwise dissipativity condition.

2. Jacobian bounds and functional inequalities

We begin with estimates for the compact-set Jacobian norm. These estimates provide a pathwise Lipschitz constant for the stochastic flow restricted to the support of the initial measure. Once such a constant is established, several functional inequalities can be transferred directly from μ_0 to the random pushforward measure μ_t. The proof of the compact-set Jacobian estimate is postponed to the final subsection.

Assumption 2 (Spatial regularity)**.** *For every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the map $x \mapsto a(x,\mu)$ belongs to $\mathcal{C}^1(\mathbb{R}^d;\mathbb{R}^d)$, and $\sigma : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$ is continuously differentiable.*

There exists $H < \infty$ such that, for all $x, y \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\|Da(x,\mu) - Da(y,\mu)\| + \left(\sum_{\ell=1}^{\infty} \|D\sigma_\ell(x) - D\sigma_\ell(y)\|^2 \right)^{1/2} \leq H|x-y|. \tag{2}$$

Since the coefficient maps are differentiable, the global Lipschitz bounds in Assumption 1 imply that, for every $x, z \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|Da(x,\mu)z| \leq L_a|z|, \qquad \langle z, Da(x,\mu)z\rangle \leq L_a|z|^2, \qquad \sum_{\ell=1}^{\infty} |D\sigma_\ell(x)z|^2 \leq L_\sigma^2|z|^2. \tag{3}$$

In particular, applying the last inequality to the standard basis of $\mathbb{R}^d$ gives

$$\sup_{x\in\mathbb{R}^d} \sum_{\ell=1}^{\infty} \|D\sigma_\ell(x)\|^2 \leq dL_\sigma^2. \tag{4}$$

2.1. **The Jacobian flow.**

Theorem 2.1 (Differentiable flow and Jacobian equation)**.** *Under Assumptions 1 and 2, the solution of (1) has a modification, still denoted by X, such that, for almost every ω and every $t \geq 0$, the map*

$$u \longmapsto T_t(\omega, u)$$

is of class $\mathcal{C}^1$. For every $u \in \mathbb{R}^d$, the Jacobian

$$J_t(u) := DT_t(u)$$

satisfies

$$\begin{cases} \mathrm{d}J_t(u) = A_t(u)J_t(u)\,\mathrm{d}t + \displaystyle\sum_{\ell=1}^{\infty} C_{\ell,t}(u)J_t(u)\,\mathrm{d}B_t^\ell, \\ J_0(u) = I, \end{cases} \tag{5}$$

where

$$A_t(u) := Da\big(X_t(u), \mu_t\big), \qquad C_{\ell,t}(u) := D\sigma_\ell\big(X_t(u)\big).$$

Proof. The continuous differentiability required in Assumption (A2)(i) of [3] follows from Assumption 2. Moreover, (2) gives Assumption (A2)(ii) with exponent $\delta = 1$, while (4) gives Assumption (A2)(iii).

Thus, (1) becomes an SDE with random spatial coefficients

$$x \longmapsto a(x, \mu_t), \qquad x \longmapsto \sigma_\ell(x), \quad \ell \geq 1.$$

Therefore, the argument of [3, Theorem 2.11] yields a modification such that $u \mapsto T_t(\omega, u)$ is of class $\mathcal{C}^1$ for almost every ω and every $t \geq 0$.

The Jacobian equation follows from [3, Corollary 2.12]. □

Definition 2.1. For a nonempty set $K \subset \mathbb{R}^d$, define

$$\mathcal{K}_t(K) := \sup_{u\in K} \|J_t(u)\|.$$

If K is compact, continuity of $u \mapsto J_t(u)$ implies $\mathcal{K}_t(K) < \infty$. This compact-set quantity is used because a corresponding supremum over all of $\mathbb{R}^d$ need not be finite. The counterexamples of Mohammed and Scheutzow [5] show that even for globally Lipschitz coefficients one may fail to have a finite global constant $\sup_{\substack{u\in\mathbb{R}^d \\ z\in\mathbb{R}^d,\ |z|=1}} |J_t(u,\omega)z|$ for a fixed (t,ω).

Theorem 2.2 (Moment bound for $\mathcal{K}_t(K)$)**.** *Suppose that Assumptions 1 and 2 hold. Then, for every nonempty compact set $K \subset \mathbb{R}^d$, every $p \geq 1$, and every $t \geq 0$,*

$$\mathbb{E}\big[\mathcal{K}_t(K)^p\big] < \infty. \tag{6}$$

The proof is given in Subsection 2.3.

2.2. **Transfer of functional inequalities.** We now turn from Jacobian estimates to their consequences for the transported measure. The key point is simple: on the convex hull of the support of μ_0, the flow $T_t(\omega,\cdot)$ is Lipschitz with constant controlled by $\kappa_t(\omega)$.

Throughout this subsection, assume that $\operatorname{supp}(\mu_0)$ is compact and set

$$K_0 := \operatorname{co}\big(\operatorname{supp}(\mu_0)\big), \qquad \kappa_t(\omega) := \mathcal{K}_t(K_0).$$

Here, $\operatorname{co}(A)$ denotes the convex hull of a set $A \subset \mathbb{R}^d$. Since K_0 is compact and convex, for every fixed $t \geq 0$ and almost every ω,

$$\operatorname{Lip}\big(T_t(\omega,\cdot)|_{K_0}\big) \leq \kappa_t(\omega). \tag{7}$$

Indeed, for $u, v \in K_0$,

$$\begin{aligned} |T_t(\omega,u) - T_t(\omega,v)| &\leq \int_0^1 \big|J_t\big(v + s(u-v),\omega\big)(u-v)\big|\,\mathrm{d}s \\ &\leq \kappa_t(\omega)|u-v|. \end{aligned}$$

Here $v + s(u-v) \in K_0$ for every $s \in [0,1]$.

Definition 2.2 (Logarithmic Sobolev inequality, [4, Chapter 5])**.** Let $\mathcal{C}_c^1(\mathbb{R}^d)$ denote the set of continuously differentiable compactly supported functions from $\mathbb{R}^d$ to $\mathbb{R}$. For $g \geq 0$ with $\int g\,\mathrm{d}\mu = 1$, define the entropy by

$$\operatorname{Ent}_\mu(g) := \int g \log g\,\mathrm{d}\mu.$$

A probability measure μ on $\mathbb{R}^d$ is said to satisfy a logarithmic Sobolev inequality with constant $C > 0$ if for every $h \in \mathcal{C}_c^1(\mathbb{R}^d)$ with $\int h^2\,\mathrm{d}\mu = 1$ one has

$$\operatorname{Ent}_\mu(h^2) \leq C \int_{\mathbb{R}^d} |\nabla h|^2\,\mathrm{d}\mu.$$

In this case, we say that μ satisfies $\operatorname{LSI}(C)$.

Proposition 2.1 (Logarithmic Sobolev inequality)**.** *Suppose that μ_0 satisfies $\operatorname{LSI}(C_0)$. Then, for every fixed $t \geq 0$, for almost every ω with $\kappa_t(\omega) > 0$, the measure $\mu_t(\omega)$ satisfies*

$$\operatorname{LSI}\big(C_0\kappa_t(\omega)^2\big).$$

If $\kappa_t(\omega) = 0$, then $\mu_t(\omega)$ is a Dirac mass and satisfies $\operatorname{LSI}(C)$ for every $C > 0$.

