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RELAXED CONDITIONS FOR CONTROLLABILITY OF STOCHASTIC SYSTEMS WITH CONTROL-DEPENDENT TERMS

This research investigated the approximate controllability of a class of second-order stochastic differential equations driven by Q-Wiener processes in a real separable Hilbert space. By employing techniques from stochastic analysis and functional analysis, we established the existence and uniqueness of mild solutions under relaxed conditions that are less stringent than the standard Lipschitz criterion. Furthermore, we proved the approximate controllability of the system using stopping time theory. Our findings contributed to the advancement of control theory for stochastic systems, particularly for second-order equations driven by Q-Wiener processes.

1. Introduction

The primary objective of this study was to investigate the approximate controllability of a second-order stochastic differential equation in a real separable Hilbert space $\mathbb{S}_2$. The equation under consideration is

$$\begin{cases} \mathrm{d}(X'(r) - H(r, X(r), U(r))) = [AX(r) + BU(r) + F(r, X(r), U(r))]\mathrm{d}r \\ \qquad\qquad +G(r, X(r), U(r))\mathrm{d}W(r), \\ X(0) = \xi_0, \quad X'(0) = \eta_0, \quad r \in [0, b]. \end{cases} \tag{1}$$

In this equation, W represents a Wiener process taking values in $\mathbb{S}_1$ with a positive, nuclear covariance operator Q on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\geq 0}, \mathbb{P})$, characterized by $\mathrm{Trace}(Q) < \infty$, and a natural filtration $\{\mathcal{F}_s\}_{s\geq 0}$ generated by W.

The operator $A : D(A) \subset \mathbb{S}_2 \to \mathbb{S}_2$ is a closed, densely defined linear operator that generates a strongly continuous cosine family on $\mathbb{S}_2$. The operator $B : \mathbb{S}_3 \to \mathbb{S}_2$ is a bounded linear operator, while G and F are measurable mappings from $[0, b] \times \mathbb{S}_2 \times \mathbb{S}_3$ to $\mathbb{S}_2$ and $\mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)$, respectively. Here, $\mathbb{S}_i$ represents a real separable Hilbert space, and $\mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)$ denotes the space of all bounded, linear, Q-Hilbert-Schmidt operators from $\mathbb{S}_0$ to $\mathbb{S}_2$. Additionally, ξ_0 and η_0 are $\mathcal{F}_0$-measurable $\mathbb{S}_2$-valued random variables independent of W, with finite $p \geq 2$ moments. The function $U : \Omega \times [0, b] \to \mathbb{S}_3$ serves as a control.

The theory is developed in infinite dimensions for several reasons. Foremost among these is its logical extension from functional analysis. First and foremost, it is a logical progression from functional analysis. In addition, parabolic equations in Hilbert spaces, as in finite dimensions, are used to characterize systems with an infinite number of degrees of freedom, including spin configurations in statistical mechanics and crystal structures in solid-state physics. In disciplines such as mathematical finance, fluid dynamics, and population biology, diffusion processes are described by infinite-dimensional parabolic equations. There is an alternative paradigm to nonlinear differential equations: second-order differential equations. Over the past two decades, there has been a substantial increase in interest in second-order differential equations (e.g., [1, 2, 3]), resulting in significant theoretical and applied progress.

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In recent years, several researchers have reported on the existence of mild solutions and the approximate controllability of second-order semilinear evolution systems in Banach spaces (see e.g., [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). Building on Da Prato's research on stochastic processes [15] and earlier works, this paper introduces a novel approach to the study of approximate controllability.

We are able to adapt our methodology to more complex systems under typical conditions by expanding and enhancing the results of prior research (e.g., [4, 6]). However, there are still a multitude of significant challenges that must be surmounted.

- Firstly, the unique characteristics of the semigroup allow us to utilize Banach's fixed point theorem across limited domains. By combining the solutions from each domain, we may deduce the existence of a solution throughout the entire domain. By resolving the cosine problem, we demonstrate that this categorization is both practicable and achievable, hence confirming the efficacy of our methodology.
- The second critical aspect is the guarantee of the uniqueness of the solutions, which is accomplished through the application of Gronwall's inequality. However, this approach frequently disregards the significance of finite factors, despite its robustness. Gronwall's inequality is a fundamental component of numerous authoritative works, which presents a significant challenge. The solution's continuity is contingent upon its unboundedness, which is a significant limitation.

The proposed equations are systematically examined in this paper, which is organized as follows:

(1) **Section 2: Families of Sine and Cosine Operators:** This section gives definitions, properties, and crucial results that are pertinent to these operators.
(2) **Section 3: Definitions and Preliminaries:** Various norms and essential concepts, including approximate controllability, are defined in this section.
(3) **Section 4: Existence, Uniqueness, and Approximate Controllability:** Here, we examine the proposed equations, verify the existence and uniqueness of mild solutions, and examine their approximate controllability.
(4) **Section 5: Existence and Uniqueness under Non-Lipschitz Conditions:** We extend the results to the case of non-Lipschitz coefficients.
(5) **Section 6: Illustrative Example:** To illustrate the applicability of our results, we present a concrete example of a second-order stochastic differential equation within a real separable Hilbert space.

2. Families of Sine and Cosine Operators

Cosine functions have a similar relationship to the second-order Cauchy problem $u'' = Au$ as C_0-semigroups do to the first-order Cauchy problem $u' = Au$. In this section, we revisit the fundamental elements of cosine families. For a deeper understanding of continuous cosine and sine families, refer to [16, 17, 18, 19].

Definition 2.1 ([16])**.** A family indexed by a single parameter, denoted $\{\mathcal{C}(s) : s \in \mathbb{R}\} \subset \mathcal{L}(\mathbb{S}_2)$, is called a strongly continuous cosine family on $\mathbb{S}_2$ if it satisfies the following conditions:

(1) $\frac{\mathcal{C}(s+r)+\mathcal{C}(s-r)}{2} = \mathcal{C}(s)\mathcal{C}(r)$, for all $s, r \in \mathbb{R}$.
(2) $\mathcal{C}(0) = I$, where I is the identity operator.
(3) $\mathcal{C}(s)y$ is a strongly continuous function of s on $\mathbb{R}$ for all $y \in \mathbb{S}_2$.

- The associated strongly continuous sine family, denoted as

$$\left\{\mathcal{S}(s) = \int_0^s C(r)\mathrm{d}r : s \in \mathbb{R}\right\} \subset \mathcal{L}(\mathbb{S}_2),$$

satisfies $\mathcal{S}(0) = 0$.

- The *generator* of a strongly continuous cosine family $\{\mathcal{C}(s) : s \in \mathbb{R}\}$ is the linear operator $A : \mathcal{D}(A) \subset \mathbb{S}_2 \to \mathbb{S}_2$ defined as

$$Ay = \lim_{\alpha \longrightarrow 0^+} 2\alpha^{-2}(\mathcal{C}(\alpha) - I), \quad \text{for all } y \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) = \left\{ y \in \mathbb{S}_2 \mid \lim_{\alpha \longrightarrow 0^+} 2\alpha^{-2}(\mathcal{C}(\alpha) - \alpha) \text{ exists} \right\}.$$

The existence of a strongly continuous cosine family generated by an operator $(A, \mathcal{D}(A))$ is characterized by a necessary and sufficient condition, akin to the Hille-Yosida generator theorem in operator semigroup theory.

Theorem 2.1 ([16]). *An operator $A \in \mathcal{L}(\mathcal{D}(A) \subseteq \mathbb{S}_2)$, closed and densely defined on $\mathbb{S}_2$, acts as the infinitesimal generator of a strongly continuous cosine family $(\mathcal{C}(s))_{s\in\mathbb{R}}$ with $\|\mathcal{C}(s)\| \leq Me^{\beta|s|}$ if and only if the resolvent $\mathcal{R}(A, k^2) = (\kappa^2 I - A)^{-1}$ exists for any $\kappa \in \mathbb{C}$, and is strongly infinitely differentiable. Moreover*

$$\left\| \frac{\mathrm{d}^m \kappa}{\mathrm{d}\kappa^m} \mathcal{R}(A, \kappa^2) \right\|_{\mathcal{L}(\mathbb{S}_2)} \leq M_\beta m! (\kappa - \beta)^{-m-1}, \quad \text{for } \mathrm{Re}(\kappa) > \beta,\ M_\beta \geq 1,\ \text{and } m \in \mathbb{N}.$$

Properties of Cosine Families. Now, we present the fundamental properties of cosine families.

Lemma 2.1 ([16]). *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a cosine family $\{\mathcal{C}(s) : s \in \mathbb{R}\}$ with corresponding sine family $\{\mathcal{S}(s) : s \in \mathbb{R}\}$.*

- *There exist constants $M_\beta \geq 1$ and $\beta \geq 0$ such that*

$$\|\mathcal{C}(s)\|_{\mathcal{L}(\mathbb{S}_2)} \leq M_\beta e^{\beta|s|}, \quad \|\mathcal{S}(s)\|_{\mathcal{L}(\mathbb{S}_2)} \leq M_\beta |s| e^{\beta|s|} \quad \text{for all } s \in \mathbb{R}.$$

- *For any $y \in \mathbb{S}_2$ and $r, s \in \mathbb{R}$*

$$\int_v^s \mathcal{S}(r)y\mathrm{d}r \in D(A), \quad \text{and} \quad A\int_v^s \mathcal{S}(r)y\mathrm{d}r = \mathcal{C}(s)y - \mathcal{C}(v)y.$$

- *There exists $M \geq 1$ such that*

$$\|\mathcal{S}(s) - \mathcal{S}(v)\|_{\mathcal{L}(\mathbb{S}_2)} \leq M \left| \int_v^s e^{\beta|r|}\mathrm{d}r \right| \text{ for all } 0 \leq v \leq s < \infty.$$

- *If $y \in \mathcal{D}(A)$, then $\mathcal{S}(s)y \in \mathcal{D}(A)$ and $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{C}(s)y = A\mathcal{S}(s)y$ with $\lim_{s\to 0^+} A\mathcal{S}(s)y = 0$.*

  Additionally

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{C}(s)y = A\mathcal{C}(s)y = \mathcal{C}(s)Ay.$$

Lemma 2.2 ([16]). *Let $\{\mathcal{C}(s) : s \in \mathbb{R}\}$ be a strongly continuous cosine family in $\mathbb{S}_2$ satisfying $\|\mathcal{C}(s)\|_{\mathcal{L}(\mathbb{S}_2)} \leq M_\beta e^{\beta|s|}$ for $s \in \mathbb{R}$, where M_β and β are positive constants, and $(A, \mathcal{D}(A))$ is its infinitesimal generator. For $\kappa \in \mathbb{C}$ with $\mathrm{Re}(\kappa) > \beta$, if $\kappa^2 \in p(A)$ (where $p(A)$ is the spectral radius), then*

$$\kappa\mathcal{R}(A, k) = \int_0^\infty e^{-\kappa r}\mathcal{C}(r)y\mathrm{d}r = \kappa(\kappa I - A)^{-1},$$
$$\mathcal{R}(A, k) = \int_0^\infty e^{-\kappa r}\mathcal{S}(r)y\mathrm{d}r = (\kappa I - A)^{-1}, \quad \text{for } y \in \mathbb{S}_2.$$

3. Preliminaries and Notations

Let's introduce a framework with three separable Hilbert spaces, $\mathbb{S}_1$, $\mathbb{S}_2$, and $\mathbb{S}_3$. Building upon key concepts from [15], consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where an $\mathbb{S}_2$-valued stochastic process $(X(r))_{r\geq 0}$ is Gaussian if, for any $n \in \mathbb{N}$ and arbitrary positive numbers $r_1, r_2, \ldots, r_n$, the $\mathbb{S}_2^n$-valued random variable $(X(r_1), \ldots, X(r_n))$ follows a Gaussian distribution.

Now, let's consider a nonnegative trace class operator Q on $\mathbb{S}_1$. A stochastic process $W(r)$ in $\mathbb{S}_1$ is a Q-Wiener process if it has continuous trajectories, independent increments, and $(W(r) - W(\tau))$ follows the Gaussian measure $\mathcal{N}(0, (r-\tau)Q)$.

This setup leads to the existence of a complete orthonormal system $\{e_k\}$ in $\mathbb{S}_1$ and a bounded sequence of nonnegative real numbers such that $Qe_n = \lambda_n e_n$. Assuming $W(r)$ is a Q -Wiener process, several properties emerge, W is Gaussian with $\mathbb{E}(W(r)) = 0$ and $\mathrm{Cov}(W(r)) = rQ$.

Now, let's introduce the subspace $\mathbb{S}_0 = Q^{1/2}(\mathbb{S}_1)$ of $\mathbb{S}_1$, forming a Hilbert space. We also consider the space of all Hilbert–Schmidt operators $\mathcal{L}^2 := \mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)$ from $\mathbb{S}_0$ into $\mathbb{S}_2$, equipped with the norm $\|\cdot\|^2_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)} := \mathrm{Trace}\,[(\cdot)Q(\cdot)^*]$.

Now, we consider three separable Hilbert spaces, denoted

$$(\mathbb{S}_i, \langle\cdot,\cdot\rangle_{\mathbb{S}_i}, \|\cdot\|_{\mathbb{S}_1})$$

for $i = 1, 2, 3$. To streamline, we use $\mathcal{L}(\mathbb{S}_i, \mathbb{S}_j)$ to denote the space of linear operators from $\mathbb{S}_i$ to $\mathbb{S}_j$. When $\mathbb{S}_i$ equals $\mathbb{S}_j$, we simplify it as $\mathcal{L}(\mathbb{S}_i)$.

In this context, let's define

- $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[0,b]}, \mathbb{P})$ a complete probability space equipped with a normal filtration generated by W and $\mathcal{F} = \mathcal{F}_b$.
- $\mathcal{B}_{\mathbb{S}_2}$: Borel sigma-algebra on the space $\mathbb{S}_2$.
- $\mathcal{P}_{\mathrm{rog}}$: Progressively measurable σ-fields on $\Omega \times [0, b]$.
- $\mathrm{L}^p_{\mathbb{S}_2}(\Omega, \mathcal{F}_s, \mathbb{P})$: the Banach space of all stochastic processes $X : \Omega \times [0, b] \to \mathbb{S}_2$ such that, for arbitrary $s \in [0, b]$, the random variable $X(s)$ is $\mathcal{F}_s$-measurable and $\mathrm{E}\,\|X(s)\|^p_{\mathbb{S}_2} < +\infty$.
- $\mathcal{S}^p_{\mathbb{S}_2}[0, b]$: the Banach space of $\mathcal{P}_{\mathrm{rog}}-$measurable and continuous stochastic processes $X : \Omega \times [0, b] \to \mathbb{S}_2$ such that $\mathrm{E}\left(\sup_{s\in[0,b]} \|X(s)\|^p_{\mathbb{S}_2}\right) < +\infty$.
- $\mathcal{H}^p_{\mathcal{L}^2}[0, b]$: the Banach space of $\mathcal{P}_{\mathrm{rog}}-$measurable processes $X : \Omega \times [0, b] \to \mathcal{L}^2 = \mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)$ such that $\mathrm{E}\left(\int_0^b \|X(s)\|^2_{\mathcal{L}^2}\,\mathrm{d}s\right)^{p/2} < +\infty$.

4. Primary Findings

Let's begin by defining the stochastic notion of mild solutions to (1).

Definition 4.1. Consider a stochastic process $X \in \mathcal{S}^p_{\mathbb{S}_2}[b_0, b]$, $b_0 = 0$. We say X constitutes a mild solution of (1) if, for any $U \in \mathcal{U}_{ad} := \mathcal{S}^p_{\mathbb{S}_3}[b_0, b]$, it adheres to the following stochastic integral equation, almost surely with respect to $\mathbb{P}$, for any $s \in [b_0, b]$,

$$\begin{aligned} X(s) &= \mathcal{C}(s - b_0)X(b_0) + \mathcal{S}(s - b_0)(X'(b_0) - H(b_0, X(b_0), U(b_0))) \\ &\quad + \int_{b_0}^{s} \mathcal{C}(s - r)H(r, X(r), U(r))\mathrm{d}r + \int_{b_0}^{s} \mathcal{S}(s - r)\left[F(r, X(r), U(r)) + BU(r)\right]\mathrm{d}r \\ &\quad + \int_{b_0}^{s} \mathcal{S}(s - r)G(r, X(r), U(r))\mathrm{d}W(r). \end{aligned}$$

After that, we establish the following property of the solution.

