Theory of Stochastic Processes
Vol. 30 (46), no. 1, 2026, pp. 1–14

LAHCEN KHAMMICH AND DRISS KIOUACH

STATIONARY DISTRIBUTION OF A STOCHASTIC $\mathcal{HLIV}$ EPIDEMIC MODEL WITH VIRAL PRODUCTION AND MEAN-REVERTING INHOMOGENEOUS GEOMETRIC BROWNIAN MOTION

In this work, we investigate a stochastic $\mathcal{HLIV}$ model incorporating viral dynamics, in which the mean-reverting inhomogeneous geometric Brownian motion process (IGBM for short) perturb the conversion rate from eclipse phase cells to infected cells. We first establish the existence and uniqueness of a positive solution and show that the model admits a unique global solution for any given initial condition. We then employ stochastic Lyapunov functions together with Fatou's lemma to derive sufficient conditions for the existence of a stationary distribution of positive solution, thereby indicating strong persistence of the disease. Finally, we conclude by summarizing the main results and implications of our study.

1. Introduction

In epidemiology, the period between a virus entering a host cell and the initiation of active viral replication is known as the eclipse phase [2]. Understanding how within-host viral dynamics influence virus transmission at the population level (between hosts) has attracted significant interest [18]. Over recent decades, mathematical modeling has emerged as a vital tool for exploring the complex nonlinear processes and intricate dynamics of infectious diseases. By analyzing mathematical models of within-host virus behavior, valuable insights can be gained to inform immunization strategies, optimize medical resource allocation, and improve efforts in disease control and eradication [13]. Numerous studies have investigated these dynamics, contributing to this growing field of research. Some human viruses, such as the human immunodeficiency virus (HIV) [16], Human $\mathcal{T}$-cell Leukemia/Lymphoma virus type I (HTLV-I) [8], hepatitis B virus (HBV) [19], and hepatitis C virus (HCV) [1], have been extensively studied in this context. In their seminal monograph, Nowak and May introduced a foundational viral infection model that has played a crucial role in advancing research on viral dynamics. Since then, numerous models have been proposed in the literature to explore the impact of the eclipse phase on viral establishment dynamics and peak viral load [7]. Notably, Bai et al. [2] developed an $\mathcal{HLIV}$ model incorporating viral production, which can be represented by the following system:

$$
\left\{
\begin{aligned}
\mathrm{d}\mathcal{H}(t) &= (\Delta - \beta\mathcal{V}\mathcal{H} - \mu_H\mathcal{H})\,\mathrm{d}t,\\
\mathrm{d}\mathcal{L}(t) &= (\beta\mathcal{V}\mathcal{H} - \delta\mathcal{L} - \mu\mathcal{L})\mathrm{d}t,\\
\mathrm{d}\mathcal{I}(t) &= (\delta\mathcal{L} - \mu_I\mathcal{I} - \alpha\mathcal{I})\,\mathrm{d}t,\\
\mathrm{d}\mathcal{V}(t) &= (b\mathcal{I} - \beta\mathcal{V}\mathcal{H} - c\mathcal{V})\mathrm{d}t.
\end{aligned}
\right.
\tag{1}
$$

Here, $\mathcal{H}$ denotes the density of uninfected target cells, $\mathcal{L}$ represents the density of eclipse-phase cells, $\mathcal{I}$ denotes the density of infected cells, and $\mathcal{V}$ denotes the concentration of virions.

2020 *Mathematics Subject Classification.* 60H10, 60J60, 92D30.
Key words and phrases. $\mathcal{HLIV}$ epidemic model, inhomogeneous geometric Brownian motion process, ergodic stationary distribution, extinction.

Parameters	Description
Δ	Recruitment rate of susceptible (target) cells into the system
β	Effective contact rate governing virus–cell transmission
$\mu_{\mathcal{H}}$	Per capita mortality rate of healthy target cells
δ	Transition rate from the eclipse phase to productive infection
μ	Baseline death rate of eclipse-phase cells
μ_I	Baseline death rate of productively infected cells
α	Additional mortality rate of infected cells due to viral pathogenicity
b	Average rate of virion production per infected cell
c	Per capita viral clearance rate

TABLE 1. List of parameters.

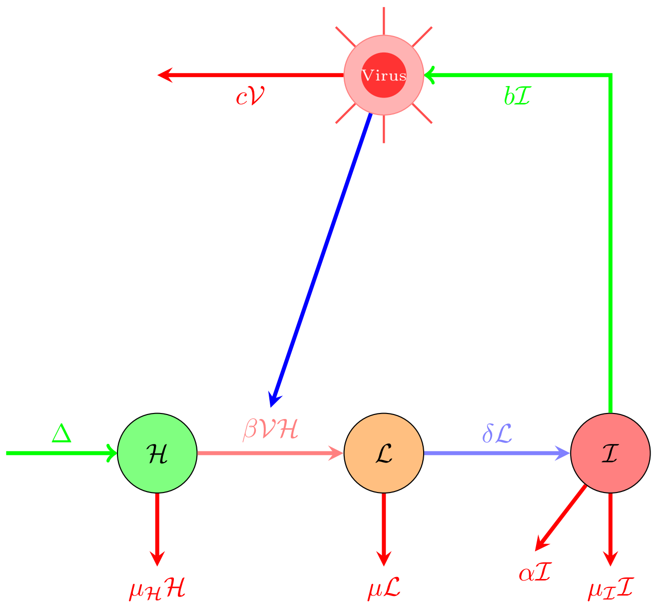

FIGURE 1. Schematic diagram for $\mathcal{HLIV}$ model.

All parameters are assumed to be positive constants, and their descriptions are provided in Table 1. The basic reproduction number $\mathcal{R}_0$ associated with system (1) is given by

$$\mathcal{R}_0 = \frac{\delta\beta\mathcal{H}_0 b}{(\alpha+\mu_{\mathcal{I}})(c+\beta\mathcal{H}_0)(\mu+\delta)},$$

where

$$\mathcal{H}_0 = \frac{\Delta}{\mu_{\mathcal{H}}}.$$

For more detail about asymptotically analysis and disease equilibrium see [11, 9].

However, modeling infectious diseases plays a fundamental role in understanding their transmission dynamics and enables decision-makers to design and implement effective control strategies to mitigate disease spread. Consequently, stochastic epidemic models have attracted considerable attention, particularly those incorporating random fluctuations in the disease transmission rate through the Ornstein–Uhlenbeck (OU) process.

The OU process captures stochastic perturbations while maintaining a bounded variance over finite time intervals $[0, t]$, making it suitable for modeling persistent random fluctuations.