Proof. Fix $t \geq 0$ and ω for which (7) holds, and write

$$T := T_t(\omega,\cdot), \qquad J(u) := J_t(u,\omega), \qquad \kappa := \kappa_t(\omega).$$

If $\kappa = 0$, then (7) shows that T is constant on K_0. Since μ_0 is supported on K_0, $\mu_t(\omega)$ is a Dirac mass and satisfies $\operatorname{LSI}(C)$ for every $C > 0$.

Suppose that $\kappa > 0$, and let $g \in \mathcal{C}_c^1(\mathbb{R}^d)$ satisfy

$$\int_{\mathbb{R}^d} g^2 \,\mathrm{d}\mu_t(\omega) = 1.$$

Choose $\chi \in \mathcal{C}_c^1(\mathbb{R}^d)$ such that $\chi = 1$ on a neighborhood of K_0, and set

$$h(u) := \chi(u) g(T(u)).$$

Since μ_0 is supported on K_0, for $u \in K_0$

$$h(u) = g(T(u)), \qquad \nabla h(u) = J(u)^{\mathsf{T}} \nabla g(T(u)).$$

Consequently,

$$\begin{aligned}
\operatorname{Ent}_{\mu_t(\omega)}(g^2) &= \operatorname{Ent}_{\mu_0}(h^2) \\
&\le C_0 \int_{K_0} \left| J(u)^{\mathsf{T}} \nabla g(T(u)) \right|^2 \mu_0(\mathrm{d}u) \\
&\le C_0 \kappa^2 \int_{K_0} |\nabla g(T(u))|^2 \, \mu_0(\mathrm{d}u) \\
&= C_0 \kappa^2 \int_{\mathbb{R}^d} |\nabla g(x)|^2 \, \mu_t(\omega, \mathrm{d}x).
\end{aligned}$$

□

For a probability measure μ and a μ-integrable function f, we write

$$\mu(f) := \int_{\mathbb{R}^d} f(x) \, \mu(\mathrm{d}x).$$

Proposition 2.2 (Pathwise concentration transfer)**.** *Suppose that μ_0 satisfies*

$$\mu_0\big(f - \mu_0(f) \ge r\big) \le \Psi(r), \qquad r \ge 0,$$

for every 1-Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, where $\Psi : [0, \infty) \to [0, 1]$ is nonincreasing.

Then, for every fixed $t \ge 0$, almost every ω, every 1-Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$, and every $r > 0$,

$$\mu_t(\omega)\big(g - \mu_t(\omega)(g) \ge r\big) \le \Psi\left(\frac{r}{\kappa_t(\omega)}\right),$$

whenever $\kappa_t(\omega) > 0$.

If $\kappa_t(\omega) = 0$, then $\mu_t(\omega)$ is a Dirac mass and the probability on the left-hand side is zero.

Proof. Fix $t \ge 0$ and ω for which (7) holds, and write

$$T := T_t(\omega, \cdot), \qquad \kappa := \kappa_t(\omega).$$

If $\kappa = 0$, then (7) shows that T is constant on K_0. Since μ_0 is supported on K_0, $\mu_t(\omega)$ is a Dirac mass.

Suppose that $\kappa > 0$, let $g : \mathbb{R}^d \to \mathbb{R}$ be 1-Lipschitz, and define on K_0

$$h(u) := \frac{g(T(u))}{\kappa}.$$

By (7), h is 1-Lipschitz on K_0. By the McShane extension theorem, there exists a 1-Lipschitz function $\widetilde{h} : \mathbb{R}^d \to \mathbb{R}$ such that $\widetilde{h} = h$ on K_0.

Since $\mu_0(K_0) = 1$,

$$\mu_0(\widetilde{h}) = \mu_0(h) = \frac{1}{\kappa} \mu_t(\omega)(g).$$

Therefore,

$$\mu_t(\omega)\big(g - \mu_t(\omega)(g) \ge r\big) = \mu_0\left(\widetilde{h} - \mu_0(\widetilde{h}) \ge \frac{r}{\kappa}\right) \le \Psi\left(\frac{r}{\kappa}\right).$$

□

Remark 2.1 (Concentration as a consequence of LSI). If μ_0 satisfies a logarithmic Sobolev inequality, then it satisfies a concentration inequality with an $\exp(-r^2/C)$ tail bound; see [4, Chapter 5].

Definition 2.3 (Talagrand transport inequality, [7])**.** Let $p \geq 1$. A probability measure $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ is said to satisfy the Talagrand transport inequality $T_p(C)$ with constant $C > 0$ if, for every $\nu \in \mathcal{P}_p(\mathbb{R}^d)$,

$$W_p(\nu, \mu)^2 \leq 2C \operatorname{Ent}(\nu \mid \mu),$$

where

$$\operatorname{Ent}(\nu \mid \mu) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \mathrm{d}\nu, & \nu \ll \mu, \\ +\infty, & \text{otherwise}. \end{cases}$$

Proposition 2.3 (Pathwise transfer of the Talagrand transport inequality)**.** *Suppose that μ_0 satisfies $T_p(C)$ for some $p \geq 1$ and $C > 0$. Then, for every fixed $t \geq 0$, for almost every ω, the measure $\mu_t(\omega)$ satisfies*

$$T_p\big(C\kappa_t(\omega)^2\big).$$

Proof. Fix $t \geq 0$ and ω for which (7) holds, and write

$$T := T_t(\omega, \cdot), \qquad \kappa := \kappa_t(\omega), \qquad \eta := \mu_t(\omega) = \mu_0 \circ T^{-1}.$$

If $\kappa = 0$, then (7) shows that T is constant on K_0. Since μ_0 is supported on K_0, η is a Dirac mass, and the assertion is immediate. Suppose that $\kappa > 0$.

Let $\lambda \in \mathcal{P}_p(\mathbb{R}^d)$. If $\lambda \not\ll \eta$, then $\operatorname{Ent}(\lambda \mid \eta) = +\infty$, and there is nothing to prove. Suppose that $\lambda \ll \eta$, write $\varphi := \frac{\mathrm{d}\lambda}{\mathrm{d}\eta}$, and define

$$\widetilde{\lambda}(\mathrm{d}u) := \varphi(T(u))\, \mu_0(\mathrm{d}u).$$

Then

$$\widetilde{\lambda} \circ T^{-1} = \lambda, \qquad \operatorname{Ent}(\widetilde{\lambda} \mid \mu_0) = \operatorname{Ent}(\lambda \mid \eta).$$

Moreover, $\widetilde{\lambda}(K_0) = \mu_0(K_0) = 1$. Hence every coupling π of $\widetilde{\lambda}$ and μ_0 is concentrated on $K_0 \times K_0$, and therefore, by (7),

$$\int |T(u) - T(v)|^p\, \pi(\mathrm{d}u, \mathrm{d}v) \leq \kappa^p \int |u - v|^p\, \pi(\mathrm{d}u, \mathrm{d}v).$$

Taking the infimum over π gives

$$W_p(\lambda, \eta) \leq \kappa W_p(\widetilde{\lambda}, \mu_0).$$

Consequently,

$$\begin{aligned} W_p(\lambda, \eta)^2 &\leq \kappa^2 W_p(\widetilde{\lambda}, \mu_0)^2 \\ &\leq 2C\kappa^2 \operatorname{Ent}(\widetilde{\lambda} \mid \mu_0) \\ &= 2C\kappa^2 \operatorname{Ent}(\lambda \mid \eta). \end{aligned}$$

□

2.3. **Proof of the compact-set Jacobian estimate.** Before proving Theorem 2.2, we prove two auxiliary lemmas.