Lemma 4.1. *Let $0 = b_0 < b_1 < b_2 = b$. Suppose $X \in \mathcal{S}^p_{\mathbb{S}_2}[b_0, b]$ is a mild solution of (1). Consider the following system:*

$$\begin{cases} \mathrm{d}(X'(r) - H(r, X(r), U(r))) = [AX(r) + BU(r) + F(r, X(r), U(r))]\mathrm{d}r \\ \qquad\qquad\qquad\qquad +G(r, X(r), U(r))\mathrm{d}W(r), \\ X(0) = \xi_0, \quad X'(0) = \eta_0, \quad r \in [b_0, b_1], \end{cases} \tag{2}$$

and

$$\begin{cases} \mathrm{d}(X'(r) - H(r, X(r), U(r))) = [AX(r) + BU(r) + F(r, X(r), U(r))]\mathrm{d}r \\ \qquad\qquad\qquad\qquad +G(r, X(r), U(r))\mathrm{d}W(r), \\ X(b_1) = \xi_1, \quad X'(b_1) = \eta_1, \quad r \in [b_1, b_2], \end{cases} \tag{3}$$

where (ξ_1, η_1) are the final conditions of (2). Then the solution of the system (2)–(3), for $s \in [b_0, b_2]$, is given by

$$\begin{aligned} X(s) &= \mathcal{C}(s - b_0)X(b_0) + \mathcal{S}(s - b_0)\left(X'(b_0) - H(b_0, X(b_0), U(b_0))\right) \\ &\quad + \int_{b_0}^{s} \mathcal{C}(s - r)H(r, X(r), U(r))\mathrm{d}r + \int_{b_0}^{s} \mathcal{S}(s - r)\left[F(r, X(r), U(r)) + BU(r)\right]\mathrm{d}r \\ &\quad + \int_{b_0}^{s} \mathcal{S}(s - r)G(r, X(r), U(r))\mathrm{d}W(r). \end{aligned} \tag{4}$$

This means that the solution of the system (2)–(3) is indeed the mild solution of (1).

Proof. In order to demonstrate this, it is necessary to establish that the solution to equation (3) on the interval $[b_1, b_2]$ may be expressed as shown in equation (4) for $s \in [b_1, b_2]$. Clearly, the solution of (3) can be expressed as

$$\begin{aligned} X(s) &= \mathcal{C}(s - b_1)X(b_1) + \mathcal{S}(s - b_1)X'(b_1) - H(b_1, X(b_1), U(b_1))) \\ &\quad + \int_{b_1}^{s} \mathcal{C}(s - r)H(r, X(r), U(r))\mathrm{d}r + \int_{b_1}^{s} \mathcal{S}(s - r)\left[F(r, X(r), U(r)) + BU(r)\right]\mathrm{d}r \\ &\quad + \int_{b_1}^{s} \mathcal{S}(s - r)G(r, X(r), U(r))\mathrm{d}W(r). \end{aligned} \tag{5}$$

Similarly, for $s \in [b_0, b_1]$, the solution of (2) can be written as

$$\begin{aligned} X(s) &= \mathcal{C}(s - b_0)X(b_0) + \mathcal{S}(s - b_0)\left(X'(b_0) - H(b_0, X(b_0), U(b_0))\right) \\ &\quad + \int_{b_0}^{s} \mathcal{C}(s - r)H(r, X(r), U(r))\mathrm{d}r + \int_{b_0}^{s} \mathcal{S}(s - r)\left[F(r, X(r), U(r)) + BU(r)\right]\mathrm{d}r \\ &\quad + \int_{b_0}^{s} \mathcal{S}(s - r)G(r, X(r), U(r))\mathrm{d}W(r). \end{aligned} \tag{6}$$

According to (6), we can express the initial conditions for (3) as

$$\begin{aligned} \xi_1 = X(b_1) &= \mathcal{C}(b_1 - b_0)X(b_0) + \mathcal{S}(b_1 - b_0)(X'(b_0) - H(b_0, X(b_0), U(b_0))) \\ &\quad + \int_{b_0}^{b_1} \mathcal{C}(b_1 - r)H(r, X(r), U(r))\mathrm{d}r + \int_{b_0}^{b_1} \mathcal{S}(b_1 - r)\left[F(r, X(r), U(r)) + BU(r)\right]\mathrm{d}r \\ &\quad + \int_{b_0}^{b_1} \mathcal{S}(b_1 - r)G(r, X(r), U(r))\mathrm{d}W(r). \end{aligned} \tag{7}$$

And

$$\begin{aligned} \eta_1 = X'(b_1) &= \mathcal{AS}(b_1 - b_0)X(b_0) + \mathcal{C}(b_1 - b_0)(X'(b_0) - H(b_0, X(b_0), U(b_0))) \\ &\quad + \int_{b_0}^{b_1} \mathcal{AS}(b_1 - r)H(r, X(r), U(r))\mathrm{d}r + \int_{b_0}^{b_1} \mathcal{C}(b_1 - r)\left[F(r, X(r), U(r)) + BU(r)\right]\mathrm{d}r \end{aligned}$$

$$+\int_{b_0}^{b_1}\mathcal{C}(b_1-r)G(r,X(r),U(r))\mathrm{d}W(r)+C(0)H(b_1,X(b_1),U(b_1)). \tag{8}$$

Next, let's combine the expressions (7)–(8) for $s\in[b_1,b_2]$

$$\begin{aligned}
&\mathcal{C}(s-b_1)X(b_1)+\mathcal{S}(s-b_1)X'(b_1)\\
&=\left[\mathcal{C}(s-b_1)\mathcal{C}(b_1-b_0)+\mathcal{A}\mathcal{S}(s-b_1)\mathcal{S}(b_1-b_0)\right]X(b_0)\\
&+\left[\mathcal{C}(s-b_1)\mathcal{S}(b_1-b_0)+\mathcal{S}(s-b_1)\mathcal{C}(b_1-b_0)\right](X'(b_0)-H(b_0,X(b_0),U(b_0)))\\
&+\int_{b_0}^{b_1}\left[\mathcal{C}(s-b_1)\mathcal{C}(b_1-r)+\mathcal{A}\mathcal{S}(s-b_1)\mathcal{S}(b_1-r)\right]H(r,X(r),U(r))\mathrm{d}r\\
&+\int_{b_0}^{b_1}\left[\mathcal{C}(s-b_1)\mathcal{S}(b_1-r)+\mathcal{S}(s-b_1))\mathcal{C}(b_1-r)\right]\left[F(r,X(r),U(r))+BU(r)\right]\mathrm{d}r\\
&+\int_{b_0}^{b_1}\left[\mathcal{C}(s-b_1)\mathcal{S}(b_1-r)+\mathcal{S}(s-b_1))\mathcal{C}(b_1-r)\right]G(r,X(r),U(r))\mathrm{d}W(r)\\
&+\mathcal{S}(s-b_1)H(b_1,X(b_1),U(b_1)).
\end{aligned} \tag{9}$$

We can rewrite the left-hand side using the identities (see [17, 19]) $\mathcal{S}(s+r)=\mathcal{C}(r)\mathcal{S}(s)+\mathcal{S}(r)\mathcal{C}(s)$ and $\mathcal{C}(s+r)=\mathcal{C}(r)\mathcal{C}(s)+A\mathcal{S}(r)\mathcal{S}(s)$, which gives

$$\begin{aligned}
&\mathcal{C}(s-b_1)X(b_1)+\mathcal{S}(s-b_1)\left(X'(b_1)-H(b_1,X(b_1),U(b_1))\right)\\
&=\mathcal{C}(s-b_0)X(b_0)+\mathcal{S}(s-b_0)\left(X'(b_0)-H(b_0,X(b_0),U(b_0))\right)r\\
&+\int_{b_0}^{b_1}\mathcal{C}(s-r)H(r,X(r),U(r))\mathrm{d}+\int_{b_0}^{b_1}\mathcal{S}(s-r)\left[F(r,X(r),U(r))+BU(r)\right]\mathrm{d}r\\
&+\int_{b_0}^{b_1}\mathcal{S}(s-r)G(r,X(r),U(r))\mathrm{d}W(r).
\end{aligned} \tag{10}$$

Therefore, by substituting equation (10) into equation (5), we obtain

$$\begin{aligned}
X(s)&=\mathcal{C}(s-b_1)X(b_1)+\mathcal{S}(s-b_1)\left(X'(b_1)-H(b_1,X(b_1),U(b_1))\right)\\
&+\int_{b_1}^{s}\mathcal{C}(s-r)H(r,X(r),U(r))\mathrm{d}r+\int_{b_1}^{s}\mathcal{S}(s-r)\left[F(r,X(r),U(r))+BU(r)\right]\mathrm{d}r\\
&+\int_{b_1}^{s}\mathcal{S}(s-r)G(r,X(r),U(r))\mathrm{d}W(r)\\
&=\mathcal{C}(s-b_0)X(b_0)+\mathcal{S}(s-b_0)\left(X'(b_0)-H(b_0,X(b_0),U(b_0))\right)\\
&+\int_{b_0}^{s}\mathcal{C}(s-r)H(r,X(r),U(r))\mathrm{d}r+\int_{b_0}^{s}\mathcal{S}(s-r)\left[F(r,X(r),U(r))+BU(r)\right]\mathrm{d}r\\
&+\int_{b_0}^{s}\mathcal{S}(s-r)G(r,X(r),U(r))\mathrm{d}W(r).
\end{aligned}$$

Thus, the solution of (2)–(3) over the whole interval $[b_0,b_2]$ matches the mild solution of (1). This completes the proof. □

Next, we delve into the stochastic counterparts of approximate controllability concepts.

(1) The controllability (stochastic) Gramian operator

$$\mathcal{P}_b:\mathrm{L}^p_{\mathbb{S}_2}(\Omega,\mathcal{F}_b,\mathbb{P})\to\mathrm{L}^p_{\mathbb{S}_2}(\Omega,\mathcal{F}_b,\mathbb{P})$$

is defined by

$$\mathcal{P}_b(0,b)h_b=\int_0^b\mathcal{S}(b-r)BB^*\mathcal{S}^*(b-r)\mathrm{E}\left(h_b|\mathcal{F}_r\right)\mathrm{d}r.$$

(2) The operator $\mathcal{G}_b(\tau,s)\in\mathcal{L}(\mathbb{S}_2,\mathbb{S}_3)$

$$\mathcal{G}_b(\tau,s)y=\int_\tau^s \mathcal{S}(s-r)BB^*\mathcal{S}^*(b-r)y\mathrm{d}r.$$

The controllability (deterministic) Gramian operator $\mathcal{G}_b(\tau,b)\in\mathcal{L}(\mathbb{S}_2,\mathbb{S}_3)$

$$\mathcal{G}_b(\tau,b)y=\int_\tau^b \mathcal{S}(b-r)BB^*\mathcal{S}^*(b-r)y\mathrm{d}r,\quad 0\leq\tau<b.$$

The lemma below gives the relationship between $\mathcal{P}_b$ and $\mathcal{G}_b$,

$$\mathrm{R}^{\rho}_{[0,b],\lambda}=\mathrm{R}\left(\lambda,-\mathcal{P}_b(0,b)\right)=\left(\lambda I+\mathcal{P}_b(0,b)\right)^{-1},\quad\text{and}\quad \mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}=\left(\lambda I+\mathcal{G}_b(0,b)\right)^{-1}.$$

Lemma 4.2. *Consider a random variable $h_b\in\mathrm{L}^p_{\mathbb{S}_2}(\Omega,\mathcal{F}_b,\mathbb{P})$ where h_b is independent of $\mathcal{F}_0$. Then, there exists a process Z in $\mathcal{H}^p_{\mathcal{L}^2}[0,b]$ such that: For every h_b, we have*

$$\mathrm{E}\left(h_b|\mathcal{F}_b\right)=h_b=\mathrm{E}h_b+\int_0^b Z(r)\mathrm{d}W(r). \tag{11}$$

Furthermore

$$\mathcal{P}_b(0,b)h_b=\mathcal{G}_b(0,b)\mathrm{E}h_b+\int_0^b \mathcal{G}_b(\tau,b)Z(\tau)\mathrm{d}W(\tau). \tag{12}$$

Finally

$$\mathrm{R}^{\mathcal{P}}_{[0,b],\lambda}h_b=\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\mathrm{E}h_b+\int_0^b \mathrm{R}^{\mathcal{G}}_{[\tau,b],\lambda}Z(\tau)\mathrm{d}W(\tau). \tag{13}$$

Proof. The proof for (11) is referenced in [20]. Let $h_b\in\mathrm{L}^p_{\mathbb{S}_2}(\Omega,\mathcal{F}_b,\mathbb{P})$. By a previous result, we ascertain the existence of $Z\in\mathcal{H}^p_{\mathcal{L}^2}[0,b]$ such that

$$\mathrm{E}\left(h_b|\mathcal{F}_s\right)=\mathrm{E}h_b+\int_0^s Z(r)\mathrm{d}W(r).$$

We then utilize the definition of the operator $\mathcal{P}_b$ and invoke the stochastic Fubini's theorem [15, Theorem 4.33] to derive the desired expression:

$$\begin{aligned}
\mathcal{P}_b(0,b)h_b&=\int_0^b \mathcal{S}(b-r)BB^*\mathcal{S}^*(b-r)\mathrm{E}\left(h_b|\mathcal{F}_r\right)\mathrm{d}r\\
&=\int_0^b \mathcal{S}(b-r)BB^*\mathcal{S}^*(b-r)\left(\mathbb{E}h_b+\int_0^r Z(\tau)\mathrm{d}W(\tau)\right)\mathrm{d}r\\
&=\int_0^b \mathcal{S}(b-r)BB^*\mathcal{S}^*(b-r)\mathbb{E}h_b+\int_0^b\left(\int_0^r \mathcal{S}(b-r)CC^*\mathcal{S}^*(b-r)Z(\tau)\mathrm{d}W(\tau)\right)\mathrm{d}r\\
&=\int_0^b \mathcal{S}(b-r)BB^*\mathcal{S}^*(b-r)\mathbb{E}h_b+\int_0^b\left(\int_\tau^b \mathcal{S}(b-r)CC^*\mathcal{S}^*(b-r)\mathrm{d}r\right)Z(\tau)\mathrm{d}W(\tau)\\
&=\mathcal{G}_b(0,b)\mathbb{E}h_b+\int_0^b \mathcal{G}_b(\tau,b)Z(\tau)\mathrm{d}W(\tau).
\end{aligned}$$

It then follows, from (11)–(12) and another prior result, that for any $h_b,h_b'\in\mathrm{L}^p_{\mathbb{S}_2}(\Omega,\mathcal{F}_b,\mathrm{P})$, there exist $Z,Z'\in\mathcal{H}^p_{\mathcal{L}^2}[0,b]$ such that

$$h_b=\mathrm{E}h_b+\int_0^b Z(\tau)dW(\tau),\quad h_b'=\mathrm{E}h_b'+\int_0^b Z'(\tau)dW(\tau),$$

and

$$\mathcal{P}_b(0,b)h_b=\mathcal{G}_b(0,b)\mathrm{E}h_b+\int_0^b \mathcal{G}_b(\tau,b)Z(\tau)dW(\tau).$$

Now, let $h_b = \mathrm{R}^{\mathcal{P}}_{[0,b],\lambda} h'_b$. Consequently

$$h'_b = \lambda h_b + \mathcal{P}_b(0,b)h_b = \lambda \mathrm{E}h_b + \lambda \int_0^b Z(\tau)W(\tau) + \mathcal{G}_b(0,b)\mathrm{E}h_b + \int_0^b \mathcal{G}_b(\tau,b)Z(\tau)dW(\tau)$$
$$= (\lambda I + \mathcal{G}_b(0,b))\,\mathrm{E}h_b + \int_0^b (\lambda I + \mathcal{G}_b(\tau,b))\,Z(\tau)dW(\tau).$$

One can easily see that

$$\mathrm{E}h'_b - (\lambda I + \mathcal{G}_b(0,b))\,\mathrm{E}h_b = 0,$$

and by the Burkholder-Davis-Gundy inequalities [21, Theorem 3.49] and Doob's Inequality, we obtain

$$Z(\tau) = (\lambda I + \mathcal{G}_b(\tau,b))^{-1} Z'(\tau).$$

Hence

$$\mathrm{R}^{\mathcal{P}}_{[0,b],\lambda} h'_b = h_b = \mathrm{E}h_b + \int_0^b Z(\tau)W(\tau)$$
$$= (\lambda I + \mathcal{G}_b(0,b))^{-1}\,\mathrm{E}h'_b + \int_0^b (\lambda I + \mathcal{G}_b(\tau,b))^{-1} Z'(\tau)dW(\tau).$$

Thus, the lemma is demonstrated. □

Definition 4.2. The system described by Equation (1) is termed approximately controllable over the interval $[0,b]$ if the range space $\bar{\mathcal{R}}(b)$ coincides with $\mathrm{L}^p_{\mathbb{S}_2}(\Omega, \mathcal{F}_{b,\mathbb{P}})$. If $\mathcal{R}(b)$ equals $\mathrm{L}^p_{\mathbb{S}_2}(\Omega, \mathcal{F}_b, \mathbb{P})$, the system is termed exactly controllable. Here, $\mathcal{R}(b)$ is defined as the set

$$\{X(b, X(0), X'(0), U) : U \in \mathcal{S}^p_{\mathbb{S}_3}[0,b]\},$$

where

$$X(b, X(0), X'(0), U) = \mathcal{C}(b)X(0) + \mathcal{S}(b)(X'(0) - H(0, X(0), U(0)))$$
$$+ \int_0^b \mathcal{S}(b-r)H(r, X(r), U(r))\mathrm{d}r + \int_0^b \mathcal{S}(b-r)\left[F(r, X(r), U(r)) + BU(r)\right]\mathrm{d}r$$
$$+ \int_0^b \mathcal{S}(b-r)G(r, X(r), U(r))\mathrm{d}W(r).$$

The linear regulator problem is defined as follows.

