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THE WINDING NUMBER OF A GAUSSIAN RANDOM FIELD

The article is devoted to the study of the winding number of a Gaussian random field on the plane. It is known that the winding number of a non-degenerate random field can be presented as a difference of two point processes. The non-degeneracy is proved. It is further proved that the corresponding point processes are simple, locally finite, stationary and mixing.

1. Introduction

There exists a significant apparatus of polymer models based on the Gaussian processes in Euclidean space. One of the most common such models is a free polymer model, where the form of a polymer chain is represented as a trajectory of a Wiener process in $\mathbb{R}^d$, $d = 2, 3$ [9]. One of the problems with such approach is the existence of the points of self-intersection for a Brownian motion, which do not exist for a real polymer. Another issue is a.s. non-differentiability of the trajectories of the Brownian motion. Which is why it was proposed by A. A. Dorogovtsev in the paper [3] to use as a model of random polymer the image of a smooth curve under an infinitely differentiable Gaussian mapping.

Let us describe the proposed model more precisely. Consider the Gaussian isotropic random field $\xi : \mathbb{R}^2 \to \mathbb{R}^3$, with independent identically distributed coordinates $\xi_i, i = 1, 2, 3$, on the probability space $(\Omega, \mathcal{F}, P)$ with covariance function

$$\mathbb{E}\xi_i(u)\xi_i(v) = e^{-||u-v||^2}.$$

It was proved in the paper [3] that the image of a unit circle under the mapping ξ is the smooth random curve without self-intersections, i.e. a random knot. Moreover, it has been proved that this curve with positive probability can have arbitrary topological type.

Since we define the winding number for $\mathbb{R}^2$-valued vector fields, we will consider the restriction of values of ξ to $\mathbb{R}^2$. Further we write

$$\xi(u) = (\xi_1(u), \xi_2(u)).$$

The subject-matter of our article is the winding number of ξ on the boundary of a bounded domain $B \subset \mathbb{R}^2$ (further denoted as $\gamma(\xi, B)$).

Further, we investigate the winding number $\gamma(\xi, B)$ as a function of the set B.

Definition 1.1. [4] The winding number $\gamma(\Psi, L)$ of a continuous vector field $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ along a continuous curve $L : [0, 1] \to \mathbb{R}^2$ is the number $\frac{1}{2\pi}(\phi(1) - \phi(0))$, where ϕ is the continuous branch of the angular function of Ψ and Ψ is not equal to 0 on the curve L.

The winding number of a continuous field along a closed curve is linked to the number of upcrossings of a fixed level by one of the components of this field by the following relation.

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Proposition 1.1. [4]

Let $\Psi(x(t), y(t)) = (P(t), Q(t))$, where Ψ is the random field, $(x(t), y(t))$ is the continuous parametrization of the closed curve Γ.

Let $P(t)$ have finite number of zeros on the interval $[0,1]$. Take $t_1, \cdots t_m$, all zeros of the function P where $Q(t)$ takes positive values. Let us denote as p the number of such zeros where the function P is increasing, respectively denote as q the number of such zeros where the function P is decreasing. Then the following equality holds:

$$\gamma(\Psi, \Gamma) = q - p.$$

If the function $P(t)$ is continuously differentiable on the interval $[0,1]$, then q is equal to the number of downcrossings of the level 0 by the function $P(t)$, conditioned on the $Q(t) > 0$, and, respectively, p is equal to the number of upcrossings conditioned on the $Q(t) > 0$.

Let us assume that L is the boundary of B, $\psi : [0,1] \to \mathbb{R}^2$ is its piecewise-smooth parametrization. Denote

$$\xi_1(\psi(t)) = x(t),$$
$$\xi_2(\psi(t)) = y(t).$$

We can use the following formula to calculate the winding number [4]:

$$\gamma(\xi, B) = \frac{1}{2\pi} \int_0^1 \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2} dt.$$

It can be seen from the formula that the winding number is defined only for vector fields without null vectors on L.