The Ornstein–Uhlenbeck process is governed by the following stochastic differential equation:

$$\mathrm{d}\delta(t) = \rho(\bar{\delta} - \delta(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B(t).$$

Here, $\bar{\delta} > 0$ denotes the long-term mean of the process, $\rho > 0$ is the mean-reversion rate, and $\sigma > 0$ represents the volatility of the stochastic perturbations driven by a standard Brownian motion $\{B(t)\}_{t\geq 0}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, namely, right continuity and completeness.

In epidemic models, the disease transmission rate δ is inherently positive. Therefore, preserving the positivity of δ is an essential requirement when introducing stochastic perturbations. Unfortunately, the Ornstein–Uhlenbeck process does not guarantee the positivity of its sample paths. To overcome this limitation, we propose a perturbation approach analogous to the Ornstein–Uhlenbeck process while ensuring the positivity of the transmission rate. Specifically, we employ the mean-reverting inhomogeneous geometric Brownian motion (IGBM) to perturb the transmission rate δ [3, 17, 4]. The IGBM process is governed by the stochastic differential equation

$$\mathrm{d}\delta(t) = \rho(\bar{\delta} - \delta(t))\,\mathrm{d}t + \sigma\delta(t)\,\mathrm{d}B(t).$$

According to [3, 15, 12], the IGBM process admits a unique ergodic stationary distribution characterized by an inverse-gamma density with shape parameter

$$\frac{\sigma^2 + 2\rho}{\sigma^2}$$

and scalar parameter

$$\frac{2\rho\bar{\delta}}{\sigma^2}.$$

Furthermore, for any π-integrable function Ψ, the following ergodic property holds:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Psi(\delta(s))ds = \int_0^\infty \Psi(x)\pi(dx).$$

In the light of what precedes, the system 1 can be rewriting as follow:

$$(2) \qquad \begin{cases} \mathrm{d}\mathcal{H}(t) = (\Delta - \beta\mathcal{V}\mathcal{H} - \mu_{\mathcal{H}}\mathcal{H})\,\mathrm{d}t, \\ \mathrm{d}\mathcal{L}(t) = (\beta\mathcal{V}\mathcal{H} - \delta\mathcal{L} - \mu\mathcal{L})\mathrm{d}t, \\ \mathrm{d}\mathcal{I}(t) = (\delta\mathcal{L} - \mu_{\mathcal{I}}\mathcal{I} - \alpha\mathcal{I})\,\mathrm{d}t, \\ \mathrm{d}\mathcal{V}(t) = (b\mathcal{I} - \beta\mathcal{V}\mathcal{H} - c\mathcal{V})\mathrm{d}t, \\ \mathrm{d}\delta(t) = \rho(\bar{\delta} - \delta(t))\mathrm{d}t + \sigma\delta(t)\mathrm{d}B(t). \end{cases}$$

The remainder of this paper is structured as follows: Section 2 focuses on demonstrating the well-posedness (see A.3) of model 2: showing that it has a unique, global-in-time, and positive solution. In Sections 3 , we present the necessary conditions for the existence of a stationary distribution , Finally, the main conclusions of the paper are discussed in Section 4.

2. Existence and uniqueness of the global positive solution

Theorem 1. *For any initial value*

$$X_0 = \big(\delta(0), \mathcal{H}(0), \mathcal{L}(0), \mathcal{I}(0), \mathcal{V}(0)\big) \in \Xi,$$

where

$$\Xi := \left\{ (\delta, \mathcal{H}, \mathcal{L}, \mathcal{I}, \mathcal{V}) \in \mathbb{R}^5_+ \,\middle|\, \mathcal{H} + \mathcal{L} + \mathcal{I} < \frac{\Delta}{\bar{\mu}},\ \mathcal{V} < \frac{b\Delta}{c\bar{\mu}} \right\},$$

and

$$\bar{\mu} = \min\{\mu_{\mathcal{H}},\, \mu,\, \mu_{\mathcal{I}}\},$$

system (2) *admits a unique global positive solution*

$$(\delta(t), \mathcal{H}(t), \mathcal{L}(t), \mathcal{I}(t), \mathcal{V}(t))$$

for all $t \geq 0$.

Proof. A detailed proof of this theorem is provided in Appendix B. □

3. Stationary distribution

This section investigates the existence of a stationary distribution for the stochastic model, which serves as a fundamental indicator of the long-term persistence of the disease.

Lemma 2 ([6, 5])**.** *Let* $\mathbb{L} \subset \mathbb{R}^d$ *be a bounded closed domain with a regular boundary* L_0. *Assume that, for any initial value* $X(0) \in \mathbb{R}^d$,

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{P}(s, X(0), \mathbb{L})\, ds > 0 \quad a.s.,$$

where $\mathbb{P}(s, X(0), \mathbb{L})$ *denotes the transition probability of the process* $X(t)$ *entering the set* $\mathbb{L}$ *at time* s. *Then the process possesses the Feller property, and consequently, system* (2) *admits at least one stationary distribution* $\eta(\cdot)$ *on* $\mathbb{R}^d$.

Define

$$\mathcal{R}_0^s = \frac{b\beta\Delta\hat{\delta}}{\mu_{\mathcal{H}}(\mu_{\mathcal{I}}+\alpha)(\beta\mathcal{H}_0+c)(\bar{\delta}+\mu)} - \frac{D}{\bar{\delta}+\mu},$$

where

$$D = \frac{1}{2}\int_0^{+\infty} |x-\bar{\delta}|\,\pi(x)\,dx, \qquad \hat{\delta} = \left(\int_0^\infty x^{1/4}\pi(x)\,dx\right)^4,$$

and π denotes the inverse-gamma invariant density with shape parameter

$$\frac{\sigma^2+2\rho}{\sigma^2}$$

and scalar parameter

$$\frac{2\rho\bar{\delta}}{\sigma^2}.$$

Theorem 3. *If* $\mathcal{R}_0^s > 1$, *then system* (2) *admits at least one ergodic stationary distribution* $\eta(\cdot)$ *in the domain* Ξ.