Definition 4.3 (Minimum Energy Principle [22, 23])**.** Minimize

$$J(U) = \mathbb{E}\,\|X(b) - h_b\|^p_{\mathbb{S}_2} + \lambda\mathbb{E}\left(\sup_{s\in[0,b]} \|U(s)\|^p_{\mathbb{S}_3}\right).$$

Over all $U \in \mathcal{S}^p_{\mathbb{S}_3}[0,b]$, where $X(b)$ is a state process, $h_b \in \mathrm{L}^p_{\mathbb{S}_2}(\Omega, \mathcal{F}_b, \mathbb{P})$, $\lambda > 0$, and $\mathbb{E}(h_b|\mathcal{F}_b) = h_b = \mathrm{E}h_b + \int_0^b Z(r)\mathrm{d}W(r)$ is as defined in Lemma 4.2.

Assumptions. To formulate and prove our main results, we require the following assumptions.

(A0) $\mathcal{C}(r), \mathcal{S}(r) : \mathbb{S}_2 \to \mathbb{S}_2$ are strongly continuous cosine families generated by the operator A, where $\mathcal{C}$ and $\mathcal{S}$ are the cosine and sine families, respectively.

(A1) $F, H : \Omega \times [0,b] \times \mathbb{S}_2 \times \mathbb{S}_3 \to \mathbb{S}_2$, and $G : \Omega \times [0,b] \times \mathbb{S}_2 \times \mathbb{S}_3 \to \mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)$ are differentiable, and

(i) The functions

$$F : \Omega \times [0,b] \times \mathbb{S}_2 \times \mathbb{S}_3 \to \mathbb{S}_2, \quad H : \Omega \times [0,b] \times \mathbb{S}_2 \times \mathbb{S}_3 \to \mathbb{S}_2,$$
$$(\omega, s, y, u) \mapsto F(\omega, s, y, u), \quad (\omega, s, y, u) \mapsto H(\omega, s, y, u)$$

are measurably mapped from $(\mathcal{P}_{\mathrm{rog}} \times \mathcal{B}_{\mathbb{S}_2} \times \mathcal{B}_{\mathbb{S}_3})$ into $(\mathbb{S}_2, \mathcal{B}_{\mathbb{S}_2})$.

(ii) The function

$$G : \Omega \times [0,b] \times \mathbb{S}_2 \times \mathbb{S}_3 \to \mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2),$$
$$(\omega, s, y, u) \mapsto G(\omega, s, y, u)$$

is measurable from $(\mathcal{P}_{\text{rog}} \times \mathcal{B}_{\mathbb{S}_2} \times \mathcal{B}_{\mathbb{S}_3})$ into $(\mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2), \mathcal{B}_{\mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)})$.

(A2) There exists a constant $C_p > 0$ such that for all $(y_1, u_1), (y_2, u_2) \in \mathbb{S}_2 \times \mathbb{S}_3$ and $(\omega, s) \in \Omega \times [0, b]$:

$$\|H(\omega, s, y, u)\|^p_{\mathbb{S}_2} + \|F(\omega, s, y, u)\|^p_{\mathbb{S}_2} + \|G(\omega, s, y, u)\|^p_{\mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)} \leq C_p \left(1 + \|y\|^p_{\mathbb{S}_2} + \|u\|^p_{\mathbb{S}_3}\right).$$

(A3) There exists a constant $L_p > 0$ such that for all $(y_1, u_1), (y_2, u_2) \in \mathbb{S}_2 \times \mathbb{S}_3$ and $(\omega, s) \in \Omega \times [0, b]$:

$$\|F(\omega, s, y_1, u_1) - F(\omega, s, y_2, u_2)\|^p_{\mathbb{S}_2} + \|H(\omega, s, y_1, u_1) - H(\omega, s, y_2, u_2)\|^p_{\mathbb{S}_2}$$
$$\leq L_p \left(\|y_1 - y_2\|^p_{\mathbb{S}_2} + \|u_1 - u_2\|^p_{\mathbb{S}_3}\right),$$

and

$$\|G(\omega, s, y_1, u_1) - G(\omega, s, y_2, u_2)\|^p_{\mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)} \leq L_p \left(\|y_1 - y_2\|^p_{\mathbb{S}_2} + \|u_1 - u_2\|^p_{\mathbb{S}_3}\right).$$

(A4) $X(0)$ and $X'(0)$ are $\mathcal{F}_0$-measurable, $\mathbb{S}_2$-valued random variables independent of W with finite $p \geq 2$ moments.

(A5) There exists a constant $\tilde{M}_\lambda(b) > 0$ such that for all $0 \leq r < b$:

$$0 < \|\lambda \mathcal{R}^{\mathcal{G}}_{[r,b]}\|_{\mathcal{L}(\mathbb{S}_3, \mathbb{S}_2)} \leq \tilde{M}_\lambda(b).$$

(A6) $\lambda \mathcal{R}^{\mathcal{G}}_{[r,b]} \underset{\lambda \to 0^+}{\longrightarrow} 0$ (converges to the zero operator as $\lambda \to 0^+$ in strong operator topology), and there exists a constant $C_p > 0$ such that for all $(y_1, u_1), (y_2, u_2) \in \mathbb{S}_2 \times \mathbb{S}_3$ and $(\omega, s) \in \Omega \times [0, b]$:

$$\|H(\omega, s, y, u)\|^p_{\mathbb{S}_2} + \|F(\omega, s, y, u)\|^p_{\mathbb{S}_2} + \|G(\omega, s, y, u)\|^p_{\mathcal{L}^2(\mathbb{S}_0, \mathbb{S}_2)} \leq C_p.$$

Let $\|B\| = \|B\|_{\mathcal{L}(\mathbb{S}_3, \mathbb{S}_2)}$, and define

$$M_C(b) = \sup_{0 \leq s \leq b} \|\mathcal{C}(s)\|_{\mathcal{L}(\mathbb{S}_2)},\ M_S(b) = \sup_{0 \leq s \leq b} \|\mathcal{S}(s)\|_{\mathcal{L}(\mathbb{S}_2)},\ M_\lambda(b) = \sup_{r \in [0,b]} \|\mathcal{R}^{\mathcal{G}}_{[r,b],\lambda}\|_{\mathcal{L}(\mathbb{S}_3, \mathbb{S}_2)}.$$

The following lemma is derived by analogy to [24, Lemma 4] and [23, Lemma 2.4].

Lemma 4.3. *Under assumptions* (A0) – (A5)*, for any arbitrary* $h_b \in \mathrm{L}^p(\Omega, \mathcal{F}_b, \mathbb{S}_2)$*, the following control*

$$U(s) = -B^* \mathcal{S}^*(b-s) \mathrm{R}^{\mathcal{G}}_{[0,b],\lambda} \left[\mathcal{C}(b) X(0) + \mathcal{S}(b)(X'(0) - H(0, X(0), U(0)) - \mathrm{E} h_b\right]$$
$$+ \int_0^s B^* \mathcal{S}^*(b-r) \mathrm{R}^{\mathcal{G}}_{[r,b],\lambda} Z(r) \mathrm{d}W(r) - \int_0^s B^* \mathcal{S}^*(b-s) \mathrm{R}^{\mathcal{G}}_{[r,b],\lambda} \mathcal{C}(b-r) H(r, X(r), U(r)) \mathrm{d}r$$
$$- \int_0^s B^* \mathcal{S}^*(b-s) \mathrm{R}^{\mathcal{G}}_{[r,b],\lambda} \mathcal{S}(b-r) F(r, X(r), U(r)) \mathrm{d}r$$
$$- \int_0^s B^* \mathcal{S}^*(b-s) \mathrm{R}^{\mathcal{G}}_{[r,b],\lambda} \mathcal{S}(b-r) G(r, X(r), U(r)) \mathrm{d}W(r), \tag{14}$$

leads the system from

$$X(s) = \mathcal{C}(s) X(0) + \mathcal{S}(s) \left[X'(0) - H(0, X(0), U(0))\right]$$
$$+ \int_0^s \mathcal{C}(s-r) H(r, X(r), U(r)) \mathrm{d}s + \int_0^s \mathcal{S}(s-r) \left[F(r, X(r), U(r)) + B U(r)\right] \mathrm{d}s$$
$$+ \int_0^s \mathcal{S}(s-r) G(r, X(r), U(r)) \mathrm{d}W(r), \tag{15}$$

to

$$
\begin{aligned}
(16)\quad X(s) &= \mathcal{C}(s)X(0) + \mathcal{S}(s)X'(0) \\
&\quad - \mathcal{G}_b(0,s)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left[\mathcal{C}(b)X(0) + \mathcal{S}(b)(X'(0) - H(0,X(0),U(0)) - \mathrm{E}h_b\right] \\
&\quad + \int_0^s \left(\mathcal{C}(s-r) - \mathcal{G}_b(r,s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)\right) H(r,X(r),U(r))\mathrm{d}r \\
&\quad + \int_0^s \left(\mathcal{S}(s-r) - \mathcal{G}_b(r,s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)\right) F(r,X(r),U(r))\mathrm{d}r \\
&\quad + \int_0^s \left(\mathcal{S}(s-r) - \mathcal{G}_b(r,s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)\right) G(r,X(r),U(r))\mathrm{d}W(r) \\
&\quad + \int_0^s \mathcal{G}_b(r,s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}Z(r)\mathrm{d}W(r).
\end{aligned}
$$

If $s = b$, we obtain

$$
\begin{aligned}
(17)\quad X(b) - h_b &= \lambda\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left(\mathcal{C}(b)X(0) + \mathcal{S}(b)(X'(0) - H(0,X(0),U(0)) - \mathrm{E}h_b\right) \\
&\quad + \int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)H(r,X(r),U(r))\mathrm{d}r + \int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)F(r,X(r),U(r))\mathrm{d}r \\
&\quad + \int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}(\mathcal{S}(b-r)G(r,X(r),U(r)) - Z(r))\mathrm{d}W(r),
\end{aligned}
$$

where $\mathbb{E}(h_b|\mathcal{F}_b) = h_b = \mathrm{E}h_b + \int_0^b Z(r)\mathrm{d}W(r)$ *(see Lemma 4.2).*

To avoid confusion, let's prove it.

Proof. Substituting (14) into (15) and applying the stochastic Fubini theorem [15, Theorem 4.33], it becomes apparent that

$$
\begin{aligned}
(18)\quad X(s) &= -\int_0^s \mathcal{S}(s-r)BB^*\mathcal{S}^*(b-r)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left[\mathcal{C}(b)X(0)\right. \\
&\qquad \left. -\mathcal{S}(b)(X'(0) - H(0,X(0),U(0)) - \mathrm{E}h_b\right]\mathrm{d}r \\
&\quad - \int_0^s \left(\int_\tau^s \mathcal{S}(s-r)BB^*\mathcal{S}^*(b-r)\mathrm{d}r\right)\mathrm{R}^{\mathcal{G}}_{[\tau,b],\lambda}\mathcal{C}(b-\tau)H(\tau,X(\tau),U(\tau))\mathrm{d}\tau \\
&\quad - \int_0^s \left(\int_\tau^s \mathcal{S}(s-r)BB^*\mathcal{S}^*(b-r)\mathrm{d}r\right)\mathrm{R}^{\mathcal{G}}_{[\tau,b],\lambda}\mathcal{S}(b-\tau)F(\tau,X(\tau),U(\tau))\mathrm{d}\tau \\
&\quad - \int_0^s \left(\int_\tau^s \mathcal{S}(s-r)BB^*\mathcal{S}^*(b-r)\mathrm{d}r\right)\mathrm{R}^{\mathcal{G}}_{[\tau,b],\lambda}(\mathcal{S}(b-\tau)G(\tau,X(\tau),U(\tau)) - Z(\tau))\mathrm{d}W(\tau) \\
&\quad + \int_0^s \mathcal{C}(s-\tau)H(\tau,X(\tau),U(\tau))\mathrm{d}\tau + \int_0^s \mathcal{S}(s-\tau)F(\tau,X(\tau),U(\tau))\mathrm{d}\tau \\
&\quad + \int_0^s \mathcal{S}(s-\tau)G(\tau,X(\tau),U(\tau))\mathrm{d}W(\tau).
\end{aligned}
$$

This completes the proof of equation (16). Rewriting equation (18) at $s = b$, we have

$$
\begin{aligned}
X(b) &= \mathcal{C}(b)X(0) + \mathcal{S}(b)(X'(0) - H(0,X(0),U(0))) \\
&\quad - \mathcal{G}_b(0,b)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left(\mathcal{C}(b)X(0) + \mathcal{S}(b)(X'(0) - H(0,X(0),U(0)) - \mathrm{E}h_b\right) \\
&\quad + \int_0^b (I - \mathcal{G}_b(r,b)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda})\mathcal{C}(b-r)H(r,X(r),U(r))\mathrm{d}r \\
&\quad + \int_0^b (I - \mathcal{G}_b(r,b)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda})\mathcal{S}(b-r)F(r,X(r),U(r))\mathrm{d}r
\end{aligned}
$$

$$+\int_0^b (I-\mathcal{G}_b(r,b))\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda})\mathcal{S}(b-r)G(r,X(r),U(r))\mathrm{d}W(r)$$
$$+\int_0^b \mathcal{G}_b(r,b)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}Z(r)\mathrm{d}r.$$

By utilizing $\lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}+\mathcal{G}_b(r,b)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}=I$, we can simplify

$$\begin{aligned}
X(b)&=\mathcal{C}(b)X(0)+\mathcal{S}(b)(X'(0)-H(0,X(0),U(0)))\\
&+(\lambda\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}-I)\left(\mathcal{C}(b)X(0)+\mathcal{S}(b)(X'(0)-H(0,X(0),U(0))-\mathrm{E}h_b\right)\\
&+\int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)H(r,X(r),U(r))\mathrm{d}r+\int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)F(r,X(r),U(r))\mathrm{d}r\\
&+\int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)G(r,X(r),U(r))\mathrm{d}W(r)+\int_0^b (I-\lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda})Z(r)\mathrm{d}r. \qquad (19)
\end{aligned}$$

Using the martingale representation 4.2, we can express $\mathbb{E}(h_b|\mathcal{F}_b)$ as

$$h_b=\mathrm{E}h_b+\int_0^b Z(r)\mathrm{d}W(r)$$

as shown in (19). This expression leads to the conclusion mentioned in (17). □

Approximate controllability. Now we announce the main result of the article.