For a given projection of a knot on the plane, let us define an overpass as an arc of the knot, that goes over at least one point of self-crossing, but never under a self-crossing point. The maximal overpass is an overpass that cannot be made any longer. Both of its endpoints are positioned before the arc goes under the self-crossing point. The bridge number of the knot K, which is denoted as $b(K)$, is the minimal possible number of maximal overpasses in the projection over all projections of K [1]. The bridge number is a knot invariant.

Since the number of upcrossings by one of components of a curve is connected to the bridge number ([1]), therefore it is related to the topological type of the resulting knot.

Proposition 1.2. *The winding number of a continuous vector field ξ on a boundary of region B is calculated as follows* [4]

$$\gamma(\xi, B) = \sum_{u \in B : \xi(u) = 0} ind(\xi)(u)$$

if ξ has no null vectors on the boundary of B, where $ind(\xi)(u)$ denotes the index of the field ξ at the point u.

If the matrix of derivatives of ξ is non-degenerate at the point u, the index can be calculated as:

$$ind(\xi)(u) = sign(det(\xi'(u))).$$

Further we will denote

Definition 1.2.

$$\gamma(\xi, B) = \sum_{u \in B : \xi(u) = 0} sign(det(\xi'(u)))$$

for an arbitrary bounded $B \in \mathcal{B}(\mathbb{R}^2)$ and a field ξ that with probability 1 has non-degenerate matrix of partial derivatives at all points $u : \xi(u) = 0$.

For the sets B with boundary Γ that consists of a finite number of closed Jordan curves this definition coincides with the standard definition of winding number[4] of the field ξ along Γ.

Lemma 1.1. *The random field ξ is almost surely twice continuously differentiable.*

Proof. The sufficient condition of C^2 differentiability is (p. 86 in [2])

$$\int_{\mathbb{R}^2} ||\lambda||^2 (\log(1+||\lambda||))^\alpha \nu(d\lambda) < \infty,$$

where $\alpha > 0$, ν is the spectral measure of ξ. We find the spectral density of ν as the inverse Fourier transform of covariance function

$$f(\lambda) = \frac{1}{2} e^{-||\lambda||^2/4}.$$

Therefore the integral has the following form

$$\frac{1}{2}\int_{\mathbb{R}^2} ||\lambda||^2 (\log(1+||\lambda||))^\alpha e^{-||\lambda||^2/4} d\lambda.$$

which converges for any $\alpha > 0$. □

To apply Definition 1.2 to the random field ξ we need to prove the following statement.

Lemma 1.2. *For any twice continuously differentiable homogeneous Gaussian random field $\xi : \mathbb{R}^2 \to \mathbb{R}^2$ with continuous density $p(x), x \in \mathbb{R}^2$, and any bounded open set $B \in \mathcal{B}(\mathbb{R}^2)$:*

$$P(\{\exists u \in B : \xi(u) = 0, det\xi'(u) = 0\}) = 0.$$

Proof. Define

$$h_\epsilon(y) = \frac{1}{\pi\epsilon^2}\mathbb{I}_{S(0,\epsilon)}(y),$$

where

$$S(0,\epsilon) = \{x \in \mathbb{R}^2 : ||x|| < \epsilon\}.$$

Then

$$\mathbb{E}\int_B h_\epsilon(\xi(x))dx = \int_B \int_{\mathbb{R}^2} h_\epsilon(y)p(y)dy \to |B|p(0), \epsilon \to 0, \tag{1}$$

where $p(y)$ is the density of the random variable $\xi(0)$ in the point y.
Assume that there exists a random event $A \subset \mathcal{F}$ such that

$$P(A) > 0, \tag{2}$$

and

$$\forall \omega \in A : \exists u \in B : \xi(\omega, u) = 0, \det(\xi'(\omega, u)) = 0.$$

Consider the integral

$$\int_B h_\epsilon(\xi(x))dx = \int_{\xi^{-1}(S(0,\epsilon))\cap B} \frac{1}{\pi\epsilon^2}dx = \frac{1}{\pi\epsilon^2}|\xi^{-1}(S(0,\epsilon))\cap B|.$$

So the integral converges for all bounded sets B.
Let us fix an $\omega \in A$. By the Taylor's formula:

$$\xi(u+v,\omega) = \xi(u,\omega) + \xi'(u,\omega)v + \frac{1}{2}((\xi''(u,\omega),v),v) + o(||v||^2), v \to 0$$

But since $\det(\xi'(u)) = 0$, there exists a vector $\alpha \in \mathbb{R}^2$, such that

$$\xi'(u)\alpha = 0,\ ||\alpha|| = 1.$$

Denote as β the vector in $\mathbb{R}^2$ orthogonal to α.