Proof. Define the following C^2 real-valued functions:

$$\begin{aligned}
W_1 &= -\ln\mathcal{L} - a_1\ln\mathcal{H} - a_2\ln\mathcal{I} - a_3\ln\mathcal{V} + \frac{a_3\beta}{\mu_{\mathcal{H}}}\mathcal{H} + \frac{a_1\beta}{c}\mathcal{V},\\
W_2 &= -\ln\mathcal{H} - \ln\mathcal{L} - \ln\mathcal{V},\\
W_3 &= -\ln\left(\frac{\Delta}{\bar{\mu}} - \mathcal{H} - \mathcal{L} - \mathcal{I}\right) - \ln\left(\frac{b\Delta}{c\bar{\mu}} - \mathcal{V}\right),\\
W_4 &= \delta - \ln\delta,\\
W_5 &= N_1W_1 + W_2 + W_3 + N_2W_4.
\end{aligned}$$

Applying Itô's formula to the above functions, we obtain

$$
\begin{aligned}
\mathscr{L}W_1 &= \left(-\frac{\beta\mathcal{V}\mathcal{H}}{\mathcal{L}} - \frac{a_1\Delta}{\mathcal{H}} - \frac{a_2\delta\mathcal{L}}{\mathcal{L}} - \frac{a_3 b\mathcal{I}}{\mathcal{V}}\right) + \mu + a_1\mu_{\mathcal{H}} + a_2(\mu_{\mathcal{I}}+\alpha) \\
&\quad + a_3(\beta\mathcal{H}_0 + c) + \delta + a_1\beta\mathcal{V} - \frac{a_3\beta^2}{\mu_{\mathcal{H}}}\mathcal{V}\mathcal{H} + \frac{a_1\beta b}{c}\mathcal{I} - \frac{a_1\beta^2}{c}\mathcal{V}\mathcal{H} - a_1\beta\mathcal{V}, \\
&\leq -4\sqrt[4]{a_1a_2a_3b\beta\Delta\delta} + \bar{\delta} + \left(\delta(t) - \bar{\delta}\right) + \mu + a_1\mu_{\mathcal{H}} + a_2(\mu_{\mathcal{I}}+\alpha) \\
&\quad + a_3(\beta\mathcal{H}_0 + c) + \frac{a_1\beta b\mathcal{I}}{c}, \\
&\leq -4\sqrt[4]{a_1a_2a_3b\beta\Delta\hat{\delta}} + \bar{\delta} + (\delta(t) - \bar{\delta})^+ + \mu + a_1\mu_{\mathcal{H}} + a_2(\mu_{\mathcal{I}}+\alpha) \\
&\quad + a_3(\beta\mathcal{H}_0 + c) + \frac{a_1\beta b\mathcal{I}}{c} + \left(4\sqrt[4]{a_1a_2a_3b\beta\Delta\hat{\delta}} - 4\sqrt[4]{a_1a_2a_3b\beta\Delta\delta}\right).
\end{aligned} \tag{3}
$$

Let

$$
a_1 = \frac{b\beta\Delta\hat{\delta}}{\mu_{\mathcal{H}}^2(\mu_{\mathcal{I}}+\alpha)(\beta\mathcal{H}_0+c)},
$$

$$
a_2 = \frac{b\beta\Delta\hat{\delta}}{\mu_{\mathcal{H}}(\mu_{\mathcal{I}}+\alpha)^2(\beta\mathcal{H}_0+c)},
$$

$$
a_3 = \frac{b\beta\Delta\hat{\delta}}{\mu_{\mathcal{H}}(\mu_{\mathcal{I}}+\alpha)(\beta\mathcal{H}_0+c)^2}.
$$

and

$$
D = \frac{1}{2}\int_0^{+\infty} |\bar{\delta} - x|\pi(x)\,dx, \qquad \hat{\delta} = \left(\int_0^{\infty} x^{\frac{1}{4}}\pi(x)\,dx\right)^4.
$$

Then, we have

$$
\begin{aligned}
\mathscr{L}W_1 &\leq -\frac{b\beta\Delta\hat{\delta}}{\mu_{\mathcal{H}}(\mu_{\mathcal{I}}+\alpha)(\beta\mathcal{H}_0+c)} + \bar{\delta} + \mu + D + 4\sqrt[4]{a_1a_2a_3b\beta\Delta}\left(\sqrt[4]{\hat{\delta}} - \sqrt[4]{\delta}\right) \\
&\quad + \left((\delta(t)-\bar{\delta})^+ - D\right) + \frac{a_1\beta b}{c}\mathcal{I}, \\
&\leq -(\bar{\delta}+\mu)\left(\frac{b\beta\Delta\hat{\delta}}{\mu_{\mathcal{H}}(\mu_{\mathcal{I}}+\alpha)(\beta\mathcal{H}_0+c)(\bar{\delta}+\mu)} - \frac{D}{\bar{\delta}+\mu} - 1\right) + \frac{a_1\beta b}{c}I \\
&\qquad + f_1(\hat{\delta}) + f_2(\delta) \\
&\leq -(\bar{\delta}+\mu)\left(\mathcal{R}_0^s - 1\right) + \frac{a_1\beta b}{c}I + f_1(\hat{\delta}) + f_2(\delta).
\end{aligned} \tag{4}
$$

For W_2, we have

$$
\begin{aligned}
\mathscr{L}W_2 &= -\frac{\Delta}{\mathcal{H}} - \frac{\beta\mathcal{V}\mathcal{H}}{\mathcal{L}} - \frac{b\mathcal{I}}{\mathcal{V}} + \beta\mathcal{H} + \beta\mathcal{V} + \delta + \mu_{\mathcal{H}} + \mu + c, \\
&\leq -\frac{\Delta}{\mathcal{H}} - \frac{\beta\mathcal{V}\mathcal{H}}{\mathcal{L}} - \frac{b\mathcal{I}}{\mathcal{V}} + \frac{\beta\Delta(b+c)}{c\bar{\mu}} + \mu_{\mathcal{H}} + \mu + c + (\delta - \bar{\delta}) + \bar{\delta}.
\end{aligned} \tag{5}
$$

For W_3, we have

$$
\begin{aligned}
\mathscr{L}W_3 &= \frac{\Delta - \mu_{\mathcal{H}}\mathcal{H} - \mu\mathcal{L} - \mu_{\mathcal{I}}\mathcal{I} - \alpha\mathcal{I}}{\frac{\Delta}{\mu} - \mathcal{H} - \mathcal{L} - \mathcal{I}} + \frac{b\mathcal{I} - \beta\mathcal{V}\mathcal{H} - c\mathcal{V}}{\frac{b\Delta}{c\bar{\mu}} - \mathcal{V}} \\
&\leq \frac{-\alpha\mathcal{I}}{\frac{\Delta}{\mu} - \mathcal{H} - \mathcal{L} - \mathcal{I}} - \frac{\beta\mathcal{V}\mathcal{H}}{\frac{b\Delta}{c\bar{\mu}} - \mathcal{V}} + \bar{\mu} + c.
\end{aligned} \tag{6}
$$

For W_4, we have

$$\mathscr{L}W_4 = \theta(\bar{\delta}-\delta) - \theta\frac{\bar{\delta}}{\delta} + \theta + \frac{\sigma^2}{2}. \tag{7}$$