Let $\lambda>0$. We first need to establish that there exists a unique mild solution to

$$\begin{cases}
X(s)=\mathcal{C}(s)X(0)+\mathcal{S}(s)(X'(0)-H(0,X(0),U(0)))\\
\quad+\int_0^s \mathcal{C}(s-r)H(r,X(r),U(r))\mathrm{d}r,\\
\quad+\int_0^s \mathcal{S}(s-r)\left[F(r,X(r),U(r))+BU(r)\right]\mathrm{d}r\\
\quad+\int_0^s \mathcal{S}(s-r)G(r,X(r),U(r))\mathrm{d}W(r),\\
U(s)=-B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\mathcal{C}(b)X(0)\\
\quad-B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left(\mathcal{S}(b)(X'(0)-H(0,X(0),U(0))-\mathrm{E}h_b\right)\\
\quad-\int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)H(r,X(r),U(r))\mathrm{d}r\\
\quad-\int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)F(r,X(r),U(r))\mathrm{d}r\\
\quad-\int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)G(r,X(r),U(r))\mathrm{d}W(r)\\
\quad+\int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}Z(r)\mathrm{d}W(r).
\end{cases} \qquad (20)$$

Theorem 4.1. *Under the assumptions that conditions* (A0) − (A5) *hold, there exists a unique mild solution* $\left(X^i,U^i\right)\in\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]$ *to the system* (20)*. Moreover, it possesses a continuous modification, and we have*

$$\mathrm{E}\left(\sup_{0\le s\le b}\|X(s),U(s)\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\le(\mu_1(b)+\mu_2(b)+\mu_3(b)b)\exp(\mu_1(b)b). \qquad (21)$$

Whereas constants are involved in the Lemma 4.6.

An a priori estimate. Without loss of generality, we prove the following estimates. Assume $(X^i,U^i)\in\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]$ are two solutions to the system (20) with $\lambda>0$.

Lemma 4.4. *Under the assumptions* (A0) − (A5)*, there exist positive constants* k_i*, for* $i=1,\cdots,4$*, such that, for every* $s\in[0,b]$*,*

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\le u\le s}||U^i(u)||^p_{\mathbb{S}_3}\right)\\
&\le k_1(b)+k_2(b)+k_3(b)\int_0^s \mathrm{E}\left(\sup_{0\le v\le r}\left(1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\mathrm{d}r, \qquad (22)
\end{aligned}$$

and

$$\mathrm{E}\left(\sup_{0\le u\le s}||U^1(u)-U^2(u)||^p_{\mathbb{S}_3}\right)$$

$$\le k_4(b)\int_0^s \mathrm{E}\left(\sup_{0\le v\le r}||(X^1(v),U^1(v))-(X^2(v),U^2(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r. \tag{23}$$

Proof. Throughout the proof, we frequently use the inequality: $x_1,\cdots,x_m\in\mathbb{S}_i, i=2,3$, $\alpha>0$,

$$\|x_1+\cdots+x_m\|^\alpha_{\mathbb{S}_i}\le \max(m^{\alpha-1},1)(\|x_1\|^\alpha_{\mathbb{S}_i}+\cdots+\|x_m\|^\alpha_{\mathbb{S}_i}). \tag{24}$$

For the proof of the first inequality in (22), we have, for every $s\in[0,b]$,

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\le u\le s}||U^1(u)||^p_{\mathbb{S}_3}\right)\\
&\le 5^{p-1}\mathrm{E}\sup_{0\le u\le s}\left\|B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left[\mathrm{E}h_b-\mathcal{C}(b)\xi_0-\mathcal{S}(b)(\eta_0-H(0,\xi_0,U^1(0))\right]\right\|^p_{\mathbb{S}_3}\\
&\quad+5^{p-1}\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)H(r,X^1(r),U^1(r))\mathrm{d}r\right\|^p_{\mathbb{S}_3}\\
&\quad+5^{p-1}\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)F(r,X^1(r),U^1(r))\mathrm{d}r\right\|^p_{\mathbb{S}_3}\\
&\quad+5^{p-1}\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)G(r,X^1(r),U^1(r))\mathrm{d}W(r)\right\|^p_{\mathbb{S}_3}\\
&\quad+5^{p-1}\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}Z(r)\mathrm{d}W(r)\right\|^p_{\mathbb{S}_3}\\
&=5^{p-1}(\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5).
\end{aligned} \tag{25}$$

For I_1, we can observe that

$$\begin{aligned}
\mathrm{I}_1&=\mathrm{E}\left(\sup_{0\le u\le s}\left\|B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left[\mathrm{E}h_b-\mathcal{C}(b)\xi_0-\mathcal{S}(b)(\eta_0-H(0,\xi_0,U^1(0))\right]\right\|^p_{\mathbb{S}_3}\right)\\
&\le 4^{p-1}\left\{M^p_\lambda(b)\,\|B\|^p\,M^p_S(b)\right\}\\
&\quad\times\mathrm{E}\left[||\mathrm{E}h_b||^p_{\mathbb{S}_2}+M^p_C(b)||\xi_0||^p_{\mathbb{S}_2}+M^p_S(b)\left(||\eta_0||^p_{\mathbb{S}_2}+C_p(1+||\xi_0||^p_{\mathbb{S}_2}+||U^1(0)||^p_{\mathbb{S}_3})\right)\right].
\end{aligned} \tag{26}$$

Now, by Hölder's inequality, for $\mathrm{I}_2+\mathrm{I}_3$, we have

$$\begin{aligned}
\mathrm{I}_2+\mathrm{I}_3&=\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)H(r,X^1(r),U^1(r))\mathrm{d}r\right\|^p_{\mathbb{S}_3}\\
&\quad+\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)F(r,X^1(r),U^1(r))\mathrm{d}r\right\|^p_{\mathbb{S}_3}\\
&\le\left\{\|B\|^p\,M^p_\lambda(b)M^p_S(b)M^p_C(b)b^{p-1}\right\}\int_0^s\mathrm{E}\left(||H(r,X^1(r),U^1(r))||^p_{\mathbb{S}_2}\right)\mathrm{d}r\\
&\quad+\left\{\|B\|^p\,M^p_\lambda(b)M^{2p}_S(b)b^{p-1}\right\}\int_0^s\mathrm{E}\left(||F(r,X^1(r),U^1(r))||^p_{\mathbb{S}_2}\right)\mathrm{d}r\\
&\le\left\{M^p_\lambda(b)\,\|B\|^p\,M^p_S(b)\,(M^p_C(b)+M^p_S(b))\,b^{p-1}\right\}C_p\\
&\quad\times\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}\left(1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\mathrm{d}r.
\end{aligned} \tag{27}$$

By applying the $L^p(p \geq 2)$ inequalities for stochastic integrals ([15, Theorem 4.37]), and then employing Hölder's inequality, we find

$$\begin{aligned}
\mathrm{I}_4 &= \mathrm{E} \sup_{0\leq u\leq s} \left\| \int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)G(r,X^1(r),U^1(r))\mathrm{d}W(r)\right\|^p_{\mathbb{S}_3} \\
&\leq \ell_p \left\{ \int_0^s \left(\mathrm{E}||B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)G(r,X^1(r),U^1(r))||^p_{\mathbb{S}_3}\right)^{\frac{2}{p}} \mathrm{d}r \right\}^{\frac{p}{2}} \\
&\leq \left\{ M^p_\lambda(b)\,\|B\|^p\, M^{2p}_S(b)\ell_p \right\} \left\{ \int_0^s \left\{ \mathrm{E}\left(\sup_{0\leq v\leq r} ||G(v,X^1(v),U^1(v))||^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)}\right)\right\}^{\frac{2}{p}} \mathrm{d}r \right\}^{\frac{p}{2}} \\
&\leq \left\{ M^p_\lambda(b)\,\|B\|^p\, M^{2p}_S(b)\ell_p \right\} C_p \left\{ \int_0^s \left\{ \mathrm{E}\left(\sup_{0\leq v\leq r} (1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3})\right)\right\}^{\frac{2}{p}} \mathrm{d}r \right\}^{\frac{p}{2}} \\
&\leq \left\{ M^p_\lambda(b)\,\|B\|^p\, M^{2p}_S(b)\ell_p b^{\frac{p}{2}-1} \right\} C_p \\
&\quad \times \int_0^s \mathrm{E}\left(\sup_{0\leq v\leq r} (1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3})\right)\mathrm{d}r. \qquad (28)
\end{aligned}$$

By applying the L^p inequalities for stochastic integrals, we have, respectively, by Hölder's inequality

$$\begin{aligned}
\mathrm{I}_5 &= \mathrm{E} \sup_{0\leq u\leq s} \left\| \int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda} Z(r)\mathrm{d}W(r)\right\|^p_{\mathbb{S}_3} \\
&\leq \left\{ \int_0^s \left(\mathrm{E}||B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda} Z(r)||^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)}\right)^{\frac{2}{p}} \mathrm{d}r \right\}^{\frac{p}{2}} \\
&\leq \left\{ M^p_\lambda(b)\,\|B\|^p\, M^{2p}_S(b)\ell_p b^{\frac{p}{2}-1} \right\} \int_0^s \mathrm{E}||Z(r)||^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)}\mathrm{d}r. \qquad (29)
\end{aligned}$$

Therefore, combining inequalities (25)–(29) yields inequality (22). For the proof of the second estimates (23), we have, for every $s \in [0,b]$,

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\leq u\leq s} ||U^1(u)-U^2(u)||^p_{\mathbb{S}_3}\right) \qquad (30) \\
&\quad \leq 3^{p-1}\mathrm{E} \sup_{0\leq u\leq s} \left\| \int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)(H(r,X^1(r),U^1(r)) \right. \\
&\qquad\qquad \left. - H(r,X^2(r),U^2(r)))\mathrm{d}r\right\|^p_{\mathbb{S}_3} \\
&\quad + 3^{p-1}\mathrm{E} \sup_{0\leq u\leq s} \left\| \int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)(F(r,X^1(r),U^1(r)) \right. \\
&\qquad\qquad \left. - F(r,X^2(r),U^2(r)))\mathrm{d}r\right\|^p_{\mathbb{S}_3} \\
&\quad + 3^{p-1}\mathrm{E} \sup_{0\leq u\leq s} \left\| \int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)(G(r,X^1(r),U^1(r)) \right. \\
&\qquad\qquad \left. - G(r,X^2(r),U^2(r)))\mathrm{d}W(r)\right\|^p_{\mathbb{S}_3} \\
&\qquad\qquad = 3^{p-1}(\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3).
\end{aligned}$$

For $\mathrm{J}_1+\mathrm{J}_2$, employing a similar method as in (27), we obtain

$$
\begin{aligned}
(31)\quad \mathrm{J}_1+\mathrm{J}_2 &= \mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)(H(r,X^1(r),U^1(r))\right.\\
&\qquad\left.-H(r,X^2(r),U^2(r)))\mathrm{d}r\right\|^p_{\mathbb{S}_3}\\
&\quad+\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)(F(r,X^1(r),U^1(r))\right.\\
&\qquad\left.-F(r,X^2(r),U^2(r)))\mathrm{d}r\right\|^p_{\mathbb{S}_3}\\
&\le\left\{M^p_\lambda(b)\,\|B\|^p\,M^p_S(b)\,(M^p_C(b)+M^p_S(b))\,b^{p-1}\right\}L_p\\
&\qquad\times\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}||(X^1(v),U^1(v))-(X^2(v),U^2(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r.
\end{aligned}
$$

For J_3, employing a similar method as in (28), we obtain

$$
\begin{aligned}
(32)\quad \mathrm{J}_3 &= \mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u B^*\mathcal{S}^*(b-u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)(G(r,X^1(r),U^1(r))\right.\\
&\qquad\left.-G(r,X^2(r),U^2(r)))\mathrm{d}W(r)\right\|^p_{\mathbb{S}_3}\\
&\le\left\{M^p_\lambda(b)\,\|B\|^p\,M^{2p}_S(b)\ell_p b^{\frac{p}{2}-1}\right\}L_p\\
&\qquad\times\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}||(X^1(v),U^1(v))-(X^2(v),U^2(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r.
\end{aligned}
$$

The constants $k_i(b)$ are given by

$$
\begin{cases}
k_1(b)=5^{p-1}\left\{4^{p-1}\left\{M^p_\lambda(b)\,\|B\|^p\,M^p_S(b)\right\}\right\}\\
\qquad\quad\times\mathrm{E}\left[M^p_C(b)||\xi_0||^p_{\mathbb{S}_2}+M^p_S(b)\left(||\eta_0||^p_{\mathbb{S}_2}+C_p(1+||\xi_0||^p_{\mathbb{S}_2}+||U^1(0)||^p_{\mathbb{S}_3})\right)\right]\\
\qquad\quad+5^{p-1}\left\{4^{p-1}\left\{M^p_\lambda(b)\,\|B\|^p\,M^p_S(b)\right\}||\mathrm{E}h_b||^p_{\mathbb{S}_2}\right\}\\
k_2(b)=5^{p-1}\left\{\left(\frac{1}{b}\right)b^{\frac{1}{2}p}M^p_\lambda(b)\,\|B\|^p\,M^{2p}_S(b)\ell_p\right\}\int_0^b\mathrm{E}||Z(r)||^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)}\mathrm{d}r,\\
k_3(b)=5^{p-1}\left\{\left(\frac{1}{b}\right)b^{\frac{1}{2}p}\,\|B\|^p\,M^p_\lambda(b)M^p_S(b)\right\}\left\{b^{\frac{1}{2}p}(M^p_C(b)+M^p_S(b))+M^p_S(b)\ell_p\right\}C_p,\\
k_4(b)=3^{p-1}\left\{\left(\frac{1}{b}\right)b^{\frac{1}{2}p}\,\|B\|^p\,M^p_\lambda(b)M^p_S(b)\right\}\left\{b^{\frac{1}{2}p}(M^p_C(b)+M^p_S(b))+M^p_S(b)\ell_p\right\}L_p.
\end{cases}
$$

Hence, plugging (31)–(32) inequalities into (30), we find (23). □

Lemma 4.5. *Under the conditions* (A0) − (A6) *there exist* $\beta_i(b)>0, i=1,\cdots,4$, *such that for any* $s\in[0,b]$,

$$
\begin{aligned}
&\mathrm{E}\left(\sup_{0\le u\le s}||X^1(u)||^p_{\mathbb{S}_2}\right)\\
(33)\qquad&\le\beta_1(b)+\beta_2(b)+\beta_3(b)\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}\left(1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\mathrm{d}r,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{E}\left(\sup_{0\le u\le s}||X^1(u)-X^2(u)||^p_{\mathbb{S}_2}\right)\\
(34)\qquad&\le\beta_4(b)\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}||(X^1(v),U^1(v))-(X^2(v),U^2(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r.
\end{aligned}
$$

Proof. Using Lemma 4.3, we can express the inequality as

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\leq u\leq s}||X^1(u)||^p_{\mathbb{S}_2}\right)\\
&\leq 5^{p-1}\mathrm{E}\left[\sup_{0\leq u\leq s}\|\mathcal{C}(u)\xi_0+\mathcal{S}(u)(\eta_0-H(0,\xi_0,U(0)))\right.\\
&\left.+\ \mathcal{G}_b(0,u)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left[\mathrm{E}h_b-\mathcal{C}(b)\xi_0-\mathcal{S}(b)(\eta_0-H(0,\xi_0,U^1(0))\right]\right\|^p_{\mathbb{S}_2}\Bigg]\\
&+5^{p-1}\mathrm{E}\sup_{0\leq u\leq s}\left\|\int_0^u(\mathcal{C}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r))H(r,X^1(r),U^1(r))\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&+5^{p-1}\mathrm{E}\sup_{0\leq u\leq s}\left\|\int_0^u(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r))F(r,X^1(r),U^1(r))\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&+5^{p-1}\mathrm{E}\sup_{0\leq u\leq s}\left\|\int_0^u(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r))G(r,X^1(r),U^1(r))\mathrm{d}W(r)\right\|^p_{\mathbb{S}_2}\\
&+5^{p-1}\mathrm{E}\sup_{0\leq u\leq s}\left\|\int_0^u\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}Z(r)\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&=5^{p-1}(\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5). \qquad (35)
\end{aligned}$$