The first lemma will be applied twice in the proof. First, it is applied to the difference $X_t(u) - X_t(v)$, which is an $\mathbb{R}^d$-valued process. Second, it is applied column by column to the Jacobian $J_t(u)$, which is an $\mathbb{R}^{d\times d}$-valued process.

Lemma 2.1. *Let Y_t be an $\mathbb{R}^d$-valued adapted continuous process solving*

$$\begin{cases} \mathrm{d}Y_t = A_t Y_t\,\mathrm{d}t + \sum_{\ell=1}^{\infty} C_{\ell,t} Y_t\,\mathrm{d}B_t^{\ell}, \\ Y_0 = y. \end{cases}$$

where $A_t \in \mathbb{R}^{d\times d}$ and $C_{\ell,t} \in \mathbb{R}^{d\times d}$ are predictable matrix-valued processes. Assume that, for some constants $L_a, L_\sigma \geq 0$, every $z \in \mathbb{R}^d$, and all $t \geq 0$,

$$\langle z, A_t z\rangle \leq L_a |z|^2, \qquad \sum_{\ell=1}^{\infty} |C_{\ell,t} z|^2 \leq L_\sigma^2 |z|^2.$$

Then, for every $q \geq 1$ and all $t \geq 0$,

$$\mathbb{E}|Y_t|^{2q} \leq |y|^{2q} \exp\left\{q\bigl(2L_a + (2q-1)L_\sigma^2\bigr)t\right\}.$$

In particular, if $J_t(u) \in \mathbb{R}^{d\times d}$ solves the Jacobian equation

$$\mathrm{d}J_t(u) = A_t(u)J_t(u)\,\mathrm{d}t + \sum_{\ell=1}^{\infty} C_{\ell,t}(u)J_t(u)\,\mathrm{d}B_t^{\ell}, \qquad J_0(u) = I,$$

and the same one-sided bounds hold for $A_t(u)$ and $C_{\ell,t}(u)$, then

$$\mathbb{E}\|J_t(u)\|^{2q} \leq d^q \exp\left\{q\bigl(2L_a + (2q-1)L_\sigma^2\bigr)t\right\}.$$

Proof. Applying Itô's formula to $|Y_t|^{2q}$ gives

$$\begin{aligned} \mathrm{d}|Y_t|^{2q} = 2q|Y_t|^{2q-2}&\langle Y_t, A_t Y_t\rangle\,\mathrm{d}t \\ &+ q|Y_t|^{2q-2}\sum_{\ell=1}^{\infty}|C_{\ell,t}Y_t|^2\,\mathrm{d}t \\ &+ 2q(q-1)|Y_t|^{2q-4}\sum_{\ell=1}^{\infty}\langle Y_t, C_{\ell,t}Y_t\rangle^2\,\mathrm{d}t + \mathrm{d}M_t, \end{aligned}$$

where

$$M_t := 2q\sum_{\ell=1}^{\infty}\int_0^t |Y_s|^{2q-2}\langle Y_s, C_{\ell,s}Y_s\rangle\,\mathrm{d}B_s^{\ell}$$

is a continuous local martingale.

The Cauchy–Schwarz inequality gives

$$\langle Y_t, C_{\ell,t}Y_t\rangle^2 \leq |Y_t|^2|C_{\ell,t}Y_t|^2.$$

Using it and the two assumptions, we get

$$\begin{aligned} &2q|Y_t|^{2q-2}\langle Y_t, A_tY_t\rangle + q|Y_t|^{2q-2}\sum_{\ell=1}^{\infty}|C_{\ell,t}Y_t|^2 \\ &\quad + 2q(q-1)|Y_t|^{2q-4}\sum_{\ell=1}^{\infty}\langle Y_t, C_{\ell,t}Y_t\rangle^2 \\ &\leq 2qL_a|Y_t|^{2q} + q|Y_t|^{2q-2}\sum_{\ell=1}^{\infty}|C_{\ell,t}Y_t|^2 \\ &\quad + 2q(q-1)|Y_t|^{2q-2}\sum_{\ell=1}^{\infty}|C_{\ell,t}Y_t|^2 \\ &= 2qL_a|Y_t|^{2q} + q(2q-1)|Y_t|^{2q-2}\sum_{\ell=1}^{\infty}|C_{\ell,t}Y_t|^2 \\ &\leq \bigl(2qL_a + q(2q-1)L_\sigma^2\bigr)|Y_t|^{2q}. \end{aligned}$$

Set

$$c_q := q\big(2L_a + (2q-1)L_\sigma^2\big).$$

For $R > 0$, define the stopping time

$$\tau_R := \inf\{s \geq 0 : \ |Y_s| \geq R\}.$$

On $[0, t \wedge \tau_R]$,

$$\begin{aligned}\sum_{\ell=1}^{\infty} \big||Y_s|^{2q-2}\langle Y_s, C_{\ell,s}Y_s\rangle\big|^2 &\leq |Y_s|^{4q-2}\sum_{\ell=1}^{\infty}|C_{\ell,s}Y_s|^2 \\ &\leq L_\sigma^2|Y_s|^{4q} \leq L_\sigma^2 R^{4q}.\end{aligned}$$

Thus $M_{t\wedge\tau_R}$ is a square-integrable martingale. Integrating the differential inequality up to $t \wedge \tau_R$ gives

$$|Y_{t\wedge\tau_R}|^{2q} \leq |y|^{2q} + c_q\int_0^t |Y_{s\wedge\tau_R}|^{2q}\,\mathrm{d}s + M_{t\wedge\tau_R}.$$

Taking expectations and applying Grönwall's lemma yields

$$\mathbb{E}|Y_{t\wedge\tau_R}|^{2q} \leq |y|^{2q}e^{c_q t}.$$

Since Y is continuous, $\tau_R \uparrow \infty$ almost surely as $R \to \infty$. Letting $R \to \infty$ and using Fatou's lemma gives

$$\mathbb{E}|Y_t|^{2q} \leq |y|^{2q}e^{c_q t}.$$

For the Jacobian estimate, let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$. The vectors $J_t(u)e_j$ are the columns of the Jacobian matrix. Since $J_t(u)$ solves the matrix SDE, each column solves the same vector linear SDE with initial value e_j:

$$\begin{cases} Y_t^{(j)} := J_t(u)e_j, \\ Y_0^{(j)} = e_j. \end{cases}$$

So the vector estimate gives

$$\mathbb{E}|J_t(u)e_j|^{2q} \leq |e_j|^{2q}e^{c_q t} = e^{c_q t}.$$

Using the Hilbert–Schmidt norm, we have

$$\|J_t(u)\|^2 = \sum_{j=1}^{d}|J_t(u)e_j|^2.$$

Set $a_j = |J_t(u)e_j|^2$. Since $q \geq 1$,

$$(a_1 + \cdots + a_d)^q \leq d^{q-1}(a_1^q + \cdots + a_d^q).$$

Therefore,

$$\mathbb{E}\|J_t(u)\|^{2q} \leq d^{q-1}\sum_{j=1}^{d}\mathbb{E}|J_t(u)e_j|^{2q} \leq d^{q-1}\cdot d\cdot e^{c_q t} = d^q e^{c_q t}.$$