Again using Taylor's formula we obtain

$$\xi(u+\alpha x,\omega)=\frac{1}{2}\xi''_\alpha(u,\omega)x^2+o(x^2),\ x\to 0,$$

where ξ''_α is the second derivative in direction α, $x\in\mathbb{R}$.
Then exists $c(\omega)\in\mathbb{R}$ such that

$$||\xi(u+\alpha x,\omega)||\le c(\omega)x^2$$

for all x small enough.

Also, since the field ξ is continuously differentiable,

$$||\xi(u+\beta y)||\le \tilde{c(\omega)}|y|.$$

Then let us consider a rectangle Q with a vertice at point u with sides $\frac{1}{2\sqrt{c}}\sqrt{\epsilon}$, $\frac{1}{2\tilde{c}}\epsilon$. For small enough ϵ the rectangle Q lies within $B\cap\xi^{-1}(S(0,\epsilon))$.
Therefore for every $\omega\in A$ the area

$$|\xi^{-1}(S(0,\epsilon))\cap B|\ge c_1(\omega)\epsilon^{3/2},$$

for all ϵ small enough.

Further

$$\frac{1}{\pi\epsilon^2}|\xi^{-1}(S(0,\epsilon))\cap B|\ge c_1(\omega)\epsilon^{-\frac{1}{2}}.$$

Therefore using Fatou's lemma we obtain

$$\mathbb{E}\int_B h_\epsilon(\xi(x))dx\ge\mathbb{E}(I_A(\omega)\int_B h_\epsilon(\xi(x))dx)\to\infty,\epsilon\to 0.$$

This means contradiction with (1). This proves that the statement (2) was wrong. □

Lemma 11.2.11 in [6] provides sufficient conditions for a random field $f:\mathbb{R}^2\to\mathbb{R}^2$ and a fixed point $u\in\mathbb{R}^2$ with probability 1 to have no points $t\in T$, such that

$$f(t)-u=det(f'(t))=0,$$

where T is a fixed compact. The difference between the known result and the Lemma 1.2 is that the uniform continuity of a conditional density of $det(f'(t))$ given $f(t)$ is not required.

Further, we will use the notation $\gamma(\xi,B)$ and $\gamma(B)$ interchangeably.

Lemma 1.3. *With probability* 1, $\gamma(\xi,\cdot)$ *is finite on the bounded sets for all twice continuously differentiable homogeneous random fields* ξ *with continuous density* $p(x),x\in\mathbb{R}^2$.

Proof. It is sufficient to prove that with probability 1, the set $\{u\in\mathbb{R}^2:\xi(u)=0,\det\xi'(u)\ne 0\}\cap S(0,n)$ is finite for every n, where

$$S(0,n)=\{x\in\mathbb{R}^2:||x||\le n\}$$

is a ball in $\mathbb{R}^2$ with radius n.

Let $A_n=S(0,n)$. Let us use the inverse mapping theorem. Since the mapping ξ is smooth with probability 1, and the set A_n is bounded, then with probability 1 for every point $x\in A_n$ such that

$$|\det\xi'(x)|>0,\ \xi(x)=0,$$

there exists a neighborhood S_x of x such that $\forall u\in S_x,u\ne x$:

$$||\xi(u)-\xi(x)||>0.$$

Any infinite bounded sequence in $\mathbb{R}^2$ should have a convergent subsequence. But if there exists x_0 which is a limit point of the set $\{x\in A_n:|\det\xi'(x)|>0,\ \xi(x)=0\}$ then

$$|\det\xi'(x_0)|=0,\ \xi(x_0)=0$$

with probability 1. This contradicts Lemma 1.2

Hence the set $\{x \in A_n : |\det \xi'(x)| > 0,\ \xi(x) = 0\}$ is no more than finite. $\square$

2. The winding number as a point process

The expectation metatheorem ([2], p. 266) provides the formula for calculating the expectation of

$$N_u = \#\{t \in B : f(t) = u, g(t) \in A \subset \mathbb{R}^k\},$$

where $f, \nabla f, g$ are a.s. continuous random fields. Yet one of the conditions of the expectation metatheorem is the non-degeneracy of distribution of the vector $(f(t), \nabla f(t), g(t))$ which does not hold in case $g = sign \det f'$.