Regarding I_1, it is evident that

$$\begin{aligned}
\mathrm{I}_1&=\mathrm{E}\left[\sup_{0\leq u\leq s}\|\mathcal{C}(u)\xi_0+\mathcal{S}(u)(\eta_0-H(0,\xi_0,U(0)))\right.\\
&\left.+\ \mathcal{G}_b(0,u)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left[\mathrm{E}h_b-\mathcal{C}(b)\xi_0-\mathcal{S}(b)(\eta_0-H(0,\xi_0,U^1(0))\right]\right\|^p_{\mathbb{S}_2}\Bigg]\\
&\leq 7^{p-1}\mathrm{E}\left[M^p_C(b)||\xi_0||^p_{\mathbb{S}_2}+M^p_S(b)\left(||\eta_0||^p_{\mathbb{S}_2}+C_p\left(1+||\xi_0||^p_{\mathbb{S}_2}+||U^1(0)||^p_{\mathbb{S}_3}\right)\right)\right]\\
&\quad+7^{p-1}\left\{M^p_\lambda(b)\,\|B\|^{2p}\,M^{2p}_S(b)b^p\right\}\\
&\quad\times\mathrm{E}\left[||\mathrm{E}h_b||^p+M^p_C(b)||\xi_0||^p_{\mathbb{S}_2}+M^p_S(b)\left(||\eta_0||^p_{\mathbb{S}_2}+C_p\left(1+||\xi_0||^p_{\mathbb{S}_2}+||U^1(0)||^p_{\mathbb{S}_3}\right)\right)\right]. \qquad (36)
\end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned}
\mathrm{I}_2+\mathrm{I}_3&=\mathrm{E}\sup_{0\leq u\leq s}\left\|\int_0^u(\mathcal{C}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r))H(r,X^1(r),U^1(r))\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&\quad+\mathrm{E}\sup_{0\leq u\leq s}\left\|\int_0^u(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r))F(r,X^1(r),U^1(r))\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&\leq 2^{p-1}\left\{M^p_C(b)b^{p-1}+M^p_\lambda(b)\,\|B\|^{2p}\,M^{2p}_S(b)M^p_C(b)b^{2p-1}\right\}C_p\\
&\quad\times\int_0^s\mathrm{E}\left(\sup_{0\leq v\leq r}(1+||X^1(v)||^p_{\mathbb{S}_2}+||U^1(v)||^p_{\mathbb{S}_3})\right)\mathrm{d}r\\
&\quad+2^{p-1}\left\{M^p_S(b)b^{p-1}+M^p_\lambda(b)\,\|B\|^{2p}\,M^{3p}_S(b)b^{2p-1}\right\}C_p\\
&\quad\times\int_0^s\mathrm{E}\left(\sup_{0\leq v\leq r}(1+||X^1(v)||^p_{\mathbb{S}_2}+||U^1(v)||^p_{\mathbb{S}_3})\right)\mathrm{d}r\\
&=2^{p-1}b^{p-1}C_p\left(M^p_C(b)+M^p_S(b)\right)\left\{b^p\,\|B\|^{2p}\,M^{2p}_S(b)M^p_\lambda(b)+1\right\}\\
&\quad\times\int_0^s\mathrm{E}\left(\sup_{0\leq v\leq r}\left(1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\mathrm{d}r. \qquad (37)
\end{aligned}$$

Using both the L^p inequalities for stochastic integrals and Hölder's inequality, we have

$$\begin{aligned}
\mathrm{I}_4 &= \mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u (\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r))G(r,X^1(r),U^1(r))\mathrm{d}W(r)\right\|^p_{\mathbb{S}_2}\\
&\le 2^{p-1}\left\{M^p_\lambda(b)\left\|B\right\|^{2p}M^{2p}_S(b)b^p+1\right\}\left\{M^p_S(b)\ell_p b^{\frac{1}{2}p-1}\right\}C_p\\
&\times\int_0^s \mathrm{E}\left(\sup_{0\le v\le r}\left(1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\mathrm{d}r. \qquad (38)
\end{aligned}$$

By the L^p inequalities for stochastic integrals, and Hölder's inequality, we have

$$\begin{aligned}
\mathrm{I}_5 &= \mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u \mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}Z(r)\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&\le\left\{M^p_\lambda(b)\left\|B\right\|^{2p}M^{2p}_S(b)\ell_p b^{\frac{3}{2}p-1}\right\}\int_0^s \mathrm{E}||Z(r)||^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)}\mathrm{d}r. \qquad (39)
\end{aligned}$$

By aggregating inequalities (35) to (39), we derive inequality (33). Now, let's examine the subsequent inequalities

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\le u\le s}\|X^1(u)-X^2(u)\|^p_{\mathbb{S}_2}\right) \qquad (40)\\
&\le 3^{p-1}\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u\left(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)\right)\right.\\
&\qquad\left.\times(H(r,X^1(r),U^1(r))-H(r,X^2(r),U^2(r)))\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&\le 3^{p-1}\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u\left(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)\right)\right.\\
&\qquad\left.\times(F(r,X^1(r),U^1(r))-F(r,X^2(r),U^2(r)))\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&+3^{p-1}\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u\left(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)\right)\right.\\
&\qquad\left.\times(G(r,X^1(r),U^1(r))-G(r,X^2(r),U^2(r)))\mathrm{d}W(r)\right\|^p_{\mathbb{S}_2}\\
&= 3^{p-1}(\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3).
\end{aligned}$$

Then, employing a similar method as in equation (37), we can proceed

$$\begin{aligned}
\mathrm{J}_1+\mathrm{J}_2 &= \mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u\left(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)\right)\right.\\
&\qquad\left.\times(H(r,X^1(r),U^1(r))-H(r,X^2(r),U^2(r)))\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&+\mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u\left(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)\right)\right.\\
&\qquad\left.\times(F(r,X^1(r),U^1(r))-F(r,X^2(r),U^2(r)))\mathrm{d}r\right\|^p_{\mathbb{S}_2}\\
&\le 2^{p-1}\left\{\left\{b^p\left\|B\right\|^{2p}M^{2p}_S(b)M^p_\lambda(b)+1\right\}(M^p_C(b)+M^p_S(b))\,b^{p-1}\right\}L_p\\
&\times\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}\|(X^1(v),U^1(v))-(X^2(v),U^2(v))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r. \qquad (41)
\end{aligned}$$

Similarly, in equation (38), we can employ a comparable method

$$\begin{aligned}
\mathrm{J}_3 &= \mathrm{E}\sup_{0\le u\le s}\left\|\int_0^u \left(\mathcal{S}(u-r)-\mathcal{G}_b(r,u)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)\right)\right.\\
&\qquad\left.\times\left(G(r,X^1(r),U^1(r))-G(r,X^2(r),U^2(r))\right)\mathrm{d}W(r)\right\|^p_{\mathbb{S}_2}\\
&\le 2^{p-1}\left\{M^p_\lambda(b)\left\|B\right\|^{2p}Mb^p+1\right\}\left\{M^p_S(b)\ell_p b^{\frac12 p-1}\right\}L_p\\
&\qquad\times\int_0^s \mathrm{E}\left(\sup_{0\le v\le r}\left\|(X^1(v),U^1(v))-(X^2(v),U^2(v))\right\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r. \qquad (42)
\end{aligned}$$

Here, $\beta_i(b)$ for $i=1,2,3,4$ are defined as follows:

$$\begin{cases}
\beta_1(b) = 5^{p-1}\left\{7^{p-1}+7^{p-1}\left\{M^p_\lambda(b)\left\|B\right\|^{2p}M^{2p}_S(b)b^p\right\}\right\}\\
\qquad\times\mathrm{E}\left[M^p_C(b)||\xi_0||^p_{\mathbb{S}_2}+M^p_S(b)\left(||\eta_0||^p_{\mathbb{S}_2}+C_p\left(1+||\xi_0||^p_{\mathbb{S}_2}+||U^1(0)||^p_{\mathbb{S}_3}\right)\right)\right]\\
\qquad+5^{p-1}\left\{7^{p-1}\left\{M^p_\lambda(b)\left\|B\right\|^{2p}M^{2p}_S(b)b^p\right\}\right\}||\mathrm{E}h_b||^p\\
\beta_2(b) = 5^{p-1}\left\{\left(\frac1b\right)b^{\frac12 p}M^p_\lambda(b)M^{2p}_S\left\|B\right\|^p\ell_p\left(\left\|B\right\|^p b^p+1\right)\right\}\mathrm{E}\int_0^b||Z(r)||^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)}\mathrm{d}r,\\
\beta_3(b) = 10^{p-1}\left\{\left(\frac1b\right)b^{\frac12 p}\left\{b^p\left\|B\right\|^{2p}M^{2p}_S(b)M^p_\lambda(b)+1\right\}\right.\\
\qquad\left.\times\left\{b^{\frac12 p}(M^p_C(b)+M^p_S(b))+\ell_pM^p_S(b)\right\}\right\}C_p,\\
\beta_4(b) = 6^{p-1}\left\{\left(\frac1b\right)b^{\frac12 p}\left\{b^p\left\|B\right\|^{2p}M^{2p}_S(b)M^p_\lambda(b)+1\right\}\right.\\
\qquad\left.\times\left\{b^{\frac12 p}(M^p_C(b)+M^p_S(b))+\ell_pM^p_S(b)\right\}\right\}L_p.
\end{cases}$$

Combining (40)–(42), we obtain (34). □

Now let us demonstrate the following straightforward lemma.

Lemma 4.6. *Under the conditions* (A0) – (A5)*, there exist positive constants* $\mu_i(b)>0$*, for* $i=1,2,3,4$*, such that for arbitrary* $u\in[0,s]$*,* $s\in[0,b]$*:*

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\le u\le s}||(X^1(u),U^1(u))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\\
&\le \mu_1(b)+\mu_2(b)+\mu_3(b)\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}\left(1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\mathrm{d}r, \qquad (43)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\le u\le s}||\left(X^1(u),U^1(u)\right)-\left(X^2(u),U^2(u)\right)||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\\
&\le \mu_4(b)\int_0^s\mathrm{E}\left(\sup_{0\le v\le s}||\left(X^1(v),U^1(v)\right)-\left(X^2(u),U^2(v)\right)||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r. \qquad (44)
\end{aligned}$$

Proof. Using (22) from Lemma 4.4 and (33) from Lemma 4.5, we obtain, for arbitrary $s\in[0,b]$:

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\le u\le s}||(X^1(u),U^1(u))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\\
&\le k_1(b)+\beta_1(b)+k_2(b)+\beta_2(b)\\
&+\left(k_3(b)+\beta_3(b)\right)\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}\left(1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\mathrm{d}r\\
&\le \mu_1(b)+\mu_2(b)+\mu_3(b)\int_0^s\mathrm{E}\left(\sup_{0\le v\le r}\left(1+||(X^1(v),U^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\mathrm{d}r.
\end{aligned}$$

For the estimates (44), we directly use $k_4(b)+\beta_4(b)=\mu_4(b)$. With

$$\begin{cases}
\mu_1(b) = 5^{p-1}\left\{7^{p-1}+7^{p-1}\left\{M_\lambda^p(b)\left\|B\right\|^{2p}M_S^{2p}(b)b^p\right\}+4^{p-1}\left\{M_\lambda^p(b)\left\|B\right\|^pM_S^p(b)\right\}\right\}\\
\qquad\times\mathrm{E}\left[M_C^p(b)||\xi_0||_{\mathbb{S}_2}^p+M_S^p(b)\left(||\eta_0||_{\mathbb{S}_2}^p+C_p(1+||\xi_0||_{\mathbb{S}_2}^p+||U^1(0)||_{\mathbb{S}_3}^p)\right)\right]\\
\qquad+5^{p-1}\left\{7^{p-1}\left\{M_\lambda^p(b)\left\|B\right\|^{2p}M_S^{2p}(b)b^p\right\}\right.\\
\qquad\left.+4^{p-1}\left\{M_\lambda^p(b)\left\|B\right\|^pM_S^p(b)\right\}\right\}||\mathrm{E}h_b||^p\\
\mu_2(b) = 5^{p-1}\left\{b^{\frac{1}{2}p-1}\ell_pM_S^{2p}M_\lambda^p(b)\left\|B\right\|^p(b+b^p+1)\right\}\int_0^b||Z(r)||_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)}^p\mathrm{d}r\\
\mu_3(b) = 5^{p-1}\left\{b^{\frac{1}{2}p-1}C_p\left(2^{p-1}+M_\lambda^p(b)\left\|B\right\|^pM_S^p(b)\left(1+2^{p-1}b^p\left\|B\right\|^pM_S^p(b)\right)\right)\right\}\\
\qquad\times\left\{b^{\frac{1}{2}p}(M_C^p(b)+M_S^p(b))+\ell_pM_S^p(b)\right\}\\
\mu_4(b) = 3^{p-1}\left\{b^{\frac{1}{2}p-1}L_p\left(2^{p-1}+M_\lambda^p(b)\left\|B\right\|^pM_S^p(b)\left(1+2^{p-1}b^p\left\|B\right\|^pM_S^p(b)\right)\right)\right\}\\
\qquad\times\left\{b^{\frac{1}{2}p}(M_C^p(b)+M_S^p(b))+\ell_pM_S^p(b)\right\}
\end{cases}$$

This completes the proof. □

Proof of Theorem 4.1. In order to prove the theorem, we need several steps.

Step 1: Establishing Uniqueness.

Let $R>0$. Define the increasing sequence of stopping times as follows:

$$\tau_R=\inf\left\{s\le b:||(X^1(s),U^1(s))-(X^2(s),U^2(s))||_{\mathbb{S}_2\times\mathbb{S}_3}\ge R\right\},$$

which represents the exit time from the open ball of radius R. If $||(X^1(s),U^1(s))-(X^2(s),U^2(s))||_{\mathbb{S}_2\times\mathbb{S}_3}<R$ for every $s\in[0,b]$, then $\tau_R=b$.
Let $(X_R^i(s),U_R^i(s))=I_{[0,\tau_R]}(s)(X^i(s),U^i(s))$. Then we have

$$\begin{aligned}
X_R^i(s) &= I_{[0,\tau_R]}(s)\mathcal{C}(s)X_R^i(0)+I_{[0,\tau_R]}(s)\mathcal{S}(s)(X_R^{i\prime}(0)-H(0,X_R^i(0),U_R^i(0)))\\
&\quad+I_{[0,\tau_R]}(s)\int_0^s\mathcal{C}(s-r)I_{[0,\tau_R]}(r)H(r,X^i(r),U^i(r))\mathrm{d}r\\
&\quad+I_{[0,\tau_R]}(s)\int_0^sI_{[0,\tau_R]}(r)\mathcal{S}(s-r)\left[F(r,X_R^i(r),U_R^i(r))+U_R^i(r)\right]\mathrm{d}r\\
&\quad+I_{[0,\tau_R]}(s)\int_0^sI_{[0,\tau_R]}(r)\mathcal{S}(s-r)G(r,X_R^i(r),U_R^i(r))\mathrm{d}W(r).
\end{aligned}$$

And

$$\begin{aligned}
U_R^i(s) &= I_{[0,\tau_R]}(s)B^*\mathcal{S}^*(b-s)\mathrm{R}_{[0,b],\lambda}^{\mathcal{G}}\left(\mathrm{E}h_b-\mathcal{S}(b)X_R^i(0)\right.\\
&\quad\left.-\mathcal{S}(b)(X_R^{i\prime}(0)-H(0,X_R^i(0),U_R^i(0)))\right)\\
&\quad-I_{[0,\tau_R]}(s)\int_0^sB^*\mathcal{S}^*(b-s)I_{[0,\tau_R]}(r)\mathrm{R}_{[r,b],\lambda}^{\mathcal{G}}\mathcal{C}(b-r)H(r,X_R^i(r),U_R^i(r))\mathrm{d}r\\
&\quad-I_{[0,\tau_R]}(s)\int_0^sB^*\mathcal{S}^*(b-s)I_{[0,\tau_R]}(r)\mathrm{R}_{[r,b],\lambda}^{\mathcal{G}}\mathcal{S}(b-r)F(r,X_R^i(r),U_R^i(r))\mathrm{d}r\\
&\quad-I_{[0,\tau_R]}(s)\int_0^sB^*\mathcal{S}^*(b-s)I_{[0,\tau_R]}(r)\mathrm{R}_{[r,b],\lambda}^{\mathcal{G}}(\mathcal{S}(b-r)G(r,X_R^i(r),U_R^i(r))\\
&\quad-Z(r))\mathrm{d}W(r).
\end{aligned}$$

