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# ITERATIONS OF A CUBIC MAP ON A ONE-DIMENSIONAL SIMPLEX

In this paper, we present a comprehensive analysis of the dynamics of a cubic operator defined by a non-stochastic matrix (containing at least one negative entry) on the one-dimensional simplex. This operator depends on two parameters. We identify all fixed points and some periodic points, and under certain conditions on the parameters, we describe the limit points. Furthermore, we determine a region in the parameter space where the corresponding operator exhibits chaotic behavior.

## 1. Introduction

Dynamical systems generated by nonlinear operators play a crucial role in addressing various challenges in biology, physics, and other scientific fields [10], [14]. Among the well-studied classes of nonlinear operators are Quadratic Stochastic Operators (QSOs) (see, for example, [3], [12]), [14]).

In this paper, we examine the dynamics of a two-parametric cubic operator defined by non-stochastic matrices on the one-dimensional simplex. For motivation of such study (see [6], [7], [8], [9], [13], [15], [16]) and for stochastic operators defined by non-stochastic matrices see ([4], [17], [18], [19]).

Let $m$ be a natural number and let $E_m = \{1, 2, \ldots, m\}$ be a finite set. Then the set of all probability distributions on the $E_m$ is

$$S^{m-1} = \left\{ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \ x_i \geq 0, \ \forall i, \ \sum_{i=1}^{m} x_i = 1 \right\},$$

which is called $m-1$-dimensional simplex.

In general, a cubic operator $V : x \in \mathbb{R}^m \to x' = V(x) \in \mathbb{R}^m$ corresponding to a matrix $\mathbb{P} = \{P_{ijk,l}\}_{i,j,k,l=1}^{m}$ is defined by:

$$V : x'_l = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k, \quad l = 1, \ldots, m. \tag{1}$$

In this paper we consider the operator (1) in case $m = 2$.

Without loss of generality we assume $P_{iij,l} = P_{iji,l} = P_{jii,l}$, $i, j, l = 1, 2$, $i \neq j$. Indeed, if this equality is not satisfied then we can introduce

$$\bar{P}_{iij,l} = \frac{1}{3} \left( P_{iij,l} + P_{iji,l} + P_{jii,l} \right), \ i, j, l = 1, 2, \ i \neq j.$$

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As a result, the operator (1), for $m = 2$ takes the form:

$$V : \begin{cases} x_1' = P_{111,1}x_1^3 + 3P_{112,1}x_1^2x_2 + 3P_{122,1}x_1x_2^2 + P_{222,1}x_2^3, \\ x_2' = P_{111,2}x_1^3 + 3P_{112,2}x_1^2x_2 + 3P_{122,2}x_1x_2^2 + P_{222,2}x_2^3. \end{cases} \tag{2}$$

**Definition 1.1.** A cubic operator is called a cubic stochastic operator (CSO) if it preserves simplex.

The following theorem gives conditions for coefficients of operator (2) to preserve the simplex $S^1$.

**Theorem 1.1.** [4] *For a cubic operator $V$ (given by (2)), to preserve the simplex $S^1$ it is **sufficient** that*

i) $\sum_{l=1}^{2} P_{ijk,l} = 1, \quad i,j,k = 1,2;$

ii) $0 \le P_{iii,l} \le 1, \quad i,l = 1,2;$

iii) $-\frac{1}{3}\sqrt[3]{P_{iii,l}^2 P_{jjj,l}} \le P_{iij,l}$, $i \ne j$, $i,j = 1,2$ *and **necessary** that the conditions (i), (ii) and*

iii') j) *if $A = 0$ then* $\left(d - 2\sqrt{ad}\right) \le 3c \le \left(d + 2\left(1 + \sqrt{(1-a)(1-d)}\right)\right)$,
$\left(a - 2\sqrt{ad}\right) \le 3b \le \left(a + 2\left(1 + \sqrt{(1-a)(1-d)}\right)\right)$;

jj) *if $A \ne 0$,*

k) *then $\Delta \le 0$;*

kk) $\Delta > 0$,

1. *then $\alpha_{01}, \alpha_{02} \notin [0,1]$;*
2. $\alpha_{01} \notin [0,1]$ *and* $\alpha_{02} \in [0,1]$,
   *$A > 0$ then $ac \ge b^2$;*
   *$A < 0$ then $bd \ge c^2$;*
3. $\alpha_{01} \in [0,1]$ *and* $\alpha_{02} \notin [0,1]$,
   *$A > 0$ then $(1-b)(1-d) \ge (1-c)^2$;*
   *$A < 0$ then $(1-a)(1-c) \ge (1-b)^2$;*
4. $\alpha_{01}, \alpha_{02} \in [0,1]$,
   *$A > 0$ then* $\begin{cases} (1-b)(1-d) \ge (1-c)^2 \\ ac \ge b^2 \end{cases}$;
   *$A < 0$ then* $\begin{cases} (1-a)(1-c) \ge (1-b)^2 \\ bd \ge c^2 \end{cases}$;

*are satisfied, where $P_{111,l} = a$, $P_{112,l} = b$, $P_{122,l} = c$, $P_{222,l} = d$, $A = a - 3b + 3c - d$, $B = b - 2c + d$, $C = c - d$, $\Delta = B^2 - AC$, $\alpha_{01} = \frac{-B-\sqrt{\Delta}}{A}$ and $\alpha_{02} = \frac{-B+\sqrt{\Delta}}{A}$.*

In this paper, we consider cubic operators, with at least one negative coefficient, which preserve the simplex $S^1$ and for each of them we comprehensively analyze corresponding dynamical system.

## 2. Dynamics of a cubic operator corresponding to a non-stochastic matrix

We consider the following CSO on the one-dimensional simplex $S^1$:

$$V = \begin{cases} x' = x^3 + (3-a)x^2y + (3-b)xy^2 + y^3, \\ y' = ax^2y + bxy^2, \end{cases} \tag{3}$$

where $a, b \in [0,4]$, $a \ne b$.