It was proven by Kuznetsov V. O. that:

Theorem 2.1. [5] *Let f be a centered Gaussian vector field on $\mathbb{R}^2$ with covariance function*

$$b_{ij}(z) = \delta_{ij} b_N(||z||),$$

where b_N has the form

$$b_N(r) = \int_0^\infty J_0(r\alpha)\mu(d\alpha),$$

where μ is a finite measure on $[0, +\infty)$ with finite 4th moment, J_0 is Bessel function of 0th order, and $\mu \neq c\delta_0$, and the condition holds

$$b_N(r) < b_N(0),\ r > 0.$$

Then the variance of the winding number f along closed piecewise-smooth curve Γ, that bounds domain T is equal to

$$\mathbb{E}\gamma(f, \Gamma)^2 = \int_{T^2} \frac{2}{||t^1 - t^2||} \Big(\frac{b_N(||t^1 - t^2||) b_N'(||t^1 - t^2||)^3}{b_N(0)^2 - b_N(||t^1 - t^2||)^2} +$$
$$+ b_N'(||t^1 - t^2||) b_N''(||t^1 - t^2||)\Big) p_{(t^1, t^2)}(0) dt^1 dt^2 - b_N''(0) p(0) S(T),$$

where $S(T)$ is the area of T,

$$p_{(t^1,t^2)}(0) = \frac{1}{4\pi^2((b_N(0))^2 - (b_N(||t^1 - t^2||))^2)},$$

and

$$p(0) = \frac{1}{2\pi b_N(0)^2}.$$

Lemma 2.1. *The covariance function $e^{-||z||^2}$ has the form described in Theorem 2.1.*

$$b(z) = e^{-||z||^2} = \int_0^\infty J_0(z\alpha)\mu(d\alpha).$$

Proof. Let us use the Fourier transform of $b(r)$

$$\hat{b}(r) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(r,z)} b(z) dz = e^{-||r||^2/4},$$

where $(\cdot, \cdot)$ denotes the inner product in space $\mathbb{R}^2$. Then the inverse Fourier transform gives us

$$b(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(r,z)} e^{-||r||^2/4} dr = \int_0^\infty \int_0^{2\pi} e^{-\alpha^2/4} e^{i\alpha||z||cos\phi} \alpha d\phi d\alpha =$$
$$= \int_0^\infty e^{-\alpha^2/4} J_0(||z||\alpha) \alpha d\alpha,$$

where we used transition to polar coordinates.

Then

$$J_0(r\alpha) = \int_0^{2\pi} e^{ir\alpha sin\theta} d\theta,$$

and it follows that

$$\mu(d\alpha) = \alpha e^{-\alpha^2/4} d\alpha.$$

All moments of the measure μ are finite .

$\square$

Therefore statement of Theorem 2.1 is applicable to the random field ξ. We conclude that the variance of $\gamma(\xi, \Gamma)$ exists and is finite for all piecewise-smooth closed curves Γ.

Let $(\mathbf{N}(\mathbb{X}), \mathcal{N})$ be the measurable space of integer-valued measures on the space $\mathbb{X}$, constructed in the following way. Let $\mathbf{N}_{<\infty}(\mathbb{X})$ be the set of all measures μ on $\mathbb{X}$ such that

$$\mu(A) \in \mathbb{N}_0 \ \forall A \in \mathcal{X},$$

where $\mathcal{X}$ is a $\sigma-$field of subsets of $\mathbb{X}$.
Then $\mathbf{N}(\mathbb{X})$ is the space of all measures on $\mathbb{X}$ that can be represented as no more than countable sum of measures from $\mathbf{N}_{<\infty}$. Further, let us define the $\sigma-$field $\mathcal{N}$

Definition 2.1. [6]

$$\mathcal{N} = \sigma(\{\mu \in \mathbf{N} : \mu(B) = k,\ B \in \mathcal{X}, k \in \mathbb{N}_0 \cup \{\infty\}\}).$$

$\mathcal{N}$ is the smallest $\sigma-$field such that $\mu \to \mu(B)$ is measurable for any $B \in \mathcal{X}$.

