Theory of Stochastic Processes
Vol. 29 (45), no. 2, 2025, pp. 30–65

KATERYNA HLYNIANA

INTERACTING PARTICLE SYSTEMS THROUGH POINT PROCESSES: STATIC STRUCTURE AND DYNAMICS

This survey develops a second-order viewpoint on point processes and on configuration-valued stochastic dynamics. We treat point processes as random counting measures on configuration space and emphasize the tools that govern first and second order: factorial moment measures, correlation functions, the pair-correlation function, Campbell–Mecke integrals, and conditional intensities. We then review algebraic classes with explicit correlation structure — determinantal, permanental/Cox, and Pfaffian point processes — highlighting how their kernels constrain repulsion or clustering through $g(r)$. The second part turns to interacting particle systems and flows, using two-point functions to compare lattice models (exclusion, voter, contact), continuum birth-death and Glauber dynamics, and the associated BBGKY-type correlation hierarchies. A central case study is one-dimensional coalescing and annihilating systems: at fixed times they form Pfaffian point processes, yielding explicit formulas for $\rho_t^{(1)}$, $\rho_t^{(2)}$ and short-range inhibition induced by collision history, and connecting to the Arratia flow. We conclude with open problems on Pfaffian models with controlled attraction, second-order classification of IPS, and multi-type extensions.

Introduction

This survey develops a second-order viewpoint on point processes and on configuration-valued stochastic dynamics. Throughout, a point process is treated as a random (simple) counting measure on a Polish space, so that the basic objects of interest — factorial moment measures, correlation functions, and Campbell–Mecke type identities — can be stated and used in a common measure-theoretic language. The guiding theme is that many qualitative features of spatial structure can be detected, compared, and sometimes computed from second-order information: intensities, covariances, and normalized two-point ratios such as the pair-correlation function

$$g(x,y) = \frac{\rho^{(2)}(x,y)}{\rho^{(1)}(x)\rho^{(1)}(y)} \qquad (x \neq y),$$

or, in stationary isotropic Euclidean settings, its radial version $g(r)$. Relative to the Poisson benchmark, $g < 1$ indicates suppression of close pairs (inhibition/repulsion), while $g > 1$ indicates an excess of close pairs (clustering). At the same time, a recurrent limitation is made explicit: in most dynamic models, second order does not evolve autonomously, since the time evolution of $\rho_t^{(2)}$ typically couples to higher orders through a correlation hierarchy.

The paper is organized in two parts.

2000 *Mathematics Subject Classification.* Primary 60G55, 60K35; Secondary 82C22, 82C21, 60G57, 60J25.

Key words and phrases. point processes, random measures, correlation functions, interacting particle systems, coalescing random walks, coalescing Brownian motions, stochastic flow with interaction.

Part I: Static point-process structure. Section 1 sets the configuration-space framework and introduces the second-order tools used throughout: factorial moment measures and correlation functions (with emphasis on first and second order), pair-correlation functions, conditional intensities, and Campbell–Mecke type integrals. These objects provide the basic interface between probabilistic constructions and statistical summaries.

Section 2 collects point-process families in which the correlation functions admit closed algebraic formulas. Determinantal point processes are characterized by determinantal identities for $\rho^{(n)}$, which imply explicit repulsive second-order structure (in particular $g \le 1$ under standard Hermitian kernel assumptions). Permanental processes are defined by permanent formulas for $\rho^{(n)}$ and exhibit clustering at second order ($g \ge 1$ in the Hermitian setting), with a Cox-process representation that makes this behaviour transparent. Pfaffian point processes are defined by Pfaffians of antisymmetric 2×2 matrix kernels; beyond their appearance in real/quaternionic random-matrix symmetry classes, they serve as the natural fixed-time algebraic description for certain coalescing and annihilating systems. The section concludes with brief remarks on further matrix-function families (immanantal, hafnian/Torontonian) motivated by quantum-optical constructions, recorded for context but not used later.

Part II: Dynamic interacting systems and flows. Section 3 turns to interacting particle systems whose time marginals can be viewed as point configurations. We begin with lattice models (exclusion, voter, contact), using one- and two-point functions to contrast hard-core constraints, neutral density with coalescing genealogies, and clustering/phase transition behaviour. We then discuss continuum birth–death dynamics on configuration space and the role of Papangelou conditional intensities and the Georgii–Nguyen–Zessin identity in connecting Gibbsian structure, dynamics, and second-order summaries.

Section 3.3 formulates the associated BBGKY-type correlation hierarchy for continuum birth–death dynamics. In this framework, the nth equation couples $\rho_t^{(n)}$ to neighbouring orders; in particular, the second-order quantities encoded by $\rho_t^{(2)}$ are typically governed by an equation involving $\rho_t^{(3)}$, so that closure at second order requires either additional integrable structure or an external approximation.

Section 4 develops the key example where such additional structure is available. In one dimension, coalescing and annihilating random walks (and their diffusive limits to coalescing Brownian motions and the Arratia flow) have fixed-time laws that are Pfaffian point processes. This Pfaffian description yields explicit formulas for $\rho_t^{(n)}$, and in particular for $\rho_t^{(1)}$, $\rho_t^{(2)}$, and the corresponding pair-correlation ratios, quantifying the short-range inhibition induced by collision histories. The section also records complementary analytic viewpoints (discrete approximations and semigroups for m-point motions) and, as presented in the text, discusses Gaussian fluctuation statements for functionals of the Arratia point measure whose covariance is expressed in terms of second-order data.