Note that in case $a > 3$ or $b > 3$ the matrix corresponding to this operator is non-stochastic.

*Remark* 2.1. In [5] the cases $a = b$ and $b = 0$ of this operator were studied.

In order for the operator (3) to preserve the simplex $S^1$, we check that its coefficients satisfy the sufficiency conditions of Theorem 1.1:

The conditions i) and ii) can be easily verified by view of the operator.

For $a, b \in [0, 4]$, we have $0 \le \frac{a}{3} \le \frac{4}{3}$, $0 \le \frac{b}{3} \le \frac{4}{3}$ and $-\frac{1}{3} \le \frac{3-a}{3} \le 1$, $-\frac{1}{3} \le \frac{3-b}{3} \le 1$ i.e. $P_{112,1} \ge -\frac{1}{3}$, $P_{122,1} \ge -\frac{1}{3}$ and $P_{112,2} \ge 0$, $P_{122,2} \ge 0$. So, the coefficients of the operator (3) satisfy condition iii).

2.1. **Fixed points.** Let $W$ be an operator defined on simplex $S^{m-1}$. A point $\mathbf{x} \in S^{m-1}$ is called a *periodic* point of $W$ if there exists an $n$ so that $W^n(\mathbf{x}) = \mathbf{x}$. The smallest positive integer $n$ satisfying the above is called the prime period or least period of the point $\mathbf{x}$. A period-one point is called a *fixed* point of $W$. Denote by $\mathrm{Fix}\,(W)$ the set of all fixed points of the operator $W$, i.e.

$$\mathrm{Fix}\,(W) = \left\{\mathbf{x} \in S^2 : W(\mathbf{x}) = \mathbf{x}\right\}.$$

Let $DW(\mathbf{x}^*) = (\partial W_i/\partial x_j)(\mathbf{x}^*)$ be a Jacobian of $W$ at the point $\mathbf{x}^*$.

**Definition 2.1.** [2] A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $DW(\mathbf{x}^*)$ has no eigenvalues on the unit circle in $\mathbb{C}$.

**Definition 2.2.** [2] A hyperbolic fixed point $\mathbf{x}^*$ is called:

i) attracting if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are in the unit disk;
ii) repelling if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are outside the closed unit disk;
iii) a saddle otherwise.

**Theorem 2.1.** *The set of fixed points of the operator* (3) *is*

$$Fix\,(V) = \begin{cases} \{e_1, \theta\} & if\ \ (a,b) \in \Omega_0, \\ \{e_1\} & if\ \ (a,b) \in \Omega_1, \\ \{e_1, \xi\} & if\ \ (a,b) \in \Omega_2, \\ \{e_1, \xi, \eta\} & if\ \ (a,b) \in \Omega_3, \end{cases}$$

*where* $e_1 = (1,0)$, $\theta = \left(\frac{2}{b}, \frac{b-2}{b}\right)$, $\xi = \left(\frac{-b+\sqrt{b^2+4(a-b)}}{2(a-b)}, \frac{2a-b-\sqrt{b^2+4(a-b)}}{2(a-b)}\right)$,

$\eta = \left(\frac{b+\sqrt{b^2+4(a-b)}}{2(b-a)}, \frac{b-2a-\sqrt{b^2+4(a-b)}}{2(b-a)}\right)$, $\Omega_0 = \{(a,b) : a = b_0,\ 2 < b \le 4\}$,

$\Omega_1 = \{(a,b) : 0 \le a \le 1,\ 0 \le b \le 2\} \cup \{(a,b) : 0 \le a < b_0,\ 2 < b \le 4\}$,

$\Omega_2 = \{(a,b) : 1 < a \le 4,\ 0 \le b \le 4\} \cup \{(a,b) : a = 1,\ 2 < b \le 4\}$,

$\Omega_3 = \{(a,b) : b_0 < a < 1,\ 2 < b \le 4\}$, $b_0 = \frac{4b-b^2}{4}$.

*Proof.* The set of fixed points of the operator (3) consists of the solutions of the system

$$\begin{cases} x^3 + (3-a)x^2y + (3-b)xy^2 + y^3 = x, \\ ax^2y + bxy^2 = y. \end{cases} \tag{4}$$

From the second equation of (4) we obtain either $y = 0$ (i.e. $e_1 = (1,0)$ is a fixed point) or

$$(a-b)\,x^2 + bx - 1 = 0. \tag{5}$$

The discriminant of the quadratic equation (5) is $D = b^2 + 4\,(a-b)$, and

if $D < 0$, then equation (5) has no real solutions;

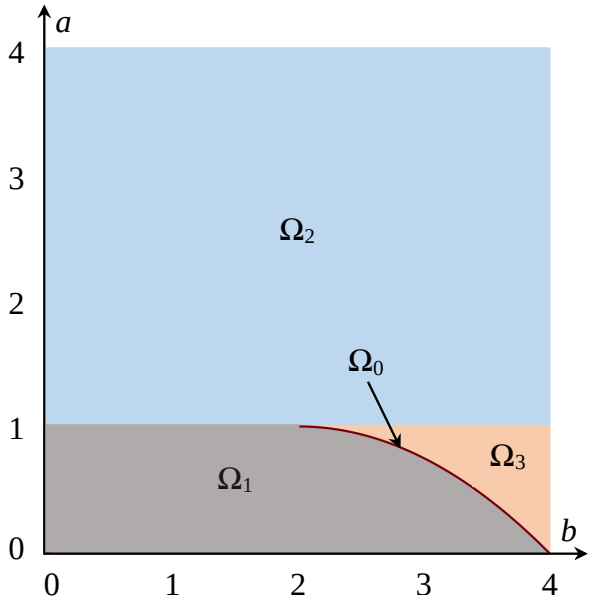


FIGURE 1. Appearance of a parametric square.

if $D = 0$, then equation (5) has a unique solution $x_0 = \frac{2}{b}$, $b \neq 0$;
if $D > 0$, then equation (5) has two real solutions:

$$x_1 = \frac{-b + \sqrt{b^2 + 4(a-b)}}{2(a-b)} \qquad \text{and} \qquad x_2 = \frac{b + \sqrt{b^2 + 4(a-b)}}{2(b-a)}.$$

Let $(a, b) \in \Omega_0$. Then, $D = 0$, which means that equation (5) has the unique solution. Furthermore,

$$2 < b \leq 4 \Rightarrow \frac{1}{2} \leq \frac{2}{b} < 1 \Rightarrow x_0 \in (0, 1).$$

Therefore, if $(a, b) \in \Omega_0$, then $Fix(V) = \{e_1, \theta\}$.