$\square$

Corollary 2.1 (Averaged Jacobian bound)**.** *Suppose that the differentiable flow exists and that its Jacobian satisfies the equation in Theorem 2.1. Assume that, for some $\alpha \in \mathbb{R}$ and $\beta \geq 0$,*

$$\langle z, Da(x,\mu)z\rangle \leq \alpha|z|^2, \qquad \sum_{\ell=1}^{\infty}|D\sigma_\ell(x)z|^2 \leq \beta^2|z|^2$$

for all $x, z \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Then, for every $t \geq 0$,

$$\mathbb{E}\int_{\mathbb{R}^d}\|J_t(u)\|^2\,\mu_0(\mathrm{d}u) \leq d\,e^{(2\alpha+\beta^2)t}.$$

Proof. Fix $u \in \mathbb{R}^d$ and set

$$\widetilde{J}_t(u) := e^{-\alpha t} J_t(u).$$

Then

$$\begin{cases} \mathrm{d}\widetilde{J}_t(u) &= \big[Da(X_t(u), \mu_t) - \alpha I\big] \widetilde{J}_t(u)\,\mathrm{d}t + \displaystyle\sum_{\ell=1}^{\infty} D\sigma_\ell(X_t(u)) \widetilde{J}_t(u)\,\mathrm{d}B_t^\ell, \\ \widetilde{J}_0(u) &= I. \end{cases}$$

For every $z \in \mathbb{R}^d$,

$$\big\langle z, \big(Da(x,\mu) - \alpha I\big) z \big\rangle \le 0,$$

while

$$\sum_{\ell=1}^{\infty} |D\sigma_\ell(x) z|^2 \le \beta^2 |z|^2.$$

Hence Lemma 2.1, applied with $q = 1$, $L_a = 0$, and $L_\sigma = \beta$, gives

$$\mathbb{E}\|\widetilde{J}_t(u)\|^2 \le d e^{\beta^2 t}.$$

Therefore,

$$\mathbb{E}\|J_t(u)\|^2 = e^{2\alpha t} \mathbb{E}\|\widetilde{J}_t(u)\|^2 \le d e^{(2\alpha + \beta^2)t}.$$

Integrating with respect to μ_0 and applying Tonelli's theorem completes the proof. □

The second lemma is needed for the difference of two Jacobians,

$$D_t := J_t(u) - J_t(v),$$

which is a matrix-valued process.

Lemma 2.2. *Let $m > 2$ be an even integer, and let D_t be an $\mathbb{R}^{d\times d}$-valued adapted continuous process satisfying*

$$\mathrm{d}D_t = A_t D_t\,\mathrm{d}t + \sum_{\ell=1}^{\infty} C_{\ell,t} D_t\,\mathrm{d}B_t^\ell + F_t\,\mathrm{d}t + \sum_{\ell=1}^{\infty} G_{\ell,t}\,\mathrm{d}B_t^\ell.$$

Here $A_t, C_{\ell,t} \in \mathbb{R}^{d\times d}$, $F_t, G_{\ell,t} \in \mathbb{R}^{d\times d}$, and the products $A_t D_t$ and $C_{\ell,t} D_t$ are matrix products. Assume that, for some constants $L_a, L_\sigma \ge 0$, every $M \in \mathbb{R}^{d\times d}$, and all $t \ge 0$,

$$\langle M, A_t M \rangle \le L_a \|M\|^2, \qquad \sum_{\ell=1}^{\infty} \|C_{\ell,t} M\|^2 \le L_\sigma^2 \|M\|^2.$$

Then there exist constants $C_m, c_m < \infty$, depending only on m, L_a, L_σ, such that, for all $t \ge 0$,

$$\mathbb{E}\|D_t\|^m \le e^{c_m t} \mathbb{E}\|D_0\|^m + C_m \int_0^t e^{c_m (t-s)} \left[\mathbb{E}\|F_s\|^m + \mathbb{E}\left(\sum_{\ell=1}^{\infty} \|G_{\ell,s}\|^2 \right)^{m/2} \right] \mathrm{d}s.$$

Proof. Set

$$g_t := \left(\sum_{\ell=1}^{\infty} \|G_{\ell,t}\|^2 \right)^{1/2} \quad \text{and} \quad U_{\ell,t} := C_{\ell,t} D_t + G_{\ell,t}.$$

Then

$$\mathrm{d}D_t = (A_t D_t + F_t)\,\mathrm{d}t + \sum_{\ell=1}^{\infty} U_{\ell,t}\,\mathrm{d}B_t^\ell.$$

Applying Itô's formula to $\|D_t\|^m$, viewed as a function on $\mathbb{R}^{d^2}$, gives

$$\begin{aligned}\mathrm{d}\|D_t\|^m &= m\|D_t\|^{m-2}\langle D_t, A_tD_t + F_t\rangle\,\mathrm{d}t \\ &\quad + \frac{m}{2}\|D_t\|^{m-2}\sum_{\ell=1}^{\infty}\|U_{\ell,t}\|^2\,\mathrm{d}t \\ &\quad + \frac{m(m-2)}{2}\|D_t\|^{m-4}\sum_{\ell=1}^{\infty}\langle D_t, U_{\ell,t}\rangle^2\,\mathrm{d}t \\ &\quad + \mathrm{d}N_t,\end{aligned}$$

where

$$N_t := m\sum_{\ell=1}^{\infty}\int_0^t \|D_s\|^{m-2}\langle D_s, U_{\ell,s}\rangle\,\mathrm{d}B_s^{\ell}$$

is a continuous local martingale.

The assumptions give

$$\sum_{\ell=1}^{\infty}\|U_{\ell,t}\|^2 = \sum_{\ell=1}^{\infty}\|C_{\ell,t}D_t + G_{\ell,t}\|^2 \le 2L_\sigma^2\|D_t\|^2 + 2g_t^2. \tag{8}$$

Combining the two diffusion terms first and using (8), we obtain

$$\begin{aligned}&\frac{m}{2}\|D_t\|^{m-2}\sum_{\ell=1}^{\infty}\|U_{\ell,t}\|^2 + \frac{m(m-2)}{2}\|D_t\|^{m-4}\sum_{\ell=1}^{\infty}\langle D_t, U_{\ell,t}\rangle^2 \\ &\le \frac{m(m-1)}{2}\|D_t\|^{m-2}\sum_{\ell=1}^{\infty}\|U_{\ell,t}\|^2 \\ &\le m(m-1)L_\sigma^2\|D_t\|^m + m(m-1)\|D_t\|^{m-2}g_t^2.\end{aligned}$$