For W_5, we have

$$\begin{aligned}
\mathscr{L}W_5 \leq & -N_1(\bar{\delta}+\mu)\left(\mathcal{R}_0^s-1\right) + N_1\frac{a_1\beta b}{c}\mathcal{I} - \frac{\Delta}{\mathcal{H}} - \frac{\beta\mathcal{V}\mathcal{H}}{\mathcal{L}} - \frac{b\mathcal{I}}{\mathcal{V}} + \frac{\beta\Delta(b+c)}{c\bar{\mu}} \\
& + \mu_{\mathcal{H}} + \mu + c - \frac{\alpha\mathcal{I}}{\frac{\Delta}{\mu} - \mathcal{H} - \mathcal{L} - \mathcal{I}} - \frac{\beta\mathcal{V}\mathcal{H}}{\frac{b\Delta}{c\bar{\mu}} - \mathcal{V}} + \bar{\mu} + c + (\delta - \bar{\delta}) \\
& + N_2\left[\theta(\bar{\delta}-\delta) - \theta\frac{\bar{\delta}}{\delta} + \theta + \frac{\sigma^2}{2}\right] + N_1 f_1(\hat{\delta}) + N_1 f_2(\delta) + \bar{\delta}, \\
& \leq -BN_1 + N_1\frac{a_1\beta b}{c}\mathcal{I} - \frac{\Delta}{\mathcal{H}} - \frac{\beta\mathcal{V}\mathcal{H}}{\mathcal{L}} - \frac{b\mathcal{I}}{\mathcal{V}} + \frac{\beta\Delta(b+c)}{c\bar{\mu}} - \frac{\alpha\mathcal{I}}{\frac{\Delta}{\mu} - \mathcal{H} - \mathcal{L} - \mathcal{I}} \\
& - \frac{\beta\mathcal{V}\mathcal{H}}{\frac{b\Delta}{c\bar{\mu}} - \mathcal{V}} + (\delta-\bar{\delta})(1-N_2\theta) - \theta N_2\frac{\bar{\delta}}{\delta} + N_2\left(\theta + \frac{\sigma^2}{2}\right) + 2c + \mu + \mu_{\mathcal{H}} + \bar{\mu} \\
& + N_1 f_1(\hat{\delta}) + N_1 f_2(\delta) + \bar{\delta}, \\
\mathscr{L}W_5 \leq & -BN_1 + N_1\frac{a_1\beta b}{c}\mathcal{I} - \frac{\Delta}{\mathcal{H}} - \frac{\beta\mathcal{V}\mathcal{H}}{\mathcal{L}} - \frac{b\mathcal{I}}{\mathcal{V}} - \frac{\alpha\mathcal{I}}{\frac{\Delta}{\mu} - \mathcal{H} - \mathcal{L} - \mathcal{I}} \\
& - \frac{\beta\mathcal{V}\mathcal{H}}{\frac{b\Delta}{c\bar{\mu}} - \mathcal{V}} + N + (1-N_2\theta)\delta - \theta N_2\frac{\bar{\delta}}{\delta} + N_1 f_1(\hat{\delta}) + N_1 f_2(\delta).
\end{aligned} \tag{8}$$

where

$$\begin{cases} B = (\bar{\delta}+\mu)\left(\mathcal{R}_0^s - 1\right), \\ N = \bar{\delta}N_2\theta + N_2\left(\theta + \frac{\sigma^2}{2}\right) + \frac{\beta\Delta(b+c)}{c\bar{\mu}} + 2c + \mu + \mu + \bar{\mu}, \end{cases}$$

Furthermore, define

$$\begin{aligned}
\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) = & -BN_1 + N_1\frac{a_1\beta b}{c}\mathcal{I} - \frac{\Delta}{\mathcal{H}} - \frac{\beta\mathcal{V}\mathcal{H}}{\mathcal{L}} - \frac{b\mathcal{I}}{\mathcal{V}} - \frac{\alpha\mathcal{I}}{\frac{\Delta}{\mu} - \mathcal{H} - \mathcal{L} - \mathcal{I}} \\
& - \frac{\beta\mathcal{V}\mathcal{H}}{\frac{b\Delta}{c\bar{\mu}} - \mathcal{V}} + N + (1-N_2\theta)\delta - \theta N_2\frac{\bar{\delta}}{\delta}.
\end{aligned} \tag{9}$$

Then, we have

$$\mathscr{L}W_5 \leq \Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) + N_1 f_1(\hat{\delta}) + N_1 f_2(\delta).$$

Now, let $\mathbb{L}$ denote the following set:

$$\begin{aligned}
\mathbb{L} = \Big\{ & (\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \epsilon \leq \mathcal{H}, \epsilon^4 \leq \mathcal{L}, \epsilon^2 \leq \mathcal{I}, \epsilon^2 \leq \mathcal{V}, \\
& \mathcal{H}+\mathcal{L}+\mathcal{I} \leq \frac{\Delta}{\bar{\mu}} - \epsilon^2, \mathcal{V} \leq \frac{b\Delta}{c\bar{\mu}} - \epsilon^4, \epsilon \leq \delta \leq \frac{1}{\epsilon} \Big\}.
\end{aligned}$$

Let

$$\Xi \backslash \mathbb{L} := \mathbb{L}^c = \bigcup_{i=1}^{8} \mathbb{L}_i^c.$$

Then,

$$\mathbb{L}_1^c = \Big\{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \mathcal{H} \in (0,\varepsilon)\Big\},$$

$$\mathbb{L}_2^c = \Big\{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \mathcal{I} \in (0,\varepsilon)\Big\},$$

$$\mathbb{L}_3^c = \Big\{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \mathcal{V} \in (0,\varepsilon^2),\, \mathcal{I} \in [\varepsilon,\infty)\Big\},$$

$$\mathbb{L}_4^c = \Big\{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \mathcal{L} \in (0,\varepsilon^4),\, \mathcal{H} \in [\varepsilon,\infty),\, \mathcal{V} \in [\varepsilon^2,\infty)\Big\},$$

$$\mathbb{L}_5^c = \Big\{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \mathcal{H}+\mathcal{L}+\mathcal{I} \in \Big(\frac{\Delta}{\bar{\mu}} - \varepsilon^2,\infty\Big),\, \mathcal{I} \in [\varepsilon,\infty)\Big\},$$

$$\mathbb{L}_6^c = \Big\{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \mathcal{V} \in \Big[\frac{b\Delta}{c\bar{\mu}},\infty\Big),\, \mathcal{H} \in [\varepsilon,\infty),\, \mathcal{V} \in [\varepsilon^2,\infty)\Big\},$$

$$\mathbb{L}_7^c = \Big\{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \delta \in (0,\varepsilon)\Big\},$$

$$\mathbb{L}_8^c = \Big\{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \Xi \mid \delta \in \Big(\frac{1}{\varepsilon},\infty\Big)\Big\}. \tag{10}$$