Now, let's consider the case where $(a, b) \in \Omega_1$. If $0 \leq a < \frac{4b-b^2}{4}$ and $0 \leq b \leq 4$, then $D < 0$, meaning equation (5) has no real solutions.

If $a = \frac{4b-b^2}{4}$ and $0 \leq b \leq 2$, then $D = 0$, meaning equation (5) has the unique solution. However,

$$0 \leq b < 2 \;\Rightarrow\; \frac{2}{b} > 1 \;\Rightarrow\; x_0 > 1 \quad \text{and} \quad b = 2 \Rightarrow x_0 = 1.$$

If $\frac{4b-b^2}{4} < a \leq 1$ and $0 \leq b \leq 2$, then $D > 0$, i.e. equation (5) has two real solutions.
If $a > b$, then

$$a < 1 \Rightarrow 4(a-b)(1-a) > 0 \Rightarrow 4(a-b) > 4a(a-b) \Rightarrow b^2 + 4(a-b) > (2a-b)^2 \Rightarrow$$
$$\sqrt{D} > 2a - b \Rightarrow -b + \sqrt{D} > 2(a-b) \Rightarrow\; x_1 > 1 \Rightarrow x_1 \notin [0, 1],$$
$$a = 1 \Rightarrow x_1 = 1,$$
$$b + \sqrt{b^2 + 4(a-b)} > 0 \;\Rightarrow\; x_2 < 0 \;\Rightarrow\; x_2 \notin [0, 1].$$

If $a < b$, then

$$a < 1 \Rightarrow 4(a-b)(1-a) < 0 \Rightarrow 4(a-b) < 4a(a-b) \Rightarrow b^2 + 4(a-b) < (2a-b)^2 \Rightarrow$$
$$\sqrt{D} < 2a - b \Rightarrow -b + \sqrt{D} < 2(a-b) \Rightarrow x_1 > 1 \Rightarrow x_1 \notin [0, 1],$$
$$0 \leq b \leq 2 \;\Rightarrow\; \frac{4b-b^2}{4} \geq \frac{b}{2} \;\Rightarrow\; a > \frac{b}{2} \;\Rightarrow\; b - 2a < 0 \;\Rightarrow$$

$$\Rightarrow \sqrt{D} > b-2a \;\Rightarrow\; b+\sqrt{D} > 2(b-a) \;\Rightarrow\; \frac{b+\sqrt{D}}{2(b-a)} > 1 \;\Rightarrow\; x_2 \notin [0,1].$$

So, if $(a,b) \in \Omega_1$, then $Fix(V) = \{e_1\}$.
Now, assume $(a,b) \in \Omega_2$. In this case, $D > 0$, so (5) has two real solutions.
If $a > b$, then $-b+\sqrt{b^2+4(a-b)} > 0 \Rightarrow x_1 > 0$ and

$$a > 1 \Rightarrow 4(a-b)(1-a) < 0 \Rightarrow 4(a-b) < 4a(a-b) \Rightarrow b^2+4(a-b) < (2a-b)^2 \Rightarrow$$

$$\sqrt{D} < 2a-b \Rightarrow -b+\sqrt{D} < 2(a-b) \Rightarrow x_1 < 1,$$

$$b+\sqrt{b^2+4(a-b)} > 0 \Rightarrow x_2 < 0.$$

If $a < b$, then $-b+\sqrt{b^2+4(a-b)} < 0 \Rightarrow x_1 > 0$.
If $a > b/2 \;\Rightarrow\; 2a-b > 0$, then

$$a > 1 \Rightarrow 4(a-b)(1-a) > 0 \Rightarrow 4(a-b) > 4a(a-b) \Rightarrow b^2+4(a-b) > (2a-b)^2 \Rightarrow$$

$$\sqrt{D} > 2a-b \Rightarrow -b+\sqrt{D} > 2(a-b) \Rightarrow x_1 < 1,$$

$$\sqrt{D} > b-2a \;\Rightarrow\; b+\sqrt{D} > 2(b-a) \;\Rightarrow\; \frac{b+\sqrt{D}}{2(b-a)} > 1 \;\Rightarrow\; x_2 > 1.$$

If $a \le b/2 \;\Rightarrow\; 2a-b \le 0$, then

$$\sqrt{D} > 2a-b \Rightarrow -b+\sqrt{D} > 2(a-b) \Rightarrow \frac{-b+\sqrt{D}}{2(a-b)} > 1 \Rightarrow x_1 < 1,$$

$$a > 1 \Rightarrow 4(a-b)(1-a) > 0 \Rightarrow 4(a-b) > 4a(a-b) \Rightarrow b^2+4(a-b) > (b-2a)^2 \Rightarrow$$

$$\sqrt{D} > b-2a \Rightarrow b+\sqrt{D} > 2(b-a) \Rightarrow x_2 > 1,$$

$$a = 1 \Rightarrow x_2 = 1.$$

Therefore, if $(a,b) \in \Omega_2$, then $Fix(V) = \{e_1, \xi\}$.