We shall also use Young's inequality with $a, b \ge 0, p, r > 1, 1/p + 1/r = 1$ in the form

$$ab \le \frac{a^p}{p} + \frac{b^r}{r}.$$

Choosing $p = \frac{m}{m-1}$ and $r = m$, we get

$$m\|D_t\|^{m-1}\|F_t\| \le (m-1)\|D_t\|^m + \|F_t\|^m.$$

Doing the same with $p = \frac{m}{m-2}$ and $r = m/2$, we obtain

$$m(m-1)\|D_t\|^{m-2}g_t^2 \le (m-2)(m-1)\|D_t\|^m + 2(m-1)g_t^m.$$

Combining the drift estimate, the estimate of the second-order terms, and Young's inequalities, the finite-variation part is bounded by

$$\begin{aligned}&mL_a\|D_t\|^m + m\|D_t\|^{m-1}\|F_t\| + m(m-1)L_\sigma^2\|D_t\|^m + m(m-1)\|D_t\|^{m-2}g_t^2 \\ &\le mL_a\|D_t\|^m + (m-1)\|D_t\|^m + \|F_t\|^m + m(m-1)L_\sigma^2\|D_t\|^m \\ &+ (m-1)(m-2)\|D_t\|^m + 2(m-1)g_t^m \\ &= \left(mL_a + m(m-1)L_\sigma^2 + (m-1)^2\right)\|D_t\|^m + \|F_t\|^m + 2(m-1)g_t^m \\ &\le c_m\|D_t\|^m + C_m(\|F_t\|^m + g_t^m),\end{aligned}$$

where one may take

$$c_m := mL_a + m(m-1)L_\sigma^2 + (m-1)^2, \qquad C_m := 2(m-1).$$

Fix $t > 0$. If

$$\mathbb{E}\|D_0\|^m + \int_0^t [\mathbb{E}\|F_s\|^m + \mathbb{E}g_s^m]\,\mathrm{d}s = +\infty, \tag{9}$$

then the asserted estimate is immediate. Hence, for the remainder of the proof, assume that the quantity in (9) is finite.

Since $m > 2$, Hölder's inequality gives

$$\int_0^t \mathbb{E}g_s^2 \,\mathrm{d}s \le \int_0^t \left(\mathbb{E}g_s^m\right)^{2/m} \mathrm{d}s \le t^{1-2/m} \left(\int_0^t \mathbb{E}g_s^m \,\mathrm{d}s\right)^{2/m} < \infty. \tag{10}$$

For $R > 0$, define

$$\tau_R := \inf\{s \ge 0 : \ \|D_s\| \ge R\}.$$

On $[0, t \wedge \tau_R]$, using (8), we have

$$\|D_s\|^{2m-4} \sum_{\ell=1}^{\infty} \langle D_s, U_{\ell,s}\rangle^2 \le \|D_s\|^{2m-2} \sum_{\ell=1}^{\infty} \|U_{\ell,s}\|^2 \le 2L_\sigma^2 R^{2m} + 2R^{2m-2} g_s^2.$$

Consequently, by (10),

$$m^2 \mathbb{E} \int_0^{t\wedge\tau_R} \|D_s\|^{2m-4} \sum_{\ell=1}^{\infty} \langle D_s, U_{\ell,s}\rangle^2 \,\mathrm{d}s \le 2m^2 L_\sigma^2 R^{2m} t + 2m^2 R^{2m-2} \int_0^t \mathbb{E}g_s^2 \,\mathrm{d}s < \infty.$$

Thus

$$\left(N_{r\wedge\tau_R}\right)_{0\le r\le t}$$

is a square-integrable martingale.

Stopping the Itô identity at τ_R, using the preceding bound on its finite-variation part, and taking expectations, we obtain, for every $0 \le r \le t$,

$$\mathbb{E}\|D_{r\wedge\tau_R}\|^m \le \mathbb{E}\|D_0\|^m + c_m \int_0^r \mathbb{E}\|D_{s\wedge\tau_R}\|^m \,\mathrm{d}s + C_m \int_0^r \left[\mathbb{E}\|F_s\|^m + \mathbb{E}g_s^m\right] \mathrm{d}s.$$

Grönwall's lemma therefore yields

$$\mathbb{E}\|D_{t\wedge\tau_R}\|^m \le e^{c_m t}\mathbb{E}\|D_0\|^m + C_m \int_0^t e^{c_m(t-s)} \left[\mathbb{E}\|F_s\|^m + \mathbb{E}g_s^m\right] \mathrm{d}s. \tag{11}$$

Finally, letting $R \to \infty$ and using Fatou's lemma on the left-hand side gives

$$\mathbb{E}\|D_t\|^m \le e^{c_m t}\mathbb{E}\|D_0\|^m + C_m \int_0^t e^{c_m(t-s)} \left[\mathbb{E}\|F_s\|^m + \mathbb{E}\left(\sum_{\ell=1}^{\infty} \|G_{\ell,s}\|^2\right)^{m/2}\right] \mathrm{d}s.$$

□

We are now ready to prove the moment estimate for $\mathcal{K}_t(K)$. The proof contains two steps. First, we obtain moment bounds for $J_t(u)$ at a fixed starting point u and for increments of the flow. Second, we combine these increment estimates with a Kolmogorov–Chentsov continuity argument to control the supremum over a compact set.

Proof of Theorem 2.2. Fix a compact set $K \subset \mathbb{R}^d$, $t \ge 0$, and $p \ge 1$. Choose an even integer $m > \max\{2p, 2d\}$. Throughout the remainder of the proof, $C_m, c_m > 0$ denote deterministic constants depending only on m, d, L_a, L_σ, H. Their values may change from line to line.

Fix $u, v \in \mathbb{R}^d$ and define the $\mathbb{R}^d$-valued process

$$\Delta X_t := X_t(u) - X_t(v).$$

Since the same random measure μ_t appears in both equations, subtraction yields

$$\mathrm{d}\Delta X_t = \big(a(X_t(u), \mu_t) - a(X_t(v), \mu_t)\big)\,\mathrm{d}t + \sum_{\ell=1}^{\infty} \big(\sigma_\ell(X_t(u)) - \sigma_\ell(X_t(v))\big)\,\mathrm{d}B_t^\ell.$$