Let

$$K = N_1\frac{a_1\beta b}{c}\frac{\Delta}{\bar{\mu}} - 2.$$

Choose $\epsilon = \frac{1}{N_1^2}$ and $N_2 = N_1^5$, where N_1 is sufficiently large so that the following inequalities hold:

$$\begin{cases} -BN_1 + N \le -2, \\ -2 + N_1\frac{a_1\beta b}{c}\epsilon \le -1, \\ -\frac{\min(\Delta,b,\beta,\alpha,N_2\theta\bar{\delta})}{\epsilon} + K \le -1, \\ (1-N_2\theta)\epsilon + K \le -1. \end{cases}$$

1^{st} case : Whenever $(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \mathbb{L}_1^c$, we have

$$\begin{aligned} \Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) &\le -\frac{\Delta}{\mathcal{H}} - 2 + N_1\frac{a_1\beta b}{c}\mathcal{I}, \\ &\le -\frac{\Delta}{\epsilon} + K, \\ &\le -\frac{\min(\Delta,b,\beta,\alpha,N_2\theta\bar{\delta})}{\epsilon} + K \le -1. \end{aligned} \tag{11}$$

2^{nd} case : Whenever $(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \mathbb{L}_2^c$, we have

$$\begin{aligned} \Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) &\le -2 + N_1\frac{a_1\beta b}{c}\mathcal{I}, \\ &\le -2 + N_1\frac{a_1\beta b}{c}\epsilon, \\ &\le -1. \end{aligned} \tag{12}$$

3^{st} case : Whenever $(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \in \mathbb{L}_3^c$, we have

$$\begin{aligned} \Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) &\le -\frac{b\mathcal{I}}{\mathcal{V}} - 2 + N_1\frac{a_1\beta b}{c}\mathcal{I} \\ &\le -\frac{b}{\epsilon} + K, \\ &\le -\frac{\min(\Delta,b,\beta,\alpha,N_2\theta\bar{\delta})}{\epsilon} + K \le -1. \end{aligned} \tag{13}$$

4^{th} case: Whenever $(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)\in\mathbb{L}_4^c$, we have

$$\begin{aligned}\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) &\leq -\frac{\beta\mathcal{V}\mathcal{H}}{\mathcal{L}}-2+N_1\frac{a_1\beta b}{c}\mathcal{I},\\ &\leq -\frac{\beta}{\epsilon}+K,\\ &\leq -\frac{\min(\Delta,b,\beta,\alpha,N_2\theta\bar{\delta})}{\epsilon}+K\leq -1.\end{aligned}\tag{14}$$

5^{th} case : Whenever $(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)\in\mathbb{L}_5^c$, we have

$$\begin{aligned}\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) &\leq -\frac{\alpha\mathcal{I}}{\frac{\Delta}{\mu}-H-L-I}-2+N_1\frac{a_1\beta b}{c}I,\\ &\leq -\frac{\alpha}{\epsilon}+K,\\ &\leq -\frac{\min(\Delta,b,\beta,\alpha,N_2\theta\bar{\delta})}{\epsilon}+K\leq -1\end{aligned}\tag{15}$$

6^{th} case : Whenever $(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)\in\mathbb{L}_5^c$, we have

$$\begin{aligned}\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) &\leq -\frac{\beta\mathcal{V}\mathcal{H}}{\frac{b\Delta}{c\bar{\mu}}-\mathcal{V}}-2+N_1\frac{a_1\beta b}{c}\mathcal{I},\\ &\leq -\frac{\beta}{\epsilon}+K,\\ &\leq -\frac{\min(\Delta,b,\beta,\alpha,N_2\theta\bar{\delta})}{\epsilon}+K\leq -1.\end{aligned}\tag{16}$$

7^{th} case : Whenever $(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)\in\mathbb{L}_5^c$, we have

$$\begin{aligned}\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) &\leq (1-N_2\theta)\delta-2+N_1\frac{a_1\beta b}{c}\mathcal{I},\\ &\leq (1-N_2\theta)\epsilon+K,\\ &\leq -1.\end{aligned}\tag{17}$$

Summarizing the seven cases presented above, we can deduce that

$$\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)\leq -1 \quad \forall(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)\in\mathbb{L}^c.$$

Alternatively, since W tends to ∞ as $|(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)|\to\infty$ or approaches the boundary of Ξ, we can ensure the existence of a point $\left(\tilde{\mathcal{H}},\tilde{\mathcal{L}},\tilde{\mathcal{I}},\tilde{\mathcal{V}},\tilde{\delta}\right)$ in the interior of Ξ at which $W\left(\left(\tilde{\mathcal{H}},\tilde{\mathcal{L}},\tilde{\mathcal{I}},\tilde{\mathcal{V}},\tilde{\delta}\right)\right)$ attains its minimum. Therefore, we can construct a non-negative C^2-function

$$W=W_5-W_5\left(\tilde{\mathcal{H}},\tilde{\mathcal{L}},\tilde{\mathcal{I}},\tilde{\mathcal{V}},\tilde{\delta}\right).$$

Then, applying Itô's formula to this function, we obtain

$$\mathcal{L}W\leq\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)+N_1f_1(\hat{\delta})+N_1f_2(\delta).$$

By taking expectations and integrating, we obtain:

$$\begin{aligned}
\frac{\mathbb{E}\,W\big(\mathcal{H}(t),\mathcal{L}(t),\mathcal{I}(t),\mathcal{V}(t),\delta(t)\big)}{t} &\geq 0 \\
&= \frac{\mathbb{E}\,W\big(\mathcal{H}(0),\mathcal{L}(0),\mathcal{I}(0),\mathcal{V}(0),\delta(0)\big)}{t} \\
&\quad + \frac{1}{t}\int_0^t \mathbb{E}\big(\mathcal{L}W\big(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{I}(\tau),\mathcal{V}(\tau),\delta(\tau)\big)\big)\,d\tau \\
&\leq \frac{\mathbb{E}\,W\big(\mathcal{H}(0),\mathcal{L}(0),\mathcal{I}(0),\mathcal{V}(0),\delta(0)\big)}{t} \\
&\quad + \frac{1}{t}\int_0^t \mathbb{E}\big(\Upsilon\big(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau)\big)\big)\,d\tau \\
&\quad + N_1\,\mathbb{E}\left(\frac{1}{t}\int_0^t \big(\delta(\tau)-\bar{\delta}\big)^+\,d\tau - D\right)
\end{aligned}$$

$$+N_1 4\sqrt[4]{a_1a_2a_3b\beta\Delta}\,\mathbb{E}\left(\int_0^{+\infty} x^{\frac{1}{4}}\pi(x)\,dx - \frac{1}{t}\int_0^t \delta(\tau)^{\frac{1}{4}}\,d\tau\right). \tag{18}$$