Definition 2.2. A point process η on the space $(\mathbb{X}, \mathcal{X})$ is a random element of the measurable space $(\mathbf{N}(\mathbb{X}), \mathcal{N}(\mathbb{X}))$.

Definition 2.3. A measure ν on metric space $\mathbb{X}$ is said to be locally finite, if

$$\nu(B) < \infty$$

for all bounded $B \in \mathcal{X}$.

We denote as $\mathbf{N}_l(\mathbb{X})$ the space of all locally finite measures on $\mathbb{X}$.

Definition 2.4. [6] A point process η on a metric space $(\mathbb{X}, \mathcal{X})$ is called locally finite if

$$\mathbb{P}(\eta(B) < \infty) = 1$$

for all bounded $B \in \mathcal{X}$

In other words, η is a measurable mapping from $(\Omega, \mathcal{F})$ into $(N(\mathbb{X}), \mathcal{N}(\mathbb{X}))$.

Definition 2.5. [6] A point process μ on $\mathbb{X}$ is said to be simple if

$$\mu(\{x\}) \leq 1 \ \forall x \in \mathbb{X}$$

with probability 1.

Let us denote as $\mathbf{N}_{ls}$ the space of all locally finite simple point measures on $\mathbb{R}^2$, and

$$\mathcal{N}_{ls} = \{A \cap \mathbf{N}_{ls} : A \in \mathcal{N}\}.$$

Definition 2.6. [6] A point process η on a measurable space $(\mathbb{X}, \mathcal{X})$ is said to be uniformly $\sigma-$finite if there exist $A_n \in \mathcal{X}, n \in \mathbb{N}$, such that

$$\bigcup_{n=1}^{\infty} A_n = \mathbb{X},$$

and

$$P(\eta(A_n) < \infty) = 1,\ n \in \mathbb{N}.$$

All locally finite point processes on metric spaces are uniformly $\sigma-$finite.

For any compact $B \subset \mathbb{R}^2$, $\gamma(\xi, B)$ can be represented as

$$\gamma(\xi, B) = \gamma^+(\xi, B) - \gamma^-(\xi, B),$$

where

$$\gamma^+(\xi, B) = card\{u \in B : \xi(u) = 0, det(\xi'(u)) > 0\},$$
$$\gamma^-(\xi, B) = card\{u \in B : \xi(u) = 0, det(\xi'(u)) < 0\}.$$

Further we will consider the field γ^+, yet the results hold for the γ^-.

Theorem 2.2. *$\gamma^+(\xi, \cdot)$ is a locally finite simple point process on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. That is, $\gamma^+(\xi, \cdot)$ is a random element of the space $(\mathbf{N}_{ls}, \mathcal{N}_{ls})$.*

Proof. It is sufficient to prove that the sets $\{\omega : \gamma^+(\xi, B) = 0\}$ are measurable, since the sets $\{\eta : \eta(B) = 0\}$ form a $\pi-$system that generates $\mathcal{N}_{ls}$.

Since the sets $\{u \in B, \omega \in \Omega : \xi(u, \omega) = 0, det(\xi'(u, \omega)) < 0\}, \{u \in B, \omega \in \Omega : \xi(u, \omega) = 0, det(\xi'(u, \omega)) > 0\}$ are measurable w.r.t. $\sigma(\mathcal{B}(\mathbb{R}^2) \times \mathcal{F})$(theorem $F.7$ in [7]), we conclude that γ^+ and γ^- are simple counting measures for all $\omega \in \Omega$. From this the statement of theorem follows. □

Proposition 2.1. [6] *A uniformly $\sigma-$finite point process η on a measurable space $(\mathbb{Y}, \mathcal{Y})$ is proper. That is, there exist random elements $X_1, \dots X_n, \dots$ in $\mathbb{Y}$ and $\mathbb{N}_0 \bigcup \{\infty\}-$ valued random variable χ such that*

$$\eta = \sum_{k=1}^{\chi} \delta_{X_k}.$$

As a locally finite point process on a metric space, γ^+ is uniformly $\sigma-$finite, therefore proper.