Define
$$\widetilde{A}_t(u,v) := \int_0^1 Da\big(X_t(v)+\theta\Delta X_t,\mu_t\big)\,\mathrm{d}\theta$$
and, for $\ell \geq 1$,
$$\widetilde{C}_{\ell,t}(u,v) := \int_0^1 D\sigma_\ell\big(X_t(v)+\theta\Delta X_t\big)\,\mathrm{d}\theta.$$
By the fundamental theorem of calculus,
$$\begin{cases} \mathrm{d}\Delta X_t = \widetilde{A}_t(u,v)\Delta X_t\,\mathrm{d}t + \sum_{\ell=1}^\infty \widetilde{C}_{\ell,t}(u,v)\Delta X_t\,\mathrm{d}B_t^\ell, \\ \Delta X_0 = u-v. \end{cases}$$
Moreover, by (3), for every $z\in\mathbb{R}^d$,
$$|\widetilde{A}_t(u,v)z| \leq L_a|z| \qquad \text{and} \qquad \langle z,\widetilde{A}_t(u,v)z\rangle \leq L_a|z|^2.$$
Jensen's inequality also gives, for every $z\in\mathbb{R}^d$,
$$\sum_{\ell=1}^\infty |\widetilde{C}_{\ell,t}(u,v)z|^2 \leq \int_0^1 \sum_{\ell=1}^\infty \big|D\sigma_\ell\big(X_t(v)+\theta\Delta X_t\big)z\big|^2\,\mathrm{d}\theta \leq L_\sigma^2|z|^2.$$
Applying Lemma 2.1 with $q=m$ gives, for all $u,v\in\mathbb{R}^d$ and $t\geq 0$,
$$\mathbb{E}|X_t(u)-X_t(v)|^{2m} \leq e^{m(2L_a+(2m-1)L_\sigma^2)t}|u-v|^{2m}. \tag{12}$$
Applying the same lemma to the Jacobian equation with $q=m/2$ and $q=m$, respectively, gives
$$\mathbb{E}\|J_t(u)\|^m \leq d^{m/2}\exp\left\{\frac{m}{2}\big(2L_a+(m-1)L_\sigma^2\big)t\right\},$$
$$\mathbb{E}\|J_t(u)\|^{2m} \leq d^m\exp\left\{m\big(2L_a+(2m-1)L_\sigma^2\big)t\right\}.$$
Consequently, after increasing the generic constants C_m, c_m if necessary,
$$\mathbb{E}\|J_t(u)\|^m \leq C_m e^{c_m t} \quad \text{and} \quad \mathbb{E}\|J_t(u)\|^{2m} \leq C_m e^{c_m t}, \tag{13}$$
uniformly in $u\in\mathbb{R}^d$. Next define the $\mathbb{R}^{d\times d}$-valued process
$$D_t := J_t(u)-J_t(v).$$
Subtracting the Jacobian equations at u and v, we obtain
$$\begin{cases} \mathrm{d}D_t = A_t(u)D_t\,\mathrm{d}t + \sum_{\ell=1}^\infty C_{\ell,t}(u)D_t\,\mathrm{d}B_t^\ell + F_t\,\mathrm{d}t + \sum_{\ell=1}^\infty G_{\ell,t}\,\mathrm{d}B_t^\ell, \\ D_0 = 0, \end{cases} \tag{14}$$
where
$$F_t := \big(A_t(u)-A_t(v)\big)J_t(v)$$
and for $\ell\geq 1$
$$G_{\ell,t} := \big(C_{\ell,t}(u)-C_{\ell,t}(v)\big)J_t(v).$$
Here, F_t and $G_{\ell,t}$, $\ell\geq 1$, are $\mathbb{R}^{d\times d}$-valued processes.

Applying (3) to the columns of any $M\in\mathbb{R}^{d\times d}$, we obtain
$$\langle M, A_t(u)M\rangle \leq L_a\|M\|^2,$$
and
$$\sum_{\ell=1}^\infty \|C_{\ell,t}(u)M\|^2 \leq L_\sigma^2\|M\|^2.$$
Similarly, applying (2) columnwise gives
$$\|F_t\| \leq H|\Delta X_t|\,\|J_t(v)\|$$

and

$$\left(\sum_{\ell=1}^{\infty}\|G_{\ell,t}\|^2\right)^{1/2} \le H|\Delta X_t|\,\|J_t(v)\|.$$

Consequently, Hölder's inequality, (12), and (13) give

$$\mathbb{E}\|F_s\|^m + \mathbb{E}\left(\sum_{\ell=1}^{\infty}\|G_{\ell,s}\|^2\right)^{m/2} \le C_m e^{c_m s}|u-v|^m, \qquad s \ge 0. \tag{15}$$

Applying Lemma 2.2 to (14), using $D_0 = 0$ and (15), and increasing C_m, c_m if necessary, we obtain

$$\mathbb{E}\|D_t\|^m \le C_m|u-v|^m \int_0^t e^{c_m(t-s)}e^{c_m s}\,\mathrm{d}s = C_m t e^{c_m t}|u-v|^m \le C_m e^{(c_m+1)t}|u-v|^m.$$

After relabelling $c_m + 1$ as c_m, we conclude that

$$\mathbb{E}\|J_t(u) - J_t(v)\|^m \le C_m e^{c_m t}|u-v|^m, \qquad u, v \in \mathbb{R}^d. \tag{16}$$

Let $Q \subset \mathbb{R}^d$ be a compact cube containing K, and choose $u_0 \in K$. Choose

$$\rho \in \left(0, 1 - \frac{d}{m}\right).$$

The Kolmogorov–Chentsov theorem [6, Theorem I.2.1 and its proof], applied after a rescaling of Q to the unit cube, yields a modification satisfying

$$\mathbb{E}\left[\sup_{\substack{u,v\in Q\\ u\neq v}} \frac{\|J_t(u) - J_t(v)\|^m}{|u-v|^{\rho m}}\right] < \infty. \tag{17}$$

Therefore,

$$\mathbb{E}\left[\sup_{u\in Q}\|J_t(u) - J_t(u_0)\|^m\right] \le \operatorname{diam}(Q)^{\rho m}\mathbb{E}\left[\sup_{\substack{u,v\in Q\\ u\neq v}} \frac{\|J_t(u) - J_t(v)\|^m}{|u-v|^{\rho m}}\right] < \infty. \tag{18}$$

For every $u \in K$,

$$\|J_t(u)\| \le \|J_t(u_0)\| + \|J_t(u) - J_t(u_0)\|.$$

Therefore,

$$\mathcal{K}_t(K) \le \|J_t(u_0)\| + \sup_{u\in Q}\|J_t(u) - J_t(u_0)\|.$$

Using (13), (18), and $(a+b)^m \le 2^{m-1}(a^m + b^m)$, we obtain

$$\mathbb{E}\mathcal{K}_t(K)^m \le 2^{m-1}\left(\mathbb{E}\|J_t(u_0)\|^m + \mathbb{E}\left[\sup_{u\in Q}\|J_t(u) - J_t(u_0)\|^m\right]\right) < \infty.$$

Since $p < m$, Jensen's inequality gives

$$\mathbb{E}\mathcal{K}_t(K)^p \le (\mathbb{E}\mathcal{K}_t(K)^m)^{p/m} < \infty.$$

$\square$

3. Dispersion

We now turn to a related but distinct interacting model of kernel type. Here the coefficients may also depend on the finite-dimensional statistic

$$\Theta_t := \int_{\mathbb{R}^d} \Phi(X_t(v))\, \mu_0(\mathrm{d}v).$$

The purpose of this section is different from that of the previous sections: rather than estimating Jacobian norms, we study the dispersion. Accordingly, the assumptions used in Section 2 are not imposed here.

$$\begin{cases} \mathrm{d}X_t(u) &= \Big[b(X_t(u),\Theta_t) + \int_{\mathbb{R}^d} F(X_t(u), X_t(v), \Theta_t)\, \mu_0(\mathrm{d}v) \Big] \mathrm{d}t \\ &\quad + \sum_{\ell=1}^{\infty} \sigma_\ell(X_t(u),\Theta_t)\, \mathrm{d}B_t^\ell, \\ X_0(u) &= u, \qquad u \in \mathbb{R}^d, \\ \mu_t &= \mu_0 \circ X_t^{-1}. \end{cases} \tag{19}$$