Utilizing a method similar to Lemma 4.6, we find

$$\begin{aligned}
&\mathrm{E}\left(\sup_{u\in[0,s]}||(X_R^1(u),U_R^1(u))-(X_R^2(u),U_R^2(u))||_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\\
&\le\mu_4(b)\int_0^s\mathrm{E}\left(\sup_{v\in[0,r]}||(X_R^1(v),U_R^2(v))-(X_R^2(v),U_R^2(v))||_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\mathrm{d}r.
\end{aligned}$$

Define $h(s)$ as

$$h(s) = \mathrm{E}\left(\sup_{u\in[0,s]} ||(X_R^1(u), U_R^1(u)) - (X_R^2(u), U_R^2(u))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right).$$

Then, from the previous inequality, we have

$$h(s) \leq \mu_4(b)\int_0^s h(r)\mathrm{d}r.$$

Since $h(s) < R^p$, both the right-hand side and left-hand side are bounded functions. Consequently, by Gronwall's Lemma,

$$0 \leq h(b) = \mathrm{E}\left(\sup_{s\in[0,b]} ||(X_R^1(s), U_R^1(s)) - (X_R^2(s), U_R^2(s))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \leq 0.$$

For all $s \in [0, b]$, it's certain that $(X_R^1(s), U_R^1(s)) = (X_R^2(s), U_R^2(s))$ with probability 1. Consequently, the processes

$$(X_R^1(s), U_R^1(s)) = (X_R^2(s), U_R^2(s)), \quad \mathbb{P}-\text{almost surely for } \omega \in \Omega,$$
$$\text{and for almost every } s \in [0, b].$$

Since $\left\{\sup_{u\in[0,s]} ||(X^1(u), U^1(u)) - (X^2(u), U^2(u))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right\} = R^p$ on the event $\{\tau_R < b\}$, it follows that

$$0 = \mathrm{E}\left(\sup_{s\in[0,\tau_R]} ||(X^1(s), U^1(s)) - (X^2(s), U^2(s))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \geq R^p\mathbb{P}(\tau_R < b).$$

Hence

$$\mathbb{P}(\tau_R < b) \leq \frac{1}{R^p}\mathrm{E}\left(\sup_{s\in[0,\tau_R]} ||(X^1(s), U^1(s)) - (X^2(s), U^2(s))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \leq \frac{0}{R^p},$$

which implies $\mathbb{P}(\tau_R < b) = 0$ for any $R > 0$. Note that τ_R increases with R, and as R tends to infinity, τ_R almost surely converges to b for all $s \in [0, b]$. Then

$$\left\{\sup_{s\in[0,b]} (X_R^1(s), U_R^1(s)) - (X_R^2(s), U_R^2(s))\right\} \longrightarrow \left\{\sup_{s\in[0,b]} (X^1(s), U^1(s)) - (X^2(s), U^2(s))\right\}$$

a.s.

By Beppo Levi's theorem (or Fatou's lemma), it follows that

$$\mathrm{E}\left(\sup_{s\in[0,b]} ||(X^1(s), U^1(s)) - (X^2(s), U^2(s))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) = 0.$$

Consequently

$$(X^1(s), U^1(s)) = (X^2(s), U^2(s)) \quad \mathbb{P}-\text{a.s. for } \omega \in \Omega, \text{ and for a.e. } s \in [0, b].$$

Considering that $(X^1(s), U^1(s))$ are solutions of Equation (20), it's straightforward to conclude that for any $s \in [0, b]$:

$$(X^1(s), U^1(s)) = (X^2(s), U^2(s)), \quad \mathbb{P}-\text{a.s.}$$

Step 2: Establishing the Boundary Estimate.

Let's assume $R > 0$. Then, we define an increasing sequence of stopping times as follows:

$$\tau_R = \inf\{s \leq b : ||(X^1(s), U^1(s))||_{\mathbb{S}_2\times\mathbb{S}_3} \geq R\}.$$

Here, τ_R represents the exit time from the open ball of radius R. If

$$||(X^1(s), U^1(s))||_{\mathbb{S}_2\times\mathbb{S}_3} < R$$

for every $s \in [0,b]$, then we understand $\tau_R = b$. Applying a modified version of Lemma 4.6, we derive

$$\begin{aligned}&\mathrm{E}\left(\sup_{u\in[0,s]}||(X_R^1(u),U_R^1(u))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\\&\leq \mu_1(b)+\mu_2(b)+\mu_3(b)\int_0^s \mathrm{E}\left(\sup_{v\in[0,r]}\left(||(X_R^1(v),U_R^1(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}+1\right)\right)\mathrm{d}r.\end{aligned}$$

Define

$$h(s)=\mathrm{E}\left(\sup_{v\in[0,s]}||(X_R^1(u),U_R^1(u))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right).$$

Using the preceding inequality, we obtain

$$h(s)\leq \mu_1(b)+\mu_2(b)+\mu_3(b)b+\mu_3(b)\int_0^s h(r)\mathrm{d}r.$$

Consequently, the left-hand side is bounded as well, leading to an application of Gronwall's inequality.

$$h(b)=\mathrm{E}\left(\sup_{s\in[0,b]}||(X_R^1(s),U_R^1(s))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\leq(\mu_1(b)+\mu_2(b)+\mu_3(b)b)e^{b\mu_3(b)}.$$

The right-hand side remains invariant to variations in R. Let's now explore the scenario as R tends towards infinity. It's crucial to note that due to the continuity of $(X^1(s),U^1(s))$, we have $\left\{\sup_{s\in[0,\tau_R]}||(X^1(s),U^1(s))||_{\mathbb{S}_2\times\mathbb{S}_3}\right\}=R^p$ on $\{\tau_R<b\}$. Consequently

$$\mathrm{E}\left(\sup_{s\in[0,\tau_R]}||(X^1(s),U^1(s))||_{\mathbb{S}_2\times\mathbb{S}_3}\right)\geq R^p\mathbb{P}(\tau_R<b).$$

As a result:

$$\mathbb{P}(\tau_R<b)\leq\frac{1}{R^p}\mathrm{E}\left(\sup_{s\in[0,\tau_R]}||(X^1(s),U^1(s))||_{\mathbb{S}_2\times\mathbb{S}_3}\right)\leq\frac{1}{R^p}(\mu_1(b)+\mu_2(b)+\mu_3(b)b)e^{b\mu_3(b)}.$$

Upon taking the limit as R tends to infinity, we observe that $\mathbb{P}(\tau_R<b)$ tends to zero. Notably, τ_R increases with R, converging almost surely to b as R approaches infinity for all $s\in[0,b]$. Consequently, we have the convergence:

$$\left\{\sup_{s\in[0,b]}((X_R^1(s),U_R^1(s)))\right\}\longrightarrow\left\{\sup_{s\in[0,b]}(X^1(s),U^1(s))\right\}\quad\text{a.s.}$$

Furthermore, applying either Beppo Levi's theorem or Fatou's lemma yields

$$\mathrm{E}\left(\sup_{u\in[0,s]}||(X^1(s),U^1(s))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\leq(\mu_1(b)+\mu_2(b)+\mu_3(b)b)e^{b\mu_3(b)}.$$

Step 3: Establishing Existence.

The proof of existence is based on the classical fixed point theorem for contractions. Thanks to uniqueness, the existence of a solution on $\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]$ will follow from the existence of a solution on an arbitrary interval $[0,b]$.

Let $\delta\in\mathbb{N}^*$, and $0=b_0<b_1<b_2<\cdots<b_\delta=b$, with $b_k=kb/\delta, k=1,\cdots,\delta$. Let

$$\mu_4(b/\delta)=3^{p-1}\left\{\left(\frac{b}{\delta}\right)^{\frac{1}{2}p-1}L_p\left(2^{p-1}+M_\lambda^p(b/\delta)\,\|B\|^p\,M_S^p(b/\delta)\right.\right.$$

$$\times \left(1+2^{p-1}\left(\frac{b}{\delta}\right)^{p}\|B\|^{p}\,M_S^p(b/\delta)\right)\Bigg\}$$
$$\times\left\{\left(\frac{b}{\delta}\right)^{\frac{1}{2}p}\left(M_C^p(b/\delta)+M_S^p(b/\delta)\right)+\ell_p M_S^p(b/\delta)\right\}.$$

Then $\mu_4(b/\delta)\longrightarrow 0$ as $\delta\longrightarrow\infty$, for any $p\geq 2$.

With δ large enough we shall prove that is a strict contraction on the complete metric space $\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1]$. The solution of the system (20) on $[0,b_1]$ is a fixed point of the mapping

$$\Lambda_\lambda:\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1]\to\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1],$$

defined as follows:

$$\begin{aligned}(45)\quad &\Lambda_\lambda(X,U)(s)\\
&=\Big(\mathcal{C}(s)X(b_0)+\mathcal{S}(s)(X(b_0)-H(b_0,X(b_0),U(b_0)))+\int_{b_0}^{s}\mathcal{C}(s-r)H(r,X(r),U(r))\mathrm{d}r\\
&\quad+\int_{b_0}^{s}\mathcal{S}(s-r)\left[F(r,X(r),U(r))+U(r)\right]\mathrm{d}r+\int_{b_0}^{s}\mathcal{S}(s-r)G(r,X(r),U(r))\mathrm{d}W(r),\\
&-B^*\mathcal{S}^*(b_1-s)\mathrm{R}^{\mathcal{G}}_{[b_0,b_1],\lambda}(b_1)\left(\mathcal{C}(b_1)X(b_0)+\mathcal{S}(b_1)\left(X(b_0)-H(b_0,X(b_0),U(b_0))\right)-\mathrm{E}h_{b_1}\right)\\
&\qquad-\int_{b_0}^{s}B^*\mathcal{S}^*(b_1-s)\mathrm{R}^{\mathcal{G}}_{[r,b_1],\lambda}(b_1)\mathcal{C}(b_1-r)H(r,X(r),U(r))\mathrm{d}r\\
&\qquad-\int_{b_0}^{s}B^*\mathcal{S}^*(b_1-s)\mathrm{R}^{\mathcal{G}}_{[r,b_1],\lambda}(b_1)\mathcal{S}(b_1-r)F(r,X(r),U(r))\mathrm{d}r\\
&\qquad-\int_{b_0}^{s}B^*\mathcal{S}^*(b_1-s)\mathrm{R}^{\mathcal{G}}_{[r,b_1],\lambda}(b_1)(\mathcal{S}(b_1-r)G(r,X(r),U(r)-Z(r))\mathrm{d}W(r)\Big).\end{aligned}$$

Utilizing Lemma 4.6, we deduce

$$\begin{aligned}&\mathbb{E}\left(\sup_{b_0\leq s\leq b_1}||\Lambda_\lambda(X(s),U(s))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\\
&\leq\mu_1(b_1)+\mu_2(b_1)+\mu_3(b_1)b_1\mathrm{E}\left(\sup_{b_0\leq s\leq b_1}\left(1+\left(\|X(s)\|^p_{\mathbb{S}_2}+\|U(s)\|^p_{\mathbb{S}_3}\right)\right)\right).\end{aligned}$$

Hence, if $(X,U)\in\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1]$, then $\Lambda_\lambda(U,X)\in\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1]$. Then the mapping Λ_λ is well-defined, meaning if $(X,U)\in\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1]$, then $\Lambda_\lambda(U,X)\in\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1]$.

Referring to 4.6, we can infer

$$\begin{aligned}&\mathrm{E}\left(\sup_{0\leq s\leq b_1}||(X^1(s),U^1(s))-(X^2(s),U^2(s))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\\
&\leq\mu_4(b_1)\int_{b_0}^{b_1}\mathrm{E}\left(\sup_{0\leq v\leq r}||(X^1(v),U^1(v))-(X^2(v),U^2(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r\\
&\leq\mu_4(b/\delta)\int_{b_0}^{b_1}\mathrm{E}\left(\sup_{0\leq v\leq r}||(X^1(v),U^1(v))-(X^2(v),U^2(v))||^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\mathrm{d}r.\end{aligned}$$

Let $\delta_0\in\mathbb{N}^*$, be such that $\mu_4(b/\delta_0)(b_1-b_0)<1$. Then Λ_λ is a strict contraction in $\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1]$ and consequently the system (20) has a unique solution $\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_1]$.

Now we consider the interval $[b_0,b_2]$. The mapping

$$\Lambda_\lambda:\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_2]\to\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[b_0,b_2],$$

is defined by $\Lambda_\lambda(X,U)(s)=\mathrm{Eq}(45)$ if $s\in[b_0,b_1]$, and if $s\in]b_1,b_2]$:

$$
\begin{aligned}
(46)\quad \Lambda_\lambda(X,U)(s) = \Big(&\mathcal{C}(s-b_1)X(b_1) + \mathcal{S}(s-b_1)\left(X'(b_1) - H(b_1, X(b_1), U(b_1))\right) \\
&+ \int_{b_1}^{s} \mathcal{C}(s-r)H(r, X(r), U(r))\mathrm{d}r \\
+ \int_{b_1}^{s} \mathcal{S}(s-r)\left[F(r, X(r), U(r)) + U(r)\right]\mathrm{d}r &+ \int_{b_1}^{s} \mathcal{S}(s-r)G(r, X(r), U(r))\mathrm{d}W(r), \\
&- B^*\mathcal{S}^*(b_2-s)\mathrm{R}^{\mathcal{G}}_{[b_1,b_2],\lambda}\left(\mathcal{C}(b_2-b_1)X(b_1)\right. \\
&+\mathcal{S}(b_2-b_1)(X'(b_1) - H(b_1, X(b_1), U(b_1)) - \mathrm{E}h_{b_2}) \\
&- \int_{b_1}^{s} B^*\mathcal{S}^*(b_2-s)\mathrm{R}^{\mathcal{G}}_{[r,b_2],\lambda}\mathcal{C}(b_2-r)H(r, X(r), U(r))\mathrm{d}r \\
&- \int_{b_1}^{s} B^*\mathcal{S}^*(b_2-s)\mathrm{R}^{\mathcal{G}}_{[r,b_2],\lambda}\mathcal{S}(b_2-r)F(r, X(r), U(r))\mathrm{d}r \\
&- \left.\int_{b_1}^{s} B^*\mathcal{S}^*(b_2-s)\mathrm{R}^{\mathcal{G}}_{[r,b_2],\lambda}\mathcal{S}(b_2-r)(G(r, X(r), U(r) - Z(r))\mathrm{d}W(r)\right).
\end{aligned}
$$

Using lemma 4.1 and repeating the same arguments, we cover the whole interval $[0, b]$. □

Theorem 4.2. *Under the assumptions* (A0) − (A6)*, the stochastic differential equation* (1) *is approximately controllable* (21)*.*

Proof. Let (X, U) be a fixed point of Λ_λ in $\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0, b]$. By Lemma 4.3, X satisfies the following equality:

$$
\begin{aligned}
X(b) - h_b &= \lambda\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left(\mathcal{C}(b)X(0) - \mathcal{S}(b)(\eta_0 - H(0, X(0), U(0)) - \mathrm{E}h_b\right) \\
&+ \int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)H\left(r, X(r), U(r)\right)\mathrm{d}r + \int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)F\left(r, X(r), U(r)\right)\mathrm{d}r \\
&+ \int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\left(\mathcal{S}(b-r)G\left(r, X(r), U(r)\right) - Z(r)\mathrm{d}W(r).\right.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathrm{E}||X(b) - h_b|| \le\ & 4^{p-1}\mathrm{E}\left\|\lambda\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left(\mathcal{C}(b)X(0) - \mathcal{S}(b)(\eta_0 - H(0, X(0), U(0)) - \mathrm{E}h_b\right)\right\|^p \\
&+ 4^{p-1}\mathrm{E}\left\|\int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)H\left(r, X(r), U(r)\right)\mathrm{d}r\right\|^p_{\mathbb{S}_2} \\
&+ 4^{p-1}\mathrm{E}\left\|\int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)F\left(r, X(r), U(r)\right)\mathrm{d}r\right\|^p_{\mathbb{S}_2} \\
&+ 4^{p-1}\mathrm{E}\left\|\int_0^b \lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\left(\mathcal{S}(b-r)(G\left(r, X(r), U(r)\right) - Z(r))\mathrm{d}W(r)\right\|^p_{\mathbb{S}_2}.\right.
\end{aligned}
$$