Finally, Section 5 turns to measure-dependent dynamics, in which drift and noise coefficients depend on the evolving empirical measure. The examples include McKean–Vlasov diffusions and Dorogovtsev-type measure-valued flows driven by stochastic flows of maps, where the measure evolves by pushforward. In this setting, second-order structure appears both in pairwise interaction effects and in covariance kernels describing fluctuations of empirical measures, as outlined in the corresponding subsections.

The paper concludes in Section 6 with open problems motivated by this second-order perspective, including questions on the admissible shapes of pair correlations in Pfaffian models, the possibility of a second-order classification of interacting particle systems, and extensions to multi-type settings where second-order structure is naturally matrix-valued.

Part I. Static Point-Process Structure

1. Configuration Spaces and Second-Order Tools

This section sets the formal background for random measures and point processes and establishes the analytical framework for the entire paper. The goal is to introduce exactly the concepts needed later: factorial moments, pair-correlation functions, conditional intensities, and Campbell–Mecke integrals. These tools will later be applied to algebraic correlation structures (determinantal, permanental, Pfaffian) and to dynamic models where second-order quantities evolve in time.

1.1. **Configuration spaces.** We first fix the measurable structure on the configuration space Γ, identified with a subspace of counting measures, so that point processes can be treated as random measures. This setup will be used later when we introduce correlation measures and Palm distributions.

Let E be a Polish space with Borel σ-algebra $\mathcal{B}(E)$; most examples below take $E = \mathbb{R}^d$. A *configuration* on E is a locally finite subset $\gamma \subset E$, meaning that for every bounded $B \in \mathcal{B}(E)$,

$$|\gamma \cap B| < \infty,$$

where $|A|$ denotes the number of elements (cardinality) of a finite set A. We denote by $\Gamma = \Gamma(E)$ the space of all such locally finite configurations.

It is standard and convenient to identify each configuration γ with the associated counting measure

$$\gamma(\cdot) \;=\; \sum_{x\in\gamma} \delta_x(\cdot).$$

Equivalently, Γ is the set of non-negative, integer-valued Radon measures on E that are locally finite and simple, that is,

$$\gamma(\{x\}) \in \{0, 1\} \quad \text{for all } x \in E.$$

We will use the random-measure viewpoint throughout; see [16, 64, 73].

The natural topology on Γ is the *vague topology*, defined as the coarsest topology that makes the maps

$$\Gamma \ni \gamma \longmapsto \int_E f \, \mathrm{d}\gamma$$

continuous for all continuous functions f with compact support. The Borel σ-algebra generated by the vague topology is denoted by $\mathcal{B}(\Gamma)$. Equivalently, $\mathcal{B}(\Gamma)$ is generated by the evaluation maps

$$\Gamma \ni \gamma \longmapsto \gamma(B), \qquad B \in \mathcal{B}(E) \text{ bounded}.$$

A *point process* on E is a measurable map

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\Gamma, \mathcal{B}(\Gamma)).$$

Equivalently, X is an integer-valued random measure on E that is almost surely locally finite and simple; we write $X(B)$ for the number of points in B.

Throughout the survey we restrict attention to simple point processes. Non-simple processes can be included by allowing atoms of mass $2, 3, \ldots$, but we do not need this here.

We also use the usual shift operator on configurations. If $E = \mathbb{R}^d$ and $z \in \mathbb{R}^d$, define

$$\theta_z \gamma := \{x - z : x \in \gamma\}.$$

A point process X is called *stationary* if $\theta_z X$ has the same distribution as X for all $z \in \mathbb{R}^d$. Stationarity is assumed in several examples below; see [15, 64, 73, 54].

Example 1.1 (Poisson point process)**.** *Our reference model is the Poisson point process, representing the absence of interaction.*

Let Λ *be a* σ*-finite Borel measure on* E *that is finite on bounded sets. A point process* X *on* E *is a Poisson point process with intensity measure* Λ *if*

- *for every bounded* $B \in \mathcal{B}(E)$*,* $X(B)$ *is Poisson with mean* $\Lambda(B)$*,*
$$X(B) \sim \mathrm{Poisson}(\Lambda(B)),$$
- *for any finite family of disjoint bounded Borel sets* $B_1, \dots, B_m$*, the random variables* $X(B_1), \dots, X(B_m)$ *are independent.*

If Λ *is diffuse, meaning* $\Lambda(\{x\}) = 0$ *for all* $x \in E$*, then* X *is simple almost surely, hence* X *takes values in* Γ*.*

In the Euclidean case $E = \mathbb{R}^d$*, the homogeneous Poisson point process with constant intensity* $\lambda > 0$ *corresponds to* $\Lambda(\mathrm{d}x) = \lambda\,\mathrm{d}x$*. It is stationary in the sense that* $\theta_z X$ *has the same distribution as* X *for all* $z \in \