Let $q \geq 1$, and let

$$\Phi : \mathbb{R}^d \to \mathbb{R}^q, \qquad b : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d, \qquad F : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d,$$

with $\sigma = (\sigma_\ell)_{\ell \geq 1} : \mathbb{R}^d \times \mathbb{R}^q \to \ell^2(\mathbb{R}^d)$. Define

$$\theta(\mu) := \int_{\mathbb{R}^d} \Phi(x)\, \mu(\mathrm{d}x), \qquad \Theta_t := \theta(\mu_t) = \int_{\mathbb{R}^d} \Phi(X_t(v))\, \mu_0(\mathrm{d}v).$$

Assumption 3 (Well-posedness of (19))**.** *The maps b, F, Φ, and σ are globally Lipschitz. More explicitly, there exists $L < \infty$ such that*

$$\begin{aligned} |\Phi(x) - \Phi(y)| &\leq L|x-y|, \\ |b(x,\theta) - b(y,\eta)| &\leq L\big(|x-y| + |\theta-\eta|\big), \\ |F(x,z,\theta) - F(y,w,\eta)| &\leq L\big(|x-y| + |z-w| + |\theta-\eta|\big), \end{aligned}$$

and

$$\sum_{\ell=1}^{\infty} |\sigma_\ell(x,\theta) - \sigma_\ell(y,\eta)|^2 \leq L^2\big(|x-y|^2 + |\theta-\eta|^2\big).$$

Under Assumption 3, equation (19) admits a unique global continuous solution.
Define the random center of mass by

$$\bar{X}_t := \int_{\mathbb{R}^d} X_t(u)\, \mu_0(\mathrm{d}u),$$

and define the dispersion at time t by

$$\mathcal{V}_t := \int_{\mathbb{R}^d} |X_t(u) - \bar{X}_t|^2\, \mu_0(\mathrm{d}u).$$

Since $\mu_t = \mu_0 \circ X_t^{-1}$, we also have

$$\mathcal{V}_t = \int_{\mathbb{R}^d} |x - \bar{X}_t|^2\, \mu_t(\mathrm{d}x) = W_2^2(\mu_t, \delta_{\bar{X}_t}).$$

We will use the elementary identity

$$2\mathcal{V}_t = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |X_t(u) - X_t(v)|^2\, \mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v). \tag{20}$$

Indeed,

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d} |X_t(u)-X_t(v)|^2\,\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v)\\
&= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \big(|X_t(u)|^2+|X_t(v)|^2-2\langle X_t(u),X_t(v)\rangle\big)\,\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v)\\
&= 2\int_{\mathbb{R}^d}|X_t(u)|^2\,\mu_0(\mathrm{d}u)-2\left|\int_{\mathbb{R}^d}X_t(u)\,\mu_0(\mathrm{d}u)\right|^2\\
&= 2\int_{\mathbb{R}^d}|X_t(u)-\bar X_t|^2\,\mu_0(\mathrm{d}u)=2\mathcal{V}_t.
\end{aligned}$$

The quantities $\bar X_t$ and $\mathcal{V}_t$ play the role of the centroid and scalar dispersion studied for random measures transported passively by Brownian flows in [8]. The difference here is that (19) is measure-dependent and interacting; under the pairwise dissipativity condition below, the dispersion satisfies exponential decay estimate.

3.1. **Dissipativity.** We assume the following dissipativity condition.

Assumption 4 (Pairwise dissipativity)**.** *There exists $\lambda>0$ such that for every $x,y\in\mathbb{R}^d$, every $\theta\in\mathbb{R}^q$, and every $\nu\in\mathcal{P}_2(\mathbb{R}^d)$,*

$$\begin{aligned}
&2\left\langle x-y,b(x,\theta)-b(y,\theta)+\int_{\mathbb{R}^d}\big[F(x,z,\theta)-F(y,z,\theta)\big]\nu(\mathrm{d}z)\right\rangle\\
&\qquad+\sum_{\ell=1}^\infty|\sigma_\ell(x,\theta)-\sigma_\ell(y,\theta)|^2\le-\lambda|x-y|^2.
\end{aligned}$$

The dissipativity condition covers several basic models.

Example 3.1 (Ornstein–Uhlenbeck SDEWI)**.** Consider

$$\begin{aligned}
\mathrm{d}X_t(u)&=-\lambda_0\big(X_t(u)-m_t\big)\,\mathrm{d}t+\sum_{\ell=1}^\infty\sigma_\ell(X_t(u))\,\mathrm{d}B_t^\ell,\\
m_t&=\int_{\mathbb{R}^d}X_t(v)\,\mu_0(\mathrm{d}v).
\end{aligned}$$

If $\sigma=(\sigma_\ell)_{\ell\ge1}:\mathbb{R}^d\to\ell^2(\mathbb{R}^d)$ satisfies

$$\left(\sum_{\ell=1}^\infty|\sigma_\ell(x)-\sigma_\ell(y)|^2\right)^{1/2}\le L_\sigma|x-y|,$$

then, for all $x,y,m\in\mathbb{R}^d$,

$$2\langle x-y,-\lambda_0(x-m)+\lambda_0(y-m)\rangle=-2\lambda_0|x-y|^2.$$

Together with the Lipschitz bound, this gives

$$\lambda=2\lambda_0-L_\sigma^2.$$

Example 3.2 (Gradient systems)**.** Take

$$b(x,\theta)=-\nabla V(x),\qquad F(x,y,\theta)=-\nabla W(x-y).$$

Assume in addition that ∇V and ∇W are globally Lipschitz. Then

$$\begin{aligned}
\mathrm{d}X_t(u)=&\left[-\nabla V(X_t(u))-\int_{\mathbb{R}^d}\nabla W(X_t(u)-X_t(v))\,\mu_0(\mathrm{d}v)\right]\mathrm{d}t\\
&+\sum_{\ell=1}^\infty\sigma_\ell(X_t(u))\,\mathrm{d}B_t^\ell.
\end{aligned}$$

If

$$\nabla^2 V \ge \rho I, \qquad \nabla^2 W \ge \rho_W I,$$

$$\left(\sum_{\ell=1}^{\infty} |\sigma_\ell(x) - \sigma_\ell(y)|^2\right)^{1/2} \le L_\sigma |x-y|,$$

then, for all $x, y, z \in \mathbb{R}^d$,

$$2\langle x-y, -\nabla V(x) + \nabla V(y)\rangle \le -2\rho|x-y|^2,$$
$$2\langle x-y, -\nabla W(x-z) + \nabla W(y-z)\rangle \le -2\rho_W|x-y|^2.$$

Together with the Lipschitz bound, this gives

$$\lambda = 2\rho + 2\rho_W - L_\sigma^2.$$

3.2. **Decay of dispersion.**

Theorem 3.1 (Exponential decay of dispersion)**.** *Assume that $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and that Assumptions 3 and 4 hold. Then, for every $t \ge 0$,*

$$\mathbb{E}\mathcal{V}_t \le e^{-\lambda t}\mathcal{V}_0.$$

Equivalently,

$$\mathbb{E}\left[W_2^2(\mu_t, \delta_{\bar{X}_t})\right] \le e^{-\lambda t} W_2^2(\mu_0, \delta_{\bar{X}_0}).$$

Proof. Fix two starting points $u, v \in \mathbb{R}^d$ and set

$$Z_t^{u,v} := X_t(u) - X_t(v).$$

We first prove that under Assumption 4

$$\mathbb{E}|Z_t^{u,v}|^2 \le e^{-\lambda t}|u-v|^2,$$

then we integrate this estimate with respect to $\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v)$.