Let us recall that

Definition 2.7. A point process μ on $\mathbb{R}^2$ is called stationary if for any shift $\theta_x, x \in \mathbb{R}^2$ the following equality holds:

$$\theta_x \mu = \mu$$

in distribution.

Lemma 2.2. *$\gamma^+(\cdot)$ is a stationary point process.*

Proof. For the equality in distribution of $\gamma(\cdot), \theta_x\gamma(\cdot)$ it is sufficient to check that for any continuous function $f : \mathbb{R}^2 \to \mathbb{R}_+$ with bounded set $\{u \in \mathbb{R}^2 : f(u) > 0\}$ the following equality holds (Proposition 2.14 in [6])

$$\int_{\mathbb{R}^2} f(u)\gamma^+(du) \overset{d}{=} \int_{\mathbb{R}^2} f(u - x)\gamma^+(du)$$

for all $x \in \mathbb{R}^2$.

Let us denote the set $\{u \in \mathbb{R}^2 : f(u) > 0\}$ as D_f. Note that D_f is open as a pre-image of an open set. Then

$$\int_{\mathbb{R}^2} f(u)\gamma^+(du) = \int_{D_f} f(u)\gamma^+(du).$$

And we need to prove

$$\int_{D_f} f(u)\gamma^+(du) \overset{d}{=} \int_{D_f} f(u - x)\gamma^+(du).$$

Since γ^+ is a stationary proper simple point process, then it is equal to

$$\gamma^+(\cdot) = \sum_{j=1}^{\infty} \delta_{T_j}(\cdot),$$

and its shift on x

$$\theta_x \gamma^+(\cdot) = \sum_{j=1}^{\infty} \delta_{T_j - x}(\cdot),$$

where T_j are random values measurable w.r.t. $\sigma-$field generated by $\{\xi(u) : u \in \mathbb{R}^2\}$. But the field ξ is homogeneous, therefore the sequence

$$\{T_j\} \stackrel{d}{=} \{T_j - x\}.$$

From this follows that

$$\int_{D_f} f(u)\gamma^+(du) = \sum_{j=1}^{\infty} f(T_j) \stackrel{d}{=} \sum_{j=1}^{\infty} f(T_j - x) = \int_{D_f} f(u-x)\gamma^+(du).$$

□

Lemma 2.3. $\gamma^+(\cdot)$ *is mixing point process. That is, for all measurable* $A, A' \in \mathcal{N}_{ls}(\mathbb{R}^2)$

$$lim_{||x||\to\infty} P(\gamma^+ \in A, \theta_x(\gamma^+ \in A')) = P(\gamma^+ \in A)P(\gamma^+ \in A').$$

Proof. Since γ^+ is simple locally finite point process, we need to prove that

$$lim_{||x||\to\infty} P(\gamma^+(B) = 0, \theta_x(\gamma^+(B') = 0)) = P(\gamma^+(B) = 0)P(\gamma^+(B') = 0).$$

It is sufficient to prove the equality for open sets $B, B' \in \mathcal{B}(\mathbb{R}^2)$. Since for the open sets B the event $\{\omega : \gamma^+(B) = 0\}$ belongs to a sigma-algebra generated by $\{\xi(r)|r \in B\}$, we need to prove that the random field ξ is strongly mixing.

It is sufficient to prove that $\xi(x)$ has continuous positive spectral density, from which follows the necessary statement (p. 73 in [8]). Recall that the spectral density

$$f(\lambda) = \frac{1}{2} e^{-||\lambda||^2/4}.$$

Since the spectral density is positive and continuous, we conclude that the random field ξ is strongly mixing. Therefore the point process $\gamma^+(\cdot)$ is mixing.

□

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