Using the strong law of large numbers [10] we have:

$$\lim_{t\to+\infty}\mathbb{E}\left(\frac{1}{t}\int_0^t \big(\delta(\tau)-\bar{\delta}\big)^+\mathrm{d}\tau - D\right) = 0 \qquad \text{a.s,}$$

and

$$\lim_{t\to+\infty}\mathbb{E}\left(\int_0^{+\infty} x^{\frac{1}{4}}\pi(x)\mathrm{d}x - \frac{1}{t}\int_0^t (\delta(\tau))^{\frac{1}{4}}\mathrm{d}\tau\right) = 0 \qquad \text{a.s.}$$

On the other hand let

$$A_0 = \sup_{(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)\in\Xi}\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)$$

and $A = \max(A_0,-1)+1$ then

$$\Upsilon(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \leq A, \quad \forall(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta)\in\Xi$$

$$\begin{aligned}
&\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}\big(\Upsilon\big(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau)\big)\big)\,d\tau \\
&= \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}\big(\Upsilon\big(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau)\big)\big)\,\mathbf{1}_{\{(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau))\in\mathbb{L}\}}\,d\tau \\
&+ \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}\big(\Upsilon\big(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau)\big)\big)\,\mathbf{1}_{\{(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau))\in\mathbb{L}^c\}}\,d\tau \\
&\leq A\,\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau))\in\mathbb{L}\}}\,d\tau \\
&\quad - \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau))\in\mathbb{L}^c\}}\,d\tau
\end{aligned}$$

$$\leq -1 + (A+1)\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau))\in\mathbb{L}\}}\,d\tau. \tag{19}$$

Therefore, we have

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(\mathcal{H}(\tau),\mathcal{L}(\tau),\mathcal{V}(\tau),\mathcal{I}(\tau),\delta(\tau))\in\mathbb{L}\}}d\tau \geq \frac{1}{A+1} > 0\,, \quad a.s.$$

Let $\mathbb{P}\left(t,\mathcal{H}(t),\mathcal{L}(t),\mathcal{V}(t),\mathcal{I}(\tau),\delta(t),\Omega\right)$ as the transition probability of

$$(\mathcal{H}(t),\mathcal{L}(t),\mathcal{V}(t),\mathcal{I}(t),\delta(t))$$

belongs to the set ω. Making the use of Fatou's lemma, we have

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{P}\left(\tau, \mathcal{H}\left(\tau\right), \mathcal{L}\left(\tau\right), \mathcal{V}\left(\tau\right), \mathcal{I}\left(\tau\right), \delta(\tau), \mathbb{L}\right) d\tau \geq \frac{1}{A+1} > 0, \quad a.s.$$

According to Lemma 2, system 2 admits at least one stationary distribution $\eta(\cdot)$ on Ξ that possesses the Feller and ergodic properties. Hence, the proof is complete. □

4. Conclusion

In this study, a stochastic HLIV model is considered, where the transition rate from eclipse-phase cells to infected cells is perturbed by the mean-reverting inhomogeneous geometric Brownian motion process. The system is shown to be well posed, admitting a unique, positive, and global-in-time solution. In addition, the existence of a stationary distribution is established to analyze the persistence of the disease.

Declarations

The authors declare that there is no conflict of interest regarding the publication of this paper.

Appendix A. Preliminaries

In this appendix, we recall some basic results and notations from stochastic differential equations that will be used throughout the paper.

A.1. **Stochastic Differential Equations.** Consider the k-dimensional stochastic differential equation

$$dX(t) = f(X(t), t)\,dt + g(X(t), t)\,dB(t), \qquad t \geq t_0, \tag{20}$$

with initial condition

$$X(t_0) = X_0 \in \mathbb{R}^k,$$

where $B(t)$ is a k-dimensional standard Brownian motion defined on the underlying complete probability space. The differential operator associated with (20) is defined by

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{k} f_i(X(t), t)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{k}\left[g^T(X(t), t)g(X(t), t)\right]_{ij}\frac{\partial^2}{\partial X_i\,\partial X_j}. \tag{21}$$

Let

$$V \in C^{2,1}\big(\mathbb{R}^k \times [t_0, \infty); \mathbb{R}_+\big).$$

Then the action of $\mathscr{L}$ on V is given by

$$\mathscr{L}V(X, t) = V_t(X, t) + V_X(X, t)f(X, t) + \frac{1}{2}\operatorname{tr}\left[g^T(X, t)V_{XX}(X, t)g(X, t)\right], \tag{22}$$

where

$$V_t = \frac{\partial V}{\partial t}, \qquad V_X = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_k}\right),$$

and

$$V_{XX} = \left(\frac{\partial^2 V}{\partial x_i\,\partial x_j}\right)_{k\times k}.$$

A.2. **Itô's Formula.** For the stochastic process $X(t) \in \mathbb{R}^k$ satisfying (20), Itô's formula yields

$$dV(X(t), t) = \mathscr{L}V(X(t), t)\,dt + V_X(X(t), t)\,g(X(t), t)\,dB(t). \tag{23}$$

A.3. **Well-Posedness.**

Definition 1 (Well-posedness in the sense of Hadamard)**.** A mathematical problem is said to be *well-posed* if the following three conditions are satisfied:

(1) **Existence:** there exists at least one solution;
(2) **Uniqueness:** the solution is unique;
(3) **Stability:** the solution depends continuously on the initial data, i.e., small perturbations in the initial conditions produce only small changes in the solution.

Appendix B. Proof of Theorem 1

Proof. Since the coefficients of system (2) satisfy the local Lipschitz condition [14], for any initial value

$$X_0 = \big(\delta(0), \mathcal{H}(0), \mathcal{L}(0), \mathcal{I}(0), \mathcal{V}(0)\big) \in \mathbb{R}_+^5,$$

there exists a unique local solution

$$X(t) = \big(\delta(t), \mathcal{H}(t), \mathcal{L}(t), \mathcal{I}(t), \mathcal{V}(t)\big)$$

on $[0, \tau_e)$, where τ_e is the explosion time [14]. To show that this solution is global, we need to prove that $\tau_e = \infty$ almost surely. To this end, let $n_0 > 0$ be sufficiently large such that every component of X_0 is lying within the interval $\left[\frac{1}{n_0}, n_0\right]$. For each integer $n > n_0$, define the stopping time as follows:

$$\tau_n = \inf\Bigg\{ t \in [0, \tau_e) : \min\{\delta(t), \mathcal{H}(t), \mathcal{L}(t), \mathcal{I}(t), \mathcal{V}(t)\} \le \frac{1}{n} \quad \text{or} \quad \max\{\delta(t), \mathcal{H}(t), \mathcal{L}(t), \mathcal{I}(t), \mathcal{V}(t)\} \ge n \Bigg\}.$$