Subtracting the equations for $X_t(u)$ and $X_t(v)$, we obtain

$$\mathrm{d}Z_t^{u,v} = G_t^{u,v}\,\mathrm{d}t + \sum_{\ell=1}^{\infty} H_{\ell,t}^{u,v}\,\mathrm{d}B_t^\ell,$$

where

$$G_t^{u,v} := b(X_t(u), \Theta_t) - b(X_t(v), \Theta_t) + \int_{\mathbb{R}^d} \left[F(X_t(u), X_t(w), \Theta_t) - F(X_t(v), X_t(w), \Theta_t)\right] \mu_0(\mathrm{d}w),$$

and for $\ell \ge 1$

$$H_{\ell,t}^{u,v} := \sigma_\ell(X_t(u), \Theta_t) - \sigma_\ell(X_t(v), \Theta_t).$$

Since $\mu_t = \mu_0 \circ X_t^{-1}$, we may rewrite the interaction term as

$$\int_{\mathbb{R}^d} F(X_t(u), X_t(w), \Theta_t)\,\mu_0(\mathrm{d}w) = \int_{\mathbb{R}^d} F(X_t(u), z, \Theta_t)\,\mu_t(\mathrm{d}z).$$

Thus

$$G_t^{u,v} = b(X_t(u), \Theta_t) - b(X_t(v), \Theta_t) + \int_{\mathbb{R}^d} \left[F(X_t(u), z, \Theta_t) - F(X_t(v), z, \Theta_t)\right] \mu_t(\mathrm{d}z).$$

Applying Itô's formula to $|Z_t^{u,v}|^2$, we obtain

$$\mathrm{d}|Z_t^{u,v}|^2 = 2\langle Z_t^{u,v}, G_t^{u,v}\rangle\,\mathrm{d}t + \sum_{\ell=1}^{\infty} |H_{\ell,t}^{u,v}|^2\,\mathrm{d}t + 2\sum_{\ell=1}^{\infty}\langle Z_t^{u,v}, H_{\ell,t}^{u,v}\rangle\,\mathrm{d}B_t^\ell.$$

Equivalently, in integral form,

$$|Z_t^{u,v}|^2 = |u-v|^2 + \int_0^t \left[2\langle Z_s^{u,v}, G_s^{u,v}\rangle + \sum_{\ell=1}^{\infty} |H_{\ell,s}^{u,v}|^2\right] \mathrm{d}s + 2\sum_{\ell=1}^{\infty}\int_0^t \langle Z_s^{u,v}, H_{\ell,s}^{u,v}\rangle \, \mathrm{d}B_s^{\ell}.$$

Applying Assumption 4 with $x = X_s(u)$, $y = X_s(v)$, $\theta = \Theta_s$, and $\nu = \mu_s$ gives, for every $s \geq 0$,

$$2\langle Z_s^{u,v}, G_s^{u,v}\rangle + \sum_{\ell=1}^{\infty} |H_{\ell,s}^{u,v}|^2 \leq -\lambda |Z_s^{u,v}|^2. \tag{21}$$

Set

$$Y_t^{u,v} := e^{\lambda t}|Z_t^{u,v}|^2.$$

Applying Itô's formula gives

$$\begin{aligned} Y_t^{u,v} &= |u-v|^2 + \int_0^t e^{\lambda s}\left[\lambda |Z_s^{u,v}|^2 + 2\langle Z_s^{u,v}, G_s^{u,v}\rangle + \sum_{\ell=1}^{\infty} |H_{\ell,s}^{u,v}|^2\right] \mathrm{d}s \\ &\quad + 2\sum_{\ell=1}^{\infty}\int_0^t e^{\lambda s}\langle Z_s^{u,v}, H_{\ell,s}^{u,v}\rangle \, \mathrm{d}B_s^{\ell}. \end{aligned} \tag{22}$$

By (21), the finite-variation integrand in (22) is nonpositive.

To make this rigorous, introduce the localization sequence

$$\tau_R := \inf\{t \geq 0 : |X_t(u)| + |X_t(v)| \geq R\}.$$

By the Lipschitz bound on σ, and since the same value of Θ_s appears in both diffusion coefficients,

$$\sum_{\ell=1}^{\infty} |H_{\ell,s}^{u,v}|^2 \leq L^2 |Z_s^{u,v}|^2.$$

On $[0, t\wedge\tau_R]$, we have $|Z_s^{u,v}| \leq |X_s(u)| + |X_s(v)| \leq R$, hence

$$\sum_{\ell=1}^{\infty} \left| e^{\lambda s}\langle Z_s^{u,v}, H_{\ell,s}^{u,v}\rangle\right|^2 \leq e^{2\lambda t} L^2 R^4.$$

Therefore the stopped stochastic integral is a square-integrable martingale.

Since the solution is global and continuous, $\tau_R \uparrow \infty$ almost surely as $R \to \infty$. Stopping (22) at τ_R and using the nonpositivity of the finite-variation term gives

$$Y_{t\wedge\tau_R}^{u,v} \leq Y_0^{u,v} + 2\sum_{\ell=1}^{\infty}\int_0^{t\wedge\tau_R} e^{\lambda s}\langle Z_s^{u,v}, H_{\ell,s}^{u,v}\rangle \, \mathrm{d}B_s^{\ell}.$$

The stopped stochastic integral is a square-integrable martingale. Taking expectations therefore gives

$$\mathbb{E}Y_{t\wedge\tau_R}^{u,v} \leq Y_0^{u,v}.$$

Letting $R \to \infty$ and using Fatou's lemma yields

$$\mathbb{E}\left[e^{\lambda t}|Z_t^{u,v}|^2\right] \leq |u-v|^2.$$

Therefore

$$\mathbb{E}|X_t(u) - X_t(v)|^2 \leq e^{-\lambda t}|u-v|^2.$$

Integrating the pairwise estimate with respect to $\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v)$ and using Tonelli's theorem gives

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{E}|X_t(u) - X_t(v)|^2 \, \mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v) \leq e^{-\lambda t}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |u-v|^2 \, \mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v).$$

By (20), this is exactly

$$2\mathbb{E}\mathcal{V}_t \leq 2e^{-\lambda t}\mathcal{V}_0.$$

Therefore

$$\mathbb{E}\mathcal{V}_t \leq e^{-\lambda t}\mathcal{V}_0.$$

Finally, since

$$\mathcal{V}_t = W_2^2(\mu_t, \delta_{\bar{X}_t})$$

pathwise, we obtain

$$\mathbb{E}\left[W_2^2(\mu_t, \delta_{\bar{X}_t})\right] \leq e^{-\lambda t}W_2^2(\mu_0, \delta_{\bar{X}_0}).$$

□

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