Finally, let $(a,b) \in \Omega_3$. In this case, $D > 0$, meaning equation (5) has two real solutions. In this situation $a < b/2 \;\Rightarrow\; 2a-b < 0$, which gives

$$-b+\sqrt{b^2+4(a-b)} < 0 \Rightarrow x_1 > 0,$$

$$\sqrt{D} > 2a-b \Rightarrow\; -b+\sqrt{D} > 2(a-b) \Rightarrow \frac{-b+\sqrt{D}}{2(a-b)} < 1 \Rightarrow x_1 < 1,$$

$$-b-\sqrt{b^2+4(a-b)} < 0 \Rightarrow x_2 > 0,$$

$$a < 1 \Rightarrow 4(a-b)(1-a) < 0 \Rightarrow 4(a-b) < 4a(a-b) \Rightarrow b^2+4(a-b) < (b-2a)^2 \Rightarrow$$

$$\sqrt{D} < b-2a \Rightarrow b+\sqrt{D} < 2(b-a) \Rightarrow x_2 < 1.$$

Therefore, if $(a,b) \in \Omega_3$, then $Fix(V) = \{e_1, \xi, \eta\}$.
The theorem is proved. □

2.2. **Characterization of Fixed Points.** We note that the second coordinate of the operator (3) can be expressed as $y' = f(y)$, where

$$f(y) = (a-b)\,y^3 + (b-2a)\,y^2 + ay \tag{6}$$

The derivative of the function (6) is given by $f'(y) = 3(a-b)y^2 + 2(b-2a)y + a$.

The fixed point $y = 0$ is classified as follows:

$$y = 0 \text{ is an/a } \begin{cases} \text{attracting}, & \text{if } a \in [0,1), \\ \text{non-hyperbolic}, & \text{if } a = 1, \\ \text{repelling}, & \text{if } a \in (1,4]. \end{cases}$$

For the fixed point $y_0 = \frac{b-2}{b}$, we observe that $|f'(y_0)| = 1$. Therefore, $y_0$ is a non-hyperbolic fixed point.

For the fixed point $y_1$, we have

$$|f'(y_1)| = \left| \frac{1}{2(a-b)} \left( b^2 + 6(a-b) + (b-2a)\sqrt{b^2 + 4(a-b)} \right) \right|. \tag{7}$$

**Case 1.** First, let's consider the case $|f'(y_1)| = 1$.

**Case 1.1.** According to (7), from the equation $f'(y_1) = 1$ and using the fact that

$$b^2 + 4(a-b) = D > 0,$$

we obtain the expression

$$\sqrt{b^2 + 4(a-b)} = 2a - b.$$

The last equation holds when

$$2a - b \geq 0 \Rightarrow a \geq \frac{b}{2},$$

and leads to the equation

$$(a-b)(1-a) = 0 \Rightarrow a = b,\ a = 1.$$

According to the definition of the operator, since $a \neq b$ and $a = 1$, $0 \leq b \leq 2$, we discard these solutions because $y_1 = 0$ in this case.

**Case 1.2.** Consider the equation $f'(y_1) = -1$. According to (7), we rewrite it as

$$b^2 + 8(a-b) = (2a-b)\sqrt{b^2 + 4(a-b)}.$$

This equation has the solutions

$$a = \varphi(b) = \frac{1}{8}\left(16 + 4b - b^2 + \sqrt{b^4 - 8b^3 + 32b^2 - 128b + 256}\right),$$

$$a = \psi(b) = \frac{1}{8}\left(16 + 4b - b^2 - \sqrt{b^4 - 8b^3 + 32b^2 - 128b + 256}\right).$$

However, since

$$\frac{8b - b^2}{8} - \frac{b}{2} = \frac{4b - b^2}{8} > 0 \Rightarrow \frac{8b - b^2}{8} > \frac{b}{2},$$

we obtain solutions satisfying the conditions

$$\left[ \begin{cases} b^2 + 8(a-b) \geq 0 \\ 2a - b \geq 0 \\ b^2 + 8(a-b) \leq 0 \\ 2a - b \leq 0 \end{cases} \Rightarrow \left[ \begin{array}{l} a \geq \frac{8b-b^2}{8} \\ a \leq \frac{b}{2} \end{array} \right. \right.$$

Among these, we only take the solution

$$a = \varphi(b) > \frac{8b - b^2}{8},$$

and discard the solutions $a = b$ and $\frac{b}{2} < \psi(b) < \frac{8b-b^2}{8}$.

Thus, when

$$(a,b) \in \Psi_{=1} = \{(a,b) : a = \varphi(b),\ 0 \le b \le 4\},$$

we have $|f'(y_1)| = 1$, i.e., $y_1$ is a non-hyperbolic fixed point.

**Case 2.** Now we consider the case

$$|f'(y_1)| < 1 \Rightarrow \begin{cases} f'(y_1) < 1 \\ f'(y_1) > -1 \end{cases}$$

**Case 2.1.** Let $a > b$. First, we find solutions of the inequality $f'(y_1) < 1$. According to (7), we get

$$\frac{1}{2(a-b)}\left(b^2 + 6(a-b) + (b-2a)\sqrt{b^2+4(a-b)}\right) < 1$$

$$\sqrt{b^2+4(a-b)} < 2a - b,$$

whose solution set is

$$\{(a,b) : a > b,\ 1 < a \le 4,\ 0 \le b \le 4\}.$$

Next, we find the solutions of $f'(y_1) > -1$. From (7):

$$b^2 + 8(a-b) > (2a-b)\sqrt{b^2+4(a-b)}.$$

Since $a > b$, then $b^2 + 8(a-b) > 0$ and $2a - b > 0$, we get

$$(b^2+8(a-b))^2 > (2a-b)^2(b^2+4(a-b)),$$

$$-4(a-b)(4a^2 + (b^2-4b-16)a - 3b^2 + 16b) > 0.$$

Since $a > b$, we have

$$4a^2 + (b^2-4b-16)a - 3b^2 + 16b < 0.$$

The roots of this inequality were found in Case 1.2 as $a = \varphi(b)$ and $a = \psi(b)$. Therefore, considering that $a > b$, the solution set is:

$$\{(a,b) : 1 < a < \varphi(b),\ 0 \le b \le 1\} \cup \{(a,b) : b < a < \varphi(b),\ 1 \le b \le 3\}.$$

Thus, when $a > b$, the solution to the inequality $|f'(y_1)| < 1$ is the following set of parameter pairs.