Where throughout this paper, we set $\inf \emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). It is easy to see that τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to\infty} \tau_n$, it holds that $\tau_\infty \le \tau_e$ a.s. In what follows, we need to verify that $\tau_\infty = \infty$ a.s.
If this assertion is violated, there exist a constant $T > 0$ and an $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \le T\} > \varepsilon$. As a result, there exists an integer $n_1 \ge n_0$ such that

$$\mathbb{P}\{\tau_n \le T\} \ge \varepsilon, \quad n \ge n_1.$$

Define a C^2-function $\Psi : \mathbb{R}_+^5 \to \mathbb{R}_+$ by

$$\begin{aligned}\Psi(\mathcal{H}, \mathcal{L}, \mathcal{I}, \mathcal{V}, \delta) = {} & (\mathcal{H} - 1 - \ln \mathcal{H}) + (\mathcal{L} - 1 - \ln \mathcal{L}) + (\mathcal{I} - 1 - \ln \mathcal{I}) + (\mathcal{V} - 1 - \ln \mathcal{V}) \\ & + (\rho)^{-1} (\delta - 1 - \ln \delta).\end{aligned}$$

Using Itô's formula, we have

$$d\Psi(\mathcal{H}, \mathcal{L}, \mathcal{I}, \mathcal{V}, \delta) = \mathscr{L}\Psi(\mathcal{H}, \mathcal{L}, \mathcal{I}, \mathcal{V}, \delta)\, dt + \sigma\rho^{-1}(\delta - 1) dB(t),$$

where

$$\begin{aligned}\mathscr{L}\Psi = {} & \left(1 - \frac{1}{\mathcal{H}}\right)[\Delta - \beta\mathcal{H}\mathcal{V} - \mu_{\mathcal{H}}\mathcal{H}] + \left(1 - \frac{1}{\mathcal{L}}\right)[\beta\mathcal{H}\mathcal{V} - \delta\mathcal{L} - \mu\mathcal{L}] \\ & + \left(1 - \frac{1}{\mathcal{I}}\right)[\delta\mathcal{L} - \mu_{\mathcal{I}}\mathcal{I} - \alpha\mathcal{I}] + \left(1 - \frac{1}{\mathcal{V}}\right)[b\mathcal{I} - \beta\mathcal{V}\mathcal{H} - c\mathcal{V}] \\ & + \rho^{-1}\left(1 - \frac{1}{\delta}\right)\big[\rho(\bar{\delta} - \delta(t))\big] + \frac{\sigma^2}{2\rho}.\end{aligned}$$

Therefore,

$$\mathscr{L}\Psi \leq \Delta + \mu_{\mathcal{H}} + \mu + \mu_{\mathcal{I}} + \alpha + c + \bar{\bar{\delta}} + 1 + \frac{\sigma^2}{2\rho} + \beta\mathcal{V} + b\mathcal{I} + \beta\mathcal{H}.$$

Furthermore, we have

$$\begin{aligned} \mathrm{d}(\mathcal{H}+\mathcal{L}+\mathcal{I}) &= [\Delta - \mu_{\mathcal{H}}\mathcal{H} - \mu\mathcal{L} - \mu_{\mathcal{I}}\mathcal{I} - \alpha\mathcal{I}]\,dt \\ &\leq [\Delta - \bar{\mu}(\mathcal{H}+\mathcal{L}+\mathcal{I})]\,dt. \end{aligned}$$

This implies that

$$\mathcal{H}(t)+\mathcal{L}(t)+\mathcal{I}(t) \leq \begin{cases} \mathcal{H}(0)+\mathcal{L}(0)+\mathcal{I}(0), & \text{if } \mathcal{H}(0)+\mathcal{L}(0)+\mathcal{I}(0) \geq \dfrac{\Delta}{\bar{\mu}}, \\ \dfrac{\Delta}{\bar{\mu}}, & \text{if } \mathcal{H}(0)+\mathcal{L}(0)+\mathcal{I}(0) < \dfrac{\Delta}{\bar{\mu}}, \end{cases} \leq A_1,$$

where

$$\bar{\mu} = \min\left(\mu_{\mathcal{H}}, \mu, \mu_{\mathcal{I}}\right), \tag{24}$$

$$A_1 = \max\left(\frac{\Delta}{\bar{\mu}}, \mathcal{H}(0)+\mathcal{L}(0)+\mathcal{I}(0)\right). \tag{25}$$

From the fourth equation of system (2), we obtain

$$\begin{aligned} d\mathcal{V} &= (b\mathcal{I} - \beta\mathcal{V}\mathcal{H} - c\mathcal{V})\;dt \\ &\leq (b\mathcal{I} - c\mathcal{V})\;dt. \end{aligned} \tag{26}$$

Hence,

$$\mathcal{V}(t) \leq \begin{cases} \mathcal{V}(0), & \text{if } \mathcal{V}(0) \geq \dfrac{b\Delta}{c\bar{\mu}}, \\ \dfrac{b\Delta}{c\bar{\mu}}, & \text{if } \mathcal{V}(0) < \dfrac{b\Delta}{c\bar{\mu}}, \end{cases} \leq A_2,$$

where

$$A_2 = \max\left(\mathcal{V}(0), \frac{b\Delta}{c\bar{\mu}}\right).$$

Substituting these bounds into the previous inequality yields

$$\mathscr{L}\Psi \leq \underbrace{\Delta + \mu_{\mathcal{H}} + \mu + \mu_{\mathcal{I}} + \alpha + c + \bar{\bar{\delta}} + 1 + \frac{\sigma^2}{2\rho} + \beta A_2 + (b+\beta)A_1}_{=:C}.$$

Hence, for all $n \geq n_0$ and $t \in [0, \tau_n)$,

$$d\Psi(\mathcal{H},\mathcal{L},\mathcal{I},\mathcal{V},\delta) \leq C\,dt + \sigma\rho^{-1}(\delta-1)\,dB(t).$$

Integrating the above inequality over the interval $[0, \tau_n \wedge T]$ and taking expectations yield

$$\begin{aligned} \mathbb{E}\big[\Psi(X(T\wedge\tau_n))\big] &\leq \Psi(X(0)) + C\,\mathbb{E}[\tau_n \wedge T] \\ &\leq \Psi(X(0)) + CT. \end{aligned} \tag{27}$$

Since $\Psi(x) \geq 0$ for all $x > 0$, we have

$$\begin{aligned} \mathbb{E}\big[\Psi(X(T\wedge\tau_n))\big] &= \mathbb{E}\big[\Psi(X(T\wedge\tau_n))\mathbf{1}_{\{\tau_n \leq T\}}\big] \\ &\quad + \mathbb{E}\big[\Psi(X(T\wedge\tau_n))\mathbf{1}_{\{\tau_n > T\}}\big] \\ &\geq \mathbb{E}\big[\Psi(X(\tau_n))\mathbf{1}_{\{\tau_n \leq T\}}\big]. \end{aligned} \tag{28}$$

Here, $\mathbf{1}_A$ denotes the indicator function of the measurable set $A \in \mathcal{F}$.