Since the functions F, G and H are uniformly bounded, $0 < ||\lambda\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}||^p \le \tilde{M}^p_\lambda(b)$ and $\lambda\mathrm{R}^{\mathcal{G}}_{[r,b]} \underset{\lambda\to 0^+}{\longrightarrow} 0$, $\lambda \to 0^+$ for all $0 \le r < b$. Therefore, by the Lebesgue dominated convergence theorem, we infer that the expectation of $\mathrm{E}||X(b) - h_b||^p \to 0$ as $\lambda \to 0^+$. This result establishes approximate controllability. □

5. Existence, Uniqueness and Approximate controllability Result with Non-Lipschitz Continuity

This section addresses the existence and uniqueness of mild solutions to equation (20) with some relaxed conditions. Along with assumptions (A0) − (A1), we introduce the following conditions (see [25, 26, 27, 28]):

(A7) There exists a function $N : [0,b] \times [0,+\infty) \to [0,+\infty)$, defined as $(s,v) \to N(s,v)$, such that:

$$\begin{aligned} &\mathrm{E}\|H(s,X(s),U(s))\|^p_{\mathbb{S}_2} + \mathrm{E}\|F(s,X(s),U(s))\|^p_{\mathbb{S}_2} + \mathrm{E}\|G(s,X(s),U(s))\|^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)} \\ &\qquad \leq N\left(s,\mathrm{E}\|X(s)\|^p_{\mathbb{S}_2}\right) + N\left(s,\mathrm{E}\|U(s)\|^p_{\mathbb{S}_2}\right) \end{aligned} \tag{47}$$

for all $s \in [0,b]$ and all $(X,U),(\bar{X},\bar{U}) \in L^p\left(\Omega,\mathcal{F}_s,\mathbb{S}_2 \times \mathbb{S}_3\right)$.

(A8) The function $N(s,v)$ is locally integrable with respect to s for each fixed $v \in [0,+\infty)$ and is continuous and non-decreasing in v for each fixed $s \in [0,b]$.

$$v(s) = \tilde{\mu}_1(b) + \tilde{\mu}_2(b) + \tilde{\mu}_3(b)\int_0^s N(r,v(r))dr$$

has a global bounded solution $\mathcal{E}(s)$ on $[0,b]$.

(A9) There is a function $K : [0,b] \times [0,+\infty) \to [0,+\infty)$ such that, for all $s \in [0,b]$ and all $(X,U),(\bar{X},\bar{U}) \in L^p\left(\Omega,\mathcal{F}_s,\mathbb{S}_2 \times \mathbb{S}_3\right)$

$$\begin{aligned} &\mathrm{E}\|H(s,X(s),U(s)) - H(s,\bar{X}(s),\bar{U}(s))\|^p_{\mathbb{S}_2} + \mathrm{E}\|F(s,X(s),U(s)) - F(s,\bar{X}(s),\bar{U}(s))\|^p_{\mathbb{S}_2} \\ &\qquad + \mathrm{E}\|G(s,X(s),U(s)) - G(s,\bar{X}(s),\bar{U}(s))\|^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)} \\ &\qquad \leq K\left(s,\mathrm{E}\|X(s) - \bar{X}(s)\|^p_{\mathbb{S}_2}\right) + K\left(s,\mathrm{E}\|U(s) - \bar{U}(s)\|^p_{\mathbb{S}_3}\right) \end{aligned}$$

(A10) The function $K(s,v)$ is locally integrable in s for each fixed $v \in [0,+\infty)$ and continuous and non-decreasing in v for each fixed $s \in [0,b]$. Additionally $K(s,0) = 0$. If a non-negative, continuous, bounded function $z(s)$ for $s \in [0,b]$ satisfies

$$\begin{cases} z(s) \leq \tilde{\mu}_3(b)\int_0^s K(r,z(r))dr, \quad s \in [0,b] \\ z(0) = 0 \end{cases}$$

for some $\tilde{\mu}_3(b) > 0$, then $z(s) = 0$ for all $s \in [0,b]$.

Interpretation of Non-Lipschitz Conditions. Conditions (A7) − (A10) generalize the classical Lipschitz and linear growth conditions, allowing for a broader class of nonlinearities. Such conditions are particularly relevant in systems with

- **Saturation or threshold effects:** In control systems where actuators saturate, the nonlinearity may grow linearly up to a point and then level off, violating global Lipschitz continuity but satisfying a condition like (A7) with $N(s,v)$ growing sublinearly.
- **Stochastic volatility in finance:** In models of asset prices, volatility may depend on the state in a non-Lipschitz manner (e.g., square-root diffusion models), where the diffusion coefficient is Hölder continuous but not Lipschitz.
- **Polynomial or exponential growth:** Systems with polynomial nonlinearities $|x|^\alpha$ for $\alpha > 1$ do not satisfy global Lipschitz conditions but can be accommodated by appropriate N and K functions.
- **Local Lipschitz with blow-up:** Many physical systems (e.g., reaction-diffusion equations) have locally Lipschitz nonlinearities that may blow up in finite time; conditions (A8) and (A10) ensure global existence by controlling growth via integral inequalities.

Mathematically, these conditions replace the linear bounds $L|x-y|$ with more general comparison functions $K(s,|x-y|)$ that may grow slower than linearly (e.g., $K(s,v) = v\log(1+v)$) or satisfy Osgood-type conditions $\int_0^1 dv/K(v) = \infty$ that guarantee uniqueness despite non-Lipschitz behavior. This framework significantly expands the class of admissible systems while maintaining rigorous controllability results.

Remark 5.1.

- If $K(s,v) = Lv$ for $v \geq 0$ and a constant $L > 0$, then condition (A10) guarantees the global Lipschitz condition.
- If K is concave with respect to the second variable for each fixed $s \geq 0$, and satisfies

$$\|F(s,x,u) - F(s,\bar{x},\bar{u})\|^p + \|H(s,x,u) - H(s,\bar{x},\bar{u})\|^p + \|G(s,x,u) - G(s,\bar{x},\bar{u})\|^p_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)} \leq K\left(s,\|x-\bar{x}\|^p\right) + K\left(s,\|u-\bar{u}\|^p\right)$$

  for all $(x,u),(\bar{x},\bar{u}) \in \mathbb{S}_2 \times \mathbb{S}_3$ and $s \geq 0$, then, by Jensen's inequality, condition (A10) is satisfied.
- Let $K(s,v) = \eta(s)\vartheta(v)$ for $s \geq 0$ and $v \geq 0$, where $\eta(s) \geq 0$ is locally integrable, and $\vartheta : [0,+\infty) \to [0,+\infty)$ is a continuous, monotone non-decreasing, and concave function with $\vartheta(0) = 0$ and $\vartheta(v) > 0$ for $v > 0$. Moreover, it holds that

$$\int_0^s \frac{1}{\vartheta(v)}\, dv = \infty.$$

  It can be readily shown that ϑ meets the requirements of assumption (A10) (see [27, Lemma 2.2]) and [28].
- It is essential to note that the Lipschitz condition is a specific instance of these broader conditions.

Example 5.1 ([29, 30])**.**

- Consider specific functions $\vartheta(\cdot)$, with $\varepsilon \in (0,1)$ being sufficiently small. We define

$$\vartheta(v) = \begin{cases} v\log(v^{-1}), & 0 \leq v \leq \varepsilon, \\ \varepsilon\log(\varepsilon^{-1}) + \vartheta(\varepsilon_-)(v-\varepsilon), & v > \varepsilon, \end{cases}$$

$$\tilde{\vartheta}(v) = \begin{cases} v\log(v^{-1})\log\log(v^{-1}), & 0 \leq v \leq \varepsilon, \\ \varepsilon\log(\varepsilon^{-1})\log\log(\varepsilon^{-1}) + \tilde{\vartheta}(\varepsilon_-)(v-\varepsilon), & v > \varepsilon. \end{cases}$$

- Here, ϑ' and $\tilde{\vartheta}'$ represent the left derivatives of ϑ and $\tilde{\vartheta}$ at ε_-. These functions are concave and non-decreasing, and they satisfy

$$\int_{0_+} \frac{dy}{\vartheta(v)} = \int_{0_+} \frac{dy}{\tilde{\vartheta}(v)} = +\infty.$$

Theorem 5.1. *Under the assumptions that conditions* (A0) – (A1)*, and* (A7) – (A10) *hold, there exists a unique mild solution* $(X^i, U^i) \in \mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]$ *to the system* (48)*. Moreover, it possesses a continuous modification, and we have*

$$\mathrm{E}\left(\sup_{0\leq s\leq b} \|X(s),U(s)\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \leq \mathcal{E}(b), \tag{48}$$

where $\mathcal{E}(b)$ *is a global solution of* $v(b)$ *on* $[0,b]$ *such that*

$$v(b) = \tilde{\mu}_1(b) + \tilde{\mu}_2(b) + \tilde{\mu}_3(b)\int_0^b N(r,v(r))dr,$$

with

$$\begin{cases}\tilde{\mu}_1(b) = 5^{p-1}\left\{7^{p-1}+7^{p-1}\left\{M_\lambda^p(b)\|B\|^{2p}M_S^{2p}(b)b^p\right\}+4^{p-1}\left\{M_\lambda^p(b)\|B\|^pM_S^p(b)\right\}\right\}\\ \qquad\times\mathrm{E}\left[M_C^p(b)\left\|\xi_0\right\|_{\mathbb{S}_2}^p+M_S^p(b)\left(\left\|\eta_0\right\|_{\mathbb{S}_2}^p+N\left(0,\left\|(\xi_0,U(0))\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\right)\right]\\ \qquad+5^{p-1}\left\{7^{p-1}\left\{M_\lambda^p(b)\|B\|^{2p}M_S^{2p}(b)b^p\right\}\right.\\ \qquad\left.+4^{p-1}\left\{M_\lambda^p(b)\|B\|^pM_S^p(b)\right\}\right\}\left\|\mathrm{E}h_b\right\|^p,\\ \tilde{\mu}_2(b) = 5^{p-1}\left\{b^{\frac{1}{2}p-1}\ell_pM_S^{2p}M_\lambda^p(b)\|B\|^p\left(b+b^p+1\right)\right\}\int_0^b\|Z(r)\|_{\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2)}^p\,\mathrm{d}r,\\ \tilde{\mu}_3(b) = 5^{p-1}\left\{b^{\frac{1}{2}p-1}\left(2^{p-1}+M_\lambda^p(b)\|B\|^pM_S^p(b)\left(1+2^{p-1}b^p\|B\|^pM_S^p(b)\right)\right)\right\}\\ \qquad\times\left\{b^{\frac{1}{2}p}\left(M_C^p(b)+M_S^p(b)\right)+\ell_pM_S^p(b)\right\},\\ \mu_4(b) = 3^{p-1}\left\{b^{\frac{1}{2}p-1}L_p\left(2^{p-1}+M_\lambda^p(b)\left\|B\right\|^pM_S^p(b)\left(1+2^{p-1}b^p\left\|B\right\|^pM_S^p(b)\right)\right)\right\}\\ \qquad\times\left\{b^{\frac{1}{2}p}(M_C^p(b)+M_S^p(b))+\ell_pM_S^p(b)\right\}.\end{cases}$$

Proof. **Step 1: Establishing Uniqueness.**

Assume $R>0$. We will define an increasing sequence of stopping times as follows:

$$\tau_R=\inf\left\{s\le b:\left\|\left(X^1(s),U^1(s)\right)-\left(X^2(s),U^2(s)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}\ge R\right\}.$$

Using a method similar to that in Theorem 4.1, and by employing hypothesis (A9), we can show,

$$\begin{aligned}&\mathrm{E}\left(\sup_{u\in[0,s]}\left\|\left(X_R^1(u),U_R^2(u)\right)-\left(X_R^2(u),U_R^2(u)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\\ &\le\tilde{\mu}_3(b)\int_0^sK\left(r,\mathrm{E}\left(\sup_{\tau\in[0,r]}\left\|X_R^1(\tau)-X_R^2(\tau)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\right)\mathrm{d}r\\ &+\tilde{\mu}_3(b)\int_0^sK\left(r,\mathrm{E}\left(\sup_{\tau\in[0,r]}\left\|U_R^1(\tau)-U_R^2(\tau)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\right)\mathrm{d}r.\end{aligned}$$

Because K is non-decreasing in v, we can simplify further:

$$\begin{aligned}&\mathrm{E}\left(\sup_{u\in[0,s]}\left\|\left(X_R^1(u),U_R^1(u)\right)-\left(X_R^2(u),U_R^2(u)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\\ \le&\tilde{\mu}_3(b)\int_0^sK\left(r,\mathrm{E}\left(\sup_{u\in[0,r]}\left\|\left(X_R^1(u),U_R^1(u)\right)-\left(X_R^2(u),U_R^2(u)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\right)dr.\end{aligned}$$

Let

$$z(s)=\mathrm{E}\left(\sup_{u\in[0,s]}\left\|\left(X_R^1(u),U_R^1(u)\right)-\left(X_R^2(u),U_R^2(u)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right).$$

Then, from the previous inequality, we have

$$z(s)\le\tilde{\mu}_3(b)\int_0^sK(r,z(r))\mathrm{d}r.$$

Since $z(s)<R^p$. Consequently, by hypothesis (A10), for all $s\in[0,b]$,

$$\left(X^1(s),U^1(s)\right)=\left(X^2(s),U^2(s)\right),\quad\mathbb{P}\text{-a.s.}$$

(The rest of the proof follows similar steps as in Theorem 4.1).

Step 2: Boundary Estimate.

Let's suppose $R>0$. Then, we introduce an increasing sequence of stopping times defined as

$$\tau_R=\inf\left\{s\le b:\left\|\left(X^1(s),U^1(s)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}\ge R\right\}.$$

Utilizing a modified version of Theorem 4.1, and considering hypothesis (A7), we derive

$$\begin{aligned}&\mathrm{E}\left(\sup_{u\in[0,s]}\left\|\left(X_R^1(u),U_R^1(u)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\\ \leq&\tilde{\mu}_1(b)+\tilde{\mu}_2(b)+\tilde{\mu}_3(b)\int_0^s N\left(r,\mathrm{E}\left(\sup_{\tau\in[0,r]}\left(\left\|\left(X_R^1(\tau),U_R^1(\tau)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\right)\right)dr.\end{aligned}$$

Now, let's define

$$v(s)=\mathrm{E}\left(\sup_{\tau\in[0,s]}\left\|\left(X_R^1(\tau),U_R^1(\tau)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right).$$

Using the previous inequality, we can conclude

$$v(s)\leq\tilde{\mu}_1(b)+\tilde{\mu}_2(b)+\tilde{\mu}_3(b)\int_0^s N(r;v(r))dr.$$

Let $\mathcal{E}(s), s\in[0,b]$, be a global solution of the equation

$$v(s)=\tilde{\mu}_1(b)+\tilde{\mu}_2(b)+\tilde{\mu}_3(b)\int_0^s N(r;v(r))dr.$$

Consequently, the left-hand side is bounded as well, leading to an application of (A8):

$$v(b)=\mathrm{E}\left(\sup_{s\in[0,b]}\left\|\left(X_R^1(s),U_R^1(s)\right)\right\|_{\mathbb{S}_2\times\mathbb{S}_3}^p\right)\leq\mathcal{E}(b),$$

with the right-hand side remains invariant to variations in R. By analogy with Theorem 4.1, we obtain the estimate (48).

Step 3: Establishing Existence.

In this Step, we introduce Picard-type approximations to (20): Let $\Lambda_\lambda:\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]\to\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]$, is defined as

$$\Lambda_\lambda\left(X_n,U_n\right)(s)=\left(X_{n+1}(s),U_{n+1}(s)\right),$$

with

$$\begin{cases}X_{n+1}(s)=\mathcal{C}(s)X(0)+\mathcal{S}(s)(X'(0)-H(0,X(0),U(0)))\\ \quad+\int_0^s\mathcal{C}(s-r)H(r,X_n(r),U_n(r))\mathrm{d}r,\\ \quad+\int_0^s\mathcal{S}(s-r)\left[F(r,X_n(r),U_n(r))+BU_{n+1}(r)\right]\mathrm{d}r\\ \quad+\int_0^s\mathcal{S}(s-r)G(r,X_n(r),U_n(r))\mathrm{d}W(r),\\ U_{n+1}(s)=-B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\mathcal{C}(b)X(0)\\ \quad-B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}\left(\mathcal{S}(b)(X'(0)-H(0,X(0),U(0))-\mathrm{E}h_b\right)\\ \quad-\int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{C}(b-r)H(r,X_n(r),U_n(r))\mathrm{d}r\\ \quad-\int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)F(r,X_n(r),U_n(r))\mathrm{d}r\\ \quad-\int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}\mathcal{S}(b-r)G(r,X_n(r),U_n(r))\mathrm{d}W(r)\\ \quad+\int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}Z(r)\mathrm{d}W(r).\end{cases}$$

Our objective in proving existence is to demonstrate that the processes $(X_n,U_n)_{n\geq0}$ converge uniformly on the time interval $[0,b]$ to a process, which we will establish as a solution.

Claim 1. Under conditions (A0) – (A1), and (A7) – (A10) the operator

$$\Lambda_\lambda:\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]\to\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b],$$

is defined as

$$\Lambda_\lambda(X,U)(s)=(X(s),U(s))$$

$$
\begin{aligned}
&= \Big(\mathcal{C}(s)X(0) + \mathcal{S}(s)(X(0) - H(0, X(0), U(0))) + \int_0^s \mathcal{S}(s-r)BU(r)\mathrm{d}r \\
&\quad + \int_0^s \mathcal{C}(s-r)H(r, X(r), U(r))\mathrm{d}r, + \int_0^s \mathcal{S}(s-r)F(r, X(r), U(r))\mathrm{d}r \\
&\quad + \int_0^s \mathcal{S}(s-r)G(r, X(r), U(r))\mathrm{d}W(r) \\
&\quad - B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[0,b],\lambda}(b)\left(\mathcal{C}(b)X(0) + \mathcal{S}(b)(X(0) - H(0, X(0), U(0))) - \mathrm{E}h_b\right) \\
&\quad - \int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}(b)\mathcal{C}(b-r)H(r, X(r), U(r))\mathrm{d}r \\
&\quad - \int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}(b)\mathcal{S}(b-r)F(r, X(r), U(r))\mathrm{d}r \\
&\quad - \int_0^s B^*\mathcal{S}^*(b-s)\mathrm{R}^{\mathcal{G}}_{[r,b],\lambda}(b)(\mathcal{S}(b-r)G(r, X(r), U(r) - Z(r))\mathrm{d}W(r))\Big).
\end{aligned}
$$

It is well-defined and continuous for $s \in [0, b]$. By Step 2, we have

$$
\mathrm{E}\left(\sup_{u\in[0,s]} \|\Lambda_\lambda(X(u), U(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \leq \tilde{\mu}_1(b) + \tilde{\mu}_2(b) + \tilde{\mu}_3(b)\int_0^s N(r, \mathcal{E}(\mathrm{b}))dr < \infty.
$$

The continuity of Λ_λ follows easily.

Claim 2. Under the conditions (A0) − (A1), and (A7) − (A8) the sequence $(X_n, U_n)_{n\geq 0}$ is bounded in the space $\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0, b]$.

For $n \geqslant 0$ we have, by the same argument as in Step 1,

$$
\begin{aligned}
\mathrm{E}\left(\sup_{u\in[0,s]} \|\Lambda_\lambda\left(X^n(u), U^n(u)\right)\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) &= \mathrm{E}\left(\sup_{u\in[0,r]} \left\|\left(X^{n+1}(u), U^{n+1}(u)\right)\right\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \\
&\leq \tilde{\mu}_1(b) + \tilde{\mu}_2(b) + \tilde{\mu}_3(b)\int_0^s N\left(r, \mathrm{E}\left(\sup_{u\in[0,r]} \|(X^n(u), U^n(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)dr.
\end{aligned}
$$

Let $\mathcal{E}(s), s \in [0, b]$, be a global solution of the equation

$$
v(s) = \tilde{\mu}_1(b) + \tilde{\mu}_2(b) + \tilde{\mu}_3(b)\int_0^s N(r, v(r))\mathrm{d}r, \quad s \in [0, b]. \tag{49}
$$

with an initial condition $u_0 = \tilde{\mu}_1(b) + \tilde{\mu}_2(b)$. We shall prove by mathematical induction that

$$
\mathrm{E}\left(\sup_{u\in[0,s]} \|(X^n(u), U^n(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \leqslant \mathcal{E}(s) \quad \text{for } s \in [0, b]. \tag{50}
$$

The inequality (50) holds for $n = 0$ due to the definition of u. Let's assume that

$$
\mathrm{E}\left(\sup_{u\in[0,s]} \|(X^n(u), U^n(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \leqslant \mathcal{E}(s) \quad \text{for } s \in [0, b].
$$

Hence, from (49), we deduce that

$$
\begin{aligned}
&\mathcal{E}(s) - \mathrm{E}\left(\sup_{u\in[0,s]} \left\|\left(X^{n+1}(u), U^{n+1}(u)\right)\right\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) \\
&\geqslant \tilde{\mu}_3(b)\int_0^s \left(N(s, v(s)) - N\left(s, \mathrm{E}\left(\sup_{u\in[0,r]} \|(X^n(u), U^n(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)\right)\mathrm{d}r \\
&\geqslant 0.
\end{aligned}
$$

These inequalities stem from our induction assumption and condition (A9).

Claim 3. Under the conditions (A0) − (A1), and (A7) − (A10), the sequence $\{X_n\}_{n\geq 0}$ forms a Cauchy sequence in $\mathcal{S}^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]$, and its limit represents a mild solution for equation (48).

Define

$$\Delta_n(s) = \sup_{m\geq n} \mathrm{E}\left(\sup_{u\in[0,s]} \|(X^m(u),U^n(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right), s\in[0,b], n\geq 0.$$

These functions Δ_n, for $n\geq 0$, are well-defined, uniformly bounded (as per Claim 2), and exhibit monotone nondecreasing behavior. As $\{\Delta_n(s)\}_{n\geq 0}$ is a monotone nonincreasing sequence for each $s\in[0,b]$, there exists a monotone nondecreasing function Δ such that

$$\lim_{n\to\infty} \Delta_n(s) = \Delta(s). \tag{51}$$

Employing an analogous reasoning to that in Claim 2, we deduce that

$$\begin{aligned}&\mathrm{E}\left(\sup_{u\in[0,s]} \|\,(X^m(u),U^m(u)) - (X^n(u),U^n(u))|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\\ \leq&\mu_3(b)\int_0^s K\left(s,\mathrm{E}\left(\sup_{u\in[0,r]} \left\|\left(X^{m-1}(u),U^{m-1}(u)\right) - \left(X^{n-1}(u),U^{n-1}(u)\right)\right\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\right)dr,\end{aligned}$$

implying

$$\Delta(s)\leq\Delta_n(s)\leq\mu_3(b)\int_0^s N\left(r,\Delta_{n-1}(r)\right)dr. \tag{52}$$

Consequently, according to (A10), Δ is identically zero. The case of a nonnegative, monotone nondecreasing function Δ satisfying (52) is discussed in [27, Lemma 2.2]. As such, we have

$$\mathrm{E}\left(\sup_{u\in[0,b]} \|(X^m(u),U^m(u)) - (X^n(u),U^n(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right)\leq\Delta_n(b),$$

and $\Delta_n(b)\to\Delta(b)=0$. Hence

$$\lim_{n,m\to\infty} \mathrm{E}\left(\sup_{u\in[0,b]} \|(X^m(u),U^m(u)) - (X^n(u),U^n(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) = 0.$$

Finally, the continuity of the operator Λ_λ ensures the last part of the lemma. The completeness of $S^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]$ guarantees the existence of a process $(X,U)\in S^p_{\mathbb{S}_2\times\mathbb{S}_3}[0,b]$ such that

$$\lim_{n\to\infty} \mathrm{E}\left(\sup_{u\in[0,b]} \|(X^n(u),U^n(u)) - (X(u),U(u))\|^p_{\mathbb{S}_2\times\mathbb{S}_3}\right) = 0.$$

Taking the limit

$$(X,U) = \lim_{n\to\infty}(X_{n+1},U_{n+1}) = \lim_{n\to\infty}\Lambda_\lambda(X_n,U_n) = \Lambda_\lambda(X,U),$$

we conclude that (X,U) represents a fixed point of Λ_λ. This completes the proof. □

Theorem 5.2. *Under the assumptions* (A0) − (A1), (A7), *and* (A8) − (A10), *the stochastic differential equation* (1) *exhibits approximate controllability.*

Proof. The proof is similar to that of Theorem 4.2. □

6. Illustrative Example

In our next example, we explore a problem inspired by [6, 17], which involves an initial boundary value problem defined as

$$
\begin{cases}
\partial\left(\frac{\partial}{\partial s}x(s,z)-h(s,x(s,z),u(s,z))\right)\partial s=\frac{\partial}{\partial z^2}x(s,z)+u(s,z)\partial s \\
\qquad\qquad +f(s,x(s,z),u(s,z))\partial s \\
\qquad +g(s,x(s,z),u(s,z)))\mathrm{d}w(s),\ \text{a.e. on } (0,b)\times[0,\pi], \\
x(0,z)=\xi(z), \qquad \frac{\partial x(0,z)}{\partial s}=\eta(z), \qquad \text{a.e. on } [0,\pi], \\
x(s,z)=0, \qquad \frac{\partial}{\partial s}x(s,z)=0, \qquad \text{a.e. on } (0,b)\times\partial[0,\pi],
\end{cases}
\tag{53}
$$

where $z\in[0,\pi],\xi(\cdot)$ and $\eta(\cdot)\in L^2\left(\Omega,\mathcal{F}_0,\mathbb{S}_2\right)$, and w is a $Q-$Wiener Process. Let $\mathbb{S}_1=\mathbb{S}_2=\mathbb{S}_3=L_2[0,\pi]$.

To further analyze this problem, we define functions

$$
f,g:\Omega\times(0,b)\times(0,\pi)\times\mathbb{S}_2\times\mathbb{S}_3\to\mathbb{S}_2, \quad h:\Omega\times(0,b)\times(0,\pi)\times\mathbb{S}_2\times\mathbb{S}_3\to\mathcal{L}^2(\mathbb{S}_0,\mathbb{S}_2).
$$

Additionally, we identify this system with an abstract form:

$$
\begin{cases}
\forall(\omega,s,z)\in\Omega\times(0,b)\times(0,\pi): & H(\omega,s,x(s),u(s))(z)=h(\omega,s,z,x(s,z),u(s,z)), \\
\forall(\omega,s,z)\in\Omega\times(0,b)\times(0,\pi): & F(\omega,s,x(s),u(s))(z)=f(\omega,s,z,x(s,z),u(s,z)), \\
\forall(\omega,s,z)\in\Omega\times(0,b)\times(0,\pi): & G(\omega,s,x(s),u(s))(z)=f_3(\omega,s,x(s,z),u(s,z)), \\
\forall(\omega,s,z)\in\Omega\times(0,b)\times(0,\pi): & X(0)(z)=\xi(z), \quad X'(0)(z)=\eta(z).
\end{cases}
$$

With these identifications, observe that (53) can be written in the abstract form:

$$
\begin{cases}
\mathrm{d}(X'(r)-H(r,X(r),U(r)))=[AX(r)+BU(r)+F(r,X(r),U(r))]\mathrm{d}r \\
\qquad\qquad +G(r,X(r),U(r))\mathrm{d}W(r), \\
X(0)=\xi_0, \quad X'(0)=\eta_0, \quad r\in[0,b],
\end{cases}
\tag{54}
$$

where $A:\mathbb{S}_2\to\mathbb{S}_2$ is defined by $Ay=\frac{\partial^2}{\partial^2 z^2}y$, with domain

$$
\mathcal{D}(A)=\left\{y\in\mathbb{S}_2: y \text{ and } \left(\frac{dy}{dz}\right) \text{ are absolutely continuous,}\right.
$$
$$
\left.\text{also } \left(\frac{d^2y}{ds^2}\right)\in\mathbb{S}_2, y(0)=y(\pi)=0\right\}.
$$

This operator A can be expressed as

$$
[Ay](z)=-\sum_{n=1}^{\infty}n^2(y,e_n)e_n(z),
$$

where $e_n(z)=\sqrt{2/\pi}\sin nz, i=1,2,\ldots\ z\in[0,\pi]$ forms an orthogonal set of eigenvalues of A. Given that $(-A)$ is positive and self-adjoint in $\mathbb{S}_2$, we can derive from [17, Example 3.3] that A serves as an infinitesimal generator of a strongly continuous cosine family $\{C(s):s\in[0,b]\}$ in $\mathbb{S}_2$, and its expression is

$$
[\mathcal{C}(s)y](z)=[C^*(s)y](z)=[\cos(A^*y)](z)=\sum_{n=1}^{\infty}\cos(ns)(y,e_n)e_n(z), \quad y\in\mathbb{S}_2,
$$

with the associated sine family $\{\mathcal{S}(s)=\int_0^b C(r)dr:s\in[0,b]\}$ given by

$$
[\mathcal{S}(s)y](z)=[S^*(s)y](z)=[\sin(Ay)](z)=\sum_{n=1}^{\infty}\frac{1}{n}\sin(ns)(y,e_n)e_n(z), \quad y\in\mathbb{S}_2.
$$

The controllability operator $\mathcal{G}_b(\tau, b) \in \mathcal{L}(\mathbb{S}_2, \mathbb{S}_3)$ is expressed as

$$[\mathcal{G}_b(\tau, s)y](z) = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\tau}^{b} \sin^2(n(b-r)))dr(y, e_n)e_n(z) \quad 0 \leq \tau < b.$$

Hence, $\mathcal{G}_b(\tau, s)$ will be greater than zero for some $\tau \in [0, b[$. Moreover, $\mathcal{G}_b$, $\mathcal{C}$, and $\mathcal{S}$ are uniformly bounded. It's not difficult to verify that there exists $\tilde{M}_\lambda(b) = 1$ such that for all $0 \leq r < b$, $0 < ||\lambda \mathrm{R}^{\mathcal{G}}_{[r,b]}||_{\mathcal{L}(\mathbb{S}_3,\mathbb{S}_2)} \leq 1$, and $\lambda \mathrm{R}^{\mathcal{G}}_{[r,b]} \to 0$, converging to zero operator as $\lambda \to 0^+$ in the strong operator topology. This is due to the positivity of $\mathcal{G}_b$, which implies the controllability of the deterministic linear system associated with (53) (see [6, 11]). Thus, with suitable choices of h, f, and g such that all the conditions of Theorem 4.1 are satisfied, the system (53) admits a unique solution. Because $\lambda \mathrm{R}^{\mathcal{G}}_{[r,b]} \to 0^+$, by Theorem 4.2, the system (53) is approximately controllable.

7. Conclusions

This research examined the approximate controllability of second-order stochastic differential equations driven by Q-Wiener processes within a real separable Hilbert space. We identified conditions for the existence and uniqueness of mild solutions that are less stringent than the traditional Lipschitz criterion.

Our main findings include:

(1) The existence and uniqueness of mild solutions for the system under relaxed conditions.
(2) The establishment of approximate controllability using fixed point theory, Picard approximation, and stopping time theory.

These results advance control theory for stochastic systems, particularly those influenced by Q-Wiener processes, while the relaxed conditions enhance the applicability of our findings. The incorporation of stopping time theory offers a solid framework for demonstrating approximate controllability.

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