**Case 2.2.** Let $a < b$. Repeating the same calculations as in Case 2.1, the solution set of $|f'(y_1)| < 1$ becomes:

$$\{(a,b) : 1 < a < b,\ 1 < b \le 2\} \cup \{(a,b) : b_0 < a < b,\ 2 < b \le 3\} \cup \\ \cup \{(a,b) : b_0 < a < \varphi(b),\ 2 < b \le 3\}.$$

Finally, the solution to the inequality $|f'(y_1)| < 1$ is:

$$\Psi_{<1} = \{(a,b) : 1 < a < \varphi(b),\ 1 < b \le 2\} \cup \{(a,b) : b_0 < a < \varphi(b),\ 2 < b \le 4\}.$$

The remaining region is:

$$\Psi_{>1} = (\Omega_2 \cup \Omega_3) \setminus (\Psi_{=1} \cup \Psi_{<1}) = \{(a,b) : \varphi(b) < a \le 4,\ 0 \le b \le 4\},$$

where $|f'(y_1)| > 1$.

For the fixed point $y_2$, we have

$$|f'(y_2)| = \left|\frac{1}{2(a-b)}\left(b^2 + 6(a-b) + (2a-b)\sqrt{b^2+4(a-b)}\right)\right|.$$

We examine $|f'(y_2)| > 1$. Since $a < \frac{b}{2}$ in $\Omega_3$, then:

$$\frac{1}{2(a-b)}\left(b^2 + 6(a-b) + (2a-b)\sqrt{b^2+4(a-b)}\right) > 1$$

$$b^2 + 6(a-b) + (2a-b)\sqrt{b^2+4(a-b)} < 2(a-b)$$

$$\sqrt{b^2+4(a-b)} < b-2a$$
$$(b-a)(1-a) > 0.$$

This final inequality holds for all $(a,b) \in \Omega_3$, hence in $\Omega_3$, $f'(y_2) > 1$.

Summarizing we can write the following proposition

**Proposition 2.1.** *The type of fixed points is characterized as follows:*

1) *The fixed point $e_1$ is attracting if $a \in [0,1)$; non-hyperbolic for $a = 1$; repelling if $a \in (1,4]$;*
2) *The fixed point $\theta$ is non-hyperbolic;*
3) *The fixed point $\xi$ is attracting if $(a,b) \in \Psi_{<1}$; non-hyperbolic for $(a,b) \in \Psi_{=1}$; repelling if $(a,b) \in \Psi_{>1}$;*
4) *The fixed point $\eta$ is repelling.*

**Proposition 2.2.** *The function $f(y)$ is strictly increasing in $(0, y_{01})$ and strictly decreasing in $(y_{01}, 1)$.*

*Proof.* It's evident that

$$f'(y) = 3(a-b)y^2 + 2(b-2a)y + a.$$

From this, we can find the critical points:

$$y_{01} = \frac{2a-b-\sqrt{a^2+b^2-ab}}{3(a-b)} \qquad \text{and} \qquad y_{02} = \frac{2a-b+\sqrt{a^2+b^2-ab}}{3(a-b)}.$$

**Case 1.** Suppose $a > b$, then

$$3a(a-b) > 0 \Rightarrow (2a-b)^2 > a^2+b^2-ab \Rightarrow 2a-b > \sqrt{a^2+b^2-ab} \Rightarrow y_{01} > 0$$

**Case 1.1.** If $a \geq 2b$, then

$$2b-a < \sqrt{a^2+b^2-ab} \Rightarrow 2a-b-\sqrt{a^2+b^2-ab} < 3(a-b) \Rightarrow y_{01} < 1,$$

$$3b(a-b) > 0 \Rightarrow \sqrt{a^2+b^2-ab} > a-2b \Rightarrow 2a-b+\sqrt{a^2+b^2-ab} > 3(a-b) \Rightarrow y_{02} > 1.$$

**Case 1.2.** If $2b > a > b$, then

$$3b(b-a) < 0 \Rightarrow (2b-a)^2 < a^2+b^2-ab \Rightarrow 2b-a < \sqrt{a^2+b^2-ab} \Rightarrow y_{01} < 1,$$

$$\sqrt{a^2+b^2-ab} > a-2b \Rightarrow 2a-b+\sqrt{a^2+b^2-ab} > 3(a-b) \Rightarrow y_{02} > 1.$$

**Case 2.** Now, let's assume $a < b$, then

$$3b(b-a) > 0 \Rightarrow 2b-a > \sqrt{a^2+b^2-ab} \Rightarrow 2a-b-\sqrt{a^2+b^2-ab} > 3(a-b) \Rightarrow y_{01} < 1.$$

**Case 2.1.** If $b/2 \leq a < b$, then

$$3a(a-b) < 0 \Rightarrow (2a-b)^2 < a^2+b^2-ab \Rightarrow 2a-b < \sqrt{a^2+b^2-ab} \Rightarrow y_{01} > 0,$$

$$2a-b+\sqrt{a^2+b^2-ab} > 0 \Rightarrow y_{02} < 0.$$

**Case 2.2.** If $a < b/2$, then

$$2a-b < 0 \Rightarrow 2a-b < \sqrt{a^2+b^2-ab} \Rightarrow y_{01} > 0,$$

$$3a(b-a) > 0 \Rightarrow a^2+b^2-ab > (b-2a)^2 \Rightarrow 2a-b+\sqrt{a^2+b^2-ab} > 0 \Rightarrow y_{02} < 0.$$

Thus, $y_{01} \in (0,1)$ and $y_{02} \notin (0,1)$. We can write the derivative $f'(y)$ in the form

$$f'(y) = 3(a-b)(y-y_{01})(y-y_{02})$$

Based on this, it's easy to verify that $f'(y) > 0$ for $y \in (0, y_{01})$ and $f'(y) < 0$ for $(y_{01}, 1)$. According to the theorem on the monotonicity of a function [5], the function $f(y)$ is strictly increasing in $(0, y_{01})$ and strictly decreasing in $(y_{01}, 1)$. □

2.3. **Dynamics for** $(a,b)\in\Omega_0$**.** In this case, the operator (3) can be expressed in the form

$$V=\begin{cases} x'=x^3+\left(3-b+\frac{b^2}{4}\right)x^2y+(3-b)xy^2+y^3,\\ y'=\frac{4b-b^2}{4}x^2y+bxy^2, \end{cases} \tag{8}$$

and the function $f(y)$ can be expressed as

$$f(y)=-\frac{b^2}{4}y^3+\frac{b^2-2b}{2}y^2+\frac{4b-b^2}{4}y \tag{9}$$

where $b\in(2,4]$.

First, we study the dynamical system generated by the function $f(y)$. Given that $f(0)=f(1)=0$, we consider below only for $y\in(0,1)$ (see Figure 2).

Let $y\in(0,y_0)$. By Proposition 2.6, we have $0<f(y)<y_0$ and

$$f(y)-y=-\frac{b^2}{4}y(y-y_0)^2<0 \implies f(y)<y$$

it follows $f^{n+1}(y)<f^n(y)$, that is the sequence $\{f^n(y)\}$ is decreasing and bounded from below. Therefore, the limit of the sequence $\{f^n(y)\}$ exists and from $\inf\limits_{y\in(0,y_0)}\{f^n(y)\}=0$, we conclude that $\lim\limits_{n\to\infty}f^n(y)=0$.

We denote by $\hat{y}_0$ the unique point satisfying the equation $f(\hat{y}_0)=y_0$, where $\hat{y}_0=\frac{b-2+\sqrt{b^2-4b+20}}{2b}$. If $y\in(\hat{y}_0,1)$, then $f(y)\in(0,y_0)$, and we have $\lim\limits_{n\to\infty}f^{n+1}(y)=\lim\limits_{n\to\infty}f^n(f(y))=0$.

Now, suppose $y\in[y_{01},\hat{y}_0]$. The proof of this case is similar to the proof of as above case and $\lim\limits_{n\to\infty}f^n(y)=y_0$ for all $y\in[y_{01},\hat{y}_0]$.

Finally, for all $y\in[0,1]$:

$$\lim_{n\to\infty}f^n(y)=\begin{cases} 0, & \text{if} \quad y\in[0,y_0)\cup(\hat{y}_0,1]\ ,\\ y_0, & \text{if} \quad y\in[y_0,\hat{y}_0]\ . \end{cases}$$

Thus, noting that $\lim\limits_{n\to\infty}x^{(n)}=1-\lim\limits_{n\to\infty}y^{(n)}$, we have proved the following:

**Theorem 2.2.** *If $(a,b)\in\Omega_0$, then for any initial point $\lambda=(x,y)\in S^1$, we have*

$$\lim_{n\to\infty}V^n(\lambda)=\begin{cases} e_1, & \text{if} \quad y\in[0,y_0)\cup(\hat{y}_0,1]\,,\\ \theta, & \text{if} \quad y\in[y_0,\hat{y}_0]\,. \end{cases}$$

2.4. **Dynamics for** $(a,b)\in\Omega_1$**.** (see Figure 2)

**Theorem 2.3.** *If $(a,b)\in\Omega_1$, then for any initial point $\lambda=(x,y)\in S^1$, we have*

$$\lim_{n\to\infty}V^n(\lambda)=e_1.$$

*Proof.* The proof of this Theorem is similar to the proof of above. □

2.5. **Dynamics for** $(a,b)\in\Omega_2$**.** In this case, as the values of $a$ and $b$ increase, the number of periodic orbits of the operator also increases.

**Lemma 2.1.** $\Pi=\left\{(a,b)\in\Omega_2:\exists x: f^3(x)=x\right\}\neq\emptyset$.

*Proof.* Let us take $a=4$ and $b=3.8$. In this case, the function (6) can be expressed as $f(y)=0.2y^3-4.2y^2+4y$. The equation $f^3(y)=y$ is a 27th-degree algebraic equation, and its solutions in the interval $[0,1]$ can be easily checked through the graph drawn using the Mappe software (see Figure 3). Thus, $\Pi\neq\emptyset$. The lemma is proved. □

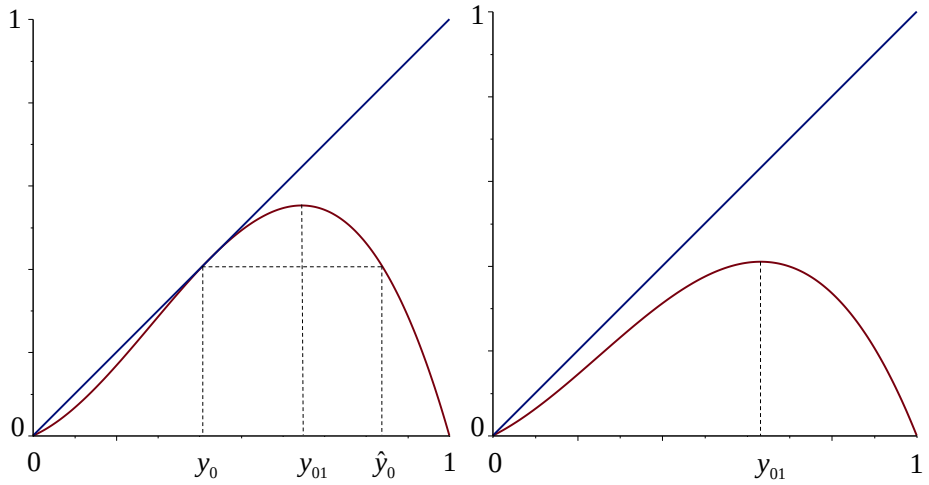


FIGURE 2.

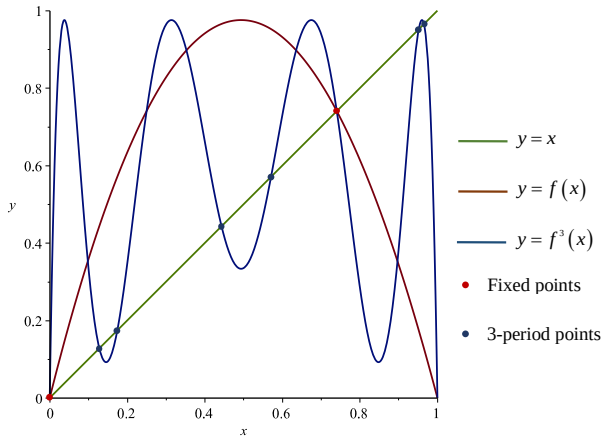


FIGURE 3.

**Theorem 2.4.** *Let $(a, b) \in \Pi$. Then the operator (3) has periodic orbits of all other periods, and the dynamical system exhibits chaos.*

*Proof.* Assume $(a, b) \in \Pi$. By the lemma, the function (6) has a periodic point of period three. By Sharkovskii's theorem, it therefore has periodic points of all other periods. Since $\lim_{n\to\infty} x^{(n)} = 1 - \lim_{n\to\infty} y^{(n)}$, it follows that the operator (3) also has periodic orbits of all other periods. The presence of chaos follows from the result in [11]. This completes the proof. □

*Remark* 2.2. For an arbitrary $(a, b) \in \Omega_2 \setminus \Pi$, determining in general whether periodic points exist or not for given parameter values requires extensive computations. Therefore, we present the following table showing the existence of periodic points for certain specific parameter values. (See the graph in Fig. 4.)

| $(a,b)$ | $Fix(f)$ | $Per_2(f)$ | $Per_3(f)$ | $Per_4(f)$ |
|---|---|---|---|---|
| $(2,4)$ | $0,\ \sqrt{2}/2$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $(1.5,4)$ | $0,\ \frac{\sqrt{10}-1}{3}$ | $\approx 0.5448,\ \approx 0.8226$ | $\emptyset$ | $\emptyset$ |
| $(3,4)$ | $0,\ \sqrt{3}-1$ | $\approx 0.4626,\ \approx 0.8608$ | $\emptyset$ | $\approx 0.4128,\ \approx 0.5469,\ \approx 0.8272,\ \approx 0.8789$ |

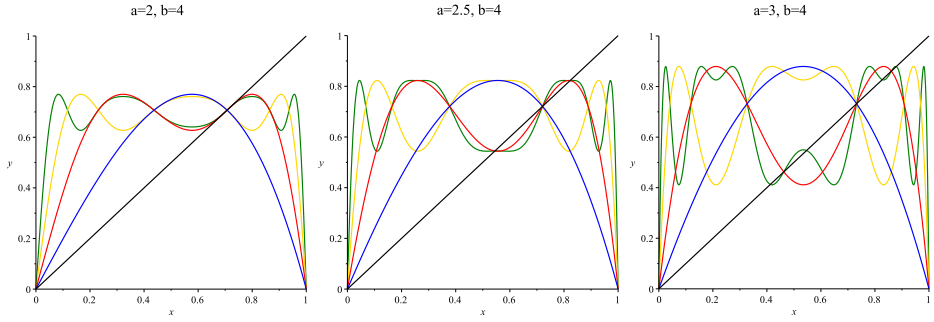


FIGURE 4. Graphs of the functions black - $y=x$, blue - $y=f(x)$, red - $y=f^2(x)$, yellow - $y=f^3(x)$, green - $y=f^4(x)$.

2.6. **Dynamics for $(a,b)\in\Omega_3$.** In this case $a\in(0,1)$, $b\in\left(2\left(1+\sqrt{1-a}\right),4\right]$. According to Theorem 2.4, the set of fixed points is $Fix(V)=\{e_1,\ \xi,\eta\}$.

**Theorem 2.5.** *If $(a,b)\in\Omega_3$, then for any initial point $\lambda=(x,y)\in S^1$, we have*

$$\lim_{n\to\infty}V^n(\lambda)=\begin{cases} e_1,\ if\ \ y\in[0,y_2)\cup(\hat{y}_2,1]\,,\\ \eta,\ \ if\ \ y\in\{y_2,\hat{y}_2\},\\ \xi,\ \ if\ \ y\in(y_2,\hat{y}_2).\end{cases}$$

*Proof.* First, we study the dynamical system generated by the function $f(y)$. Given that $f(0)=f(1)=0$, we consider below only for $y\in(0,1)$.

We denote by $\hat{y}_1\in(0,1)$ the unique point satisfying the equation $f(\hat{y}_1)=y_1$, and by $\hat{y}_2\in(0,1)$ the unique point satisfying the equation $f(\hat{y}_2)=y_2$.

Let $b\in\left(2\left(1+\sqrt{1-a}\right),\min\left\{4,\frac{4a-9-(3-2a)\sqrt{9-4a}}{2(a-2)}\right\}\right]$ (See Fig 5.1). In this case, $0<y_2<y_1\le y_{01}\le\hat{y}_1<\hat{y}_2<1$. Specifically $y_1=y_{01}$ in $b=\frac{4a-9-(3-2a)\sqrt{9-4a}}{2(a-2)}$.

First, suppose $y\in(0,y_2)$. By Proposition 2.6, we have $0<f(y)<y_2$ and

$$f(y)-y=(\alpha-4)\,y\,(y-y_1)\,(y-y_2)<0\quad\Longrightarrow\quad f(y)<y$$

it follows $f^{n+1}(y)<f^n(y)$, that is the sequence $\{f^n(y)\}$ is decreasing and bounded from below. Therefore, the limit of the sequence $\{f^n(y)\}$ exists and from $\inf\limits_{y\in(0,y_0)}\{f^n(y)\}=0$, we conclude that $\lim\limits_{n\to\infty}f^n(y)=0$. If $y\in(\hat{y}_2,1)$, then $f(y)\in(0,y_2)$, and we have

$$\lim_{n\to\infty}f^{n+1}(y)=\lim_{n\to\infty}f^n(f(y))=0.$$

Now, let's assume $y\in(y_2,y_1)$. By Proposition 2.6, we have $y_2<f(y)<y_1$ and

$$f(y)-y=(\alpha-4)\,y\,(y-y_1)\,(y-y_2)>0\quad\Longrightarrow\quad f(y)>y$$

it follows $f^{n+1}(y)>f^n(y)$, that is the sequence $\{f^n(y)\}$ is increasing and bounded from above. Therefore, the limit of the sequence $\{f^n(y)\}$ exists and from $\sup\limits_{y\in(y_2,y_1)}\{f^n(y)\}=$

$y_1$, we conclude that $\lim_{n\to\infty} f^n(y) = y_1$. If $y \in (\hat{y}_1, \hat{y}_2)$, then $f(y) \in (y_2, y_1)$, and we have $\lim_{n\to\infty} f^{n+1}(y) = \lim_{n\to\infty} f^n(f(y)) = y_1$.

Next, let's assume $y \in (y_1, \hat{y}_1)$. The proof of this case is the same as the case $y \in (0, y_2)$ which was considered above and $\lim_{n\to\infty} f^n(y) = y_1$.

So

$$\lim_{n\to\infty} f^n(y) = \begin{cases} 0, & if \ \ y \in [0, y_2) \cup (\hat{y}_2, 1] \ , \\ y_2, & if \ \ y \in \{y_2, \hat{y}_2\} \, , \\ y_1, & if \ \ y \in (y_2, \hat{y}_2) \, , \end{cases}$$

for $b \in \left(2\left(1+\sqrt{1-a}\right), \min\left\{4, \frac{4a-9-(3-2a)\sqrt{9-4a}}{2(a-2)}\right\}\right]$.

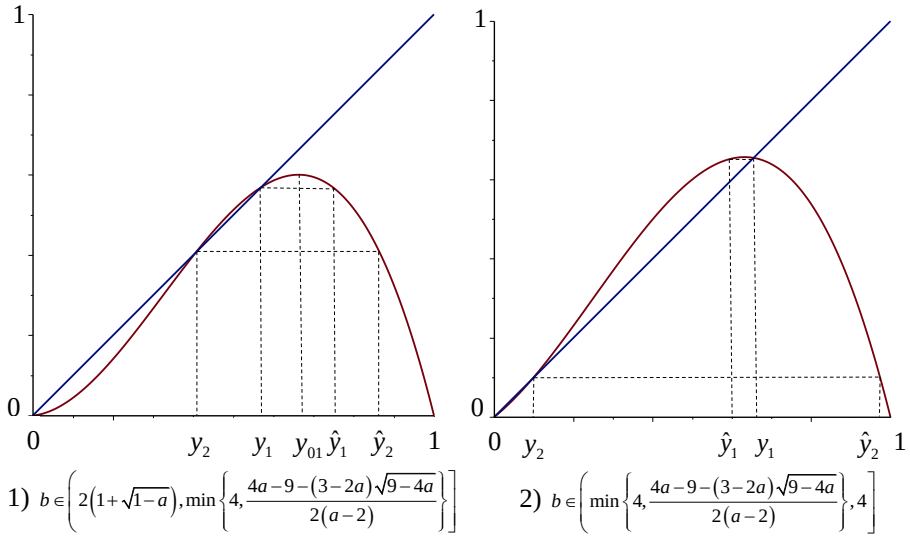


FIGURE 5.

Let $b \in \left(\min\left\{4, \frac{4a-9-(3-2a)\sqrt{9-4a}}{2(a-2)}\right\}, 4\right]$ (See Fig 5.2). In this case, $0 < y_2 < \hat{y}_1 \leq y_{01} \leq y_1 < \hat{y}_2 < 1$. Suppose $y \in (0, y_2)$. The proof for this case is analogous to that for $b \in \left(2\left(1+\sqrt{1-a}\right), \min\left\{4, \frac{4a-9-(3-2a)\sqrt{9-4a}}{2(a-2)}\right\}\right]$ with $y \in (0, y_2)$. For $y \in (0, y_2) \cup (\hat{y}_2, 1)$, we have $\lim_{n\to\infty} f^n(y) = 0$.

Let's assume $y \in [\hat{y}_1, y_1]$. Then $f^2$ maps the interval $[\hat{y}_1, y_1]$ inside $[y_{01}, y_1]$. It follows that $\lim_{n\to\infty} f^n(y) = y_1$ for all $y \in [\hat{y}_1, y_1]$. Now, assume that $y \in (y_2, \hat{y}_1)$, then there exists $k > 0$ such that $f^k(y) \in [\hat{y}_1, y_1]$ holds, and as a result we get $\lim_{n\to\infty} f^{k+n}(y) = y_1$. As before, $f(y)$ maps the interval $(y_1, \hat{y}_2)$ onto $(y_2, \hat{y}_1)$, so the $\lim_{n\to\infty} f^n(y) = y_1$ here as well.

So

$$\lim_{n\to\infty} f^n(y) = \begin{cases} 0, & if \ \ y \in [0, y_2) \cup (\hat{y}_2, 1] \ , \\ y_2, & if \ \ y \in \{y_2, \hat{y}_2\} \, , \\ y_1, & if \ \ y \in (y_2, \hat{y}_2) \, , \end{cases}$$

for $b \in \left(\min\left\{4, \frac{4a-9-(3-2a)\sqrt{9-4a}}{2(a-2)}\right\}, 4\right]$.

Finally, noting that $\lim\limits_{n\to\infty} x^{(n)} = 1 - \lim\limits_{n\to\infty} y^{(n)}$, we get

$$\lim_{n\to\infty} V^n(\lambda) = \begin{cases} e_1, \ if \ \ y \in [0, y_2) \cup (\hat{y}_2, 1], \\ \eta, \ \ if \ \ y \in \{y_2, \hat{y}_2\}, \\ \xi, \ \ if \ \ y \in (y_2, \hat{y}_2). \end{cases}$$

The theorem is proved. □

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