Observe that, for every ω satisfying $\tau_n(\omega) \leq T$, at least one component of $X(\tau_n)$ equals either n or $\frac{1}{n}$. Consequently,

$$\Psi(X(\tau_n)) \geq (n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 - \ln \frac{1}{n}\right) \wedge \rho^{-1} (n - 1 - \ln n) \wedge \rho^{-1} \left(\frac{1}{n} - 1 - \ln \frac{1}{n}\right).$$

Therefore,

$$\mathbb{E}\big[\Psi(X(\tau_n))\mathbf{1}_{\{\tau_n \leq T\}}\big] \geq \mathbb{P}(\tau_n \leq T)\Big[(n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 - \ln \frac{1}{n}\right) \wedge \rho^{-1} (n - 1 - \ln n) \wedge \rho^{-1} \left(\frac{1}{n} - 1 - \ln \frac{1}{n}\right)\Big]. \tag{29}$$

Combining (27), (28), and (29), we obtain

$$\Psi(X(0)) + CT \geq \varepsilon\Big[(n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 - \ln \frac{1}{n}\right) \wedge \rho^{-1} (n - 1 - \ln n) \wedge \rho^{-1} \left(\frac{1}{n} - 1 - \ln \frac{1}{n}\right)\Big].$$

Letting $n \to \infty$ yields a contradiction. Therefore,

$$\tau_\infty = \infty, \qquad \text{a.s.}$$

This completes the proof.

□

References

1. R. Avendano, L. Esteva, J. A. Flores, J. L. Fuentes Allen, G. Gómez, and Je. López-Estrada, *A mathematical model for the dynamics of hepatitis C*, Computational and Mathematical Methods in Medicine **4** (2001), no. 2, 109–118. https://doi.org/10.1080/10273660290003777
2. Fan Bai, Krystin E. S. Huff, and Linda J. S. Allen, *The effect of delay in viral production in within-host models during early infection*, Journal of Biological Dynamics **13** (2018), no. sup1, 47–73. https://doi.org/10.1080/17513758.2018.1498984
3. Aziza Belmaati, Driss Kiouach, and Salim El Azami El-idrissi, *Asymptotic comportment of a stochastic SIQR model with mean-reverting inhomogeneous geometric Brownian motion*, Communications in Mathematical Biology and Neuroscience **2023** (2023), Article ID 114. https://doi.org/10.28919/cmbn/8195
4. Sudipto Bhattacharya, *Project valuation with mean-reverting cash flow streams*, The Journal of Finance **33** (1978), no. 5, 1317–1331. https://doi.org/10.1111/j.1540-6261.1978.tb03422.x
5. Nguyen Thanh Dieu, *Asymptotic properties of a stochastic SIR epidemic model with Beddington–DeAngelis incidence rate*, Journal of Dynamics and Differential Equations **30** (2017), no. 1, 93–106. https://doi.org/10.1007/s10884-016-9532-8
6. Nguyen Huu Du, Dang Hai Nguyen, and G. George Yin, *Conditions for permanence and ergodicity of certain stochastic predator–prey models*, Journal of Applied Probability **53** (2016), no. 1, 187–202. https://doi.org/10.1017/jpr.2015.18
7. A. M. Elaiw, *Global properties of a class of virus infection models with multitarget cells*, Nonlinear Dynamics **69** (2011), no. 1–2, 423–435. https://doi.org/10.1007/s11071-011-0275-0
8. Subhas Khajanchi, Sovan Bera, and Tapan Kumar Roy, *Mathematical analysis of the global dynamics of a HTLV-I infection model, considering the role of cytotoxic T-lymphocytes*, Mathematics and Computers in Simulation **180** (2021), 354–378. https://doi.org/10.1016/j.matcom.2020.09.009
9. A. Lahrouz, L. Omari, D. Kiouach, and A. Belmaâti, *Deterministic and stochastic stability of a mathematical model of smoking*, Statistics & Probability Letters **81** (2011), no. 8, 1276–1284. https://doi.org/10.1016/j.spl.2011.03.029
10. R.SH Liptser, *A strong law of large numbers for local martingales*, Stochastics **3** (1980), no. 1–4, 217–228. https://doi.org/10.1080/17442508008833146

11. Qun Liu, *Stationary distribution and extinction of a stochastic HLIV model with viral production and Ornstein–Uhlenbeck process*, Communications in Nonlinear Science and Numerical Simulation **119** (2023), 107111. https://doi.org/10.1016/j.cnsns.2023.107111
12. Qun Liu and Daqing Jiang, *Stationary distribution and extinction of a stochastic SIR model with nonlinear perturbation*, Applied Mathematics Letters **73** (2017), 8–15. https://doi.org/10.1016/j.aml.2017.04.021
13. M.L. Mann Manyombe, J. Mbang, and G. Chendjou, *Stability and Hopf bifurcation of a CTL-inclusive HIV-1 infection model with both viral and cellular infections, and three delays*, Chaos, Solitons & Fractals **144** (2021), 110695. https://doi.org/10.1016/j.chaos.2021.110695
14. X Mao, *Stochastic differential equations and their applications*, Horwood series in mathematics & applications, Horwood Publishing, Chichester, England, January 1997 (en).
15. Ahmed Nafidi, Ghizlane Moutabir, and Ramón Gutiérrez-Sánchez, *Stochastic Brennan–Schwartz diffusion process: Statistical computation and application*, Mathematics **7** (2019), no. 11, 1062. https://doi.org/10.3390/math7111062
16. Alan S. Perelson and Patrick W. Nelson, *Mathematical analysis of HIV-1 dynamics in vivo*, SIAM Review **41** (1999), no. 1, 3–44. https://doi.org/10.1137/s0036144598335107
17. Sudipto Sarkar, *The effect of mean reversion on investment under uncertainty*, Journal of Economic Dynamics and Control **28** (2003), no. 2, 377–396. https://doi.org/10.1016/s0165-1889(02)00181-1
18. Peng Wu, Zhaosheng Feng, and Xuebing Zhang, *Global dynamics of an age-space structured HIV/AIDS model with viral load-dependent infection and conversion rates*, Journal of Computational and Applied Mathematics **412** (2022), 114309. https://doi.org/10.1016/j.cam.2022.114309
19. Noura Yousfi, Khalid Hattaf, and Abdessamad Tridane, *Modeling the adaptive immune response in HBV infection*, Journal of Mathematical Biology **63** (2011), no. 5, 933–957. https://doi.org/10.1007/s00285-010-0397-x

E-mail address: lahcen.khammich@usmba.ac.ma
URL: https://orcid.org/0009-0003-6689-0376

URL: https://orcid.org/0000-0002-1497-3028

L2MASI Laboratory, Faculty of sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Fez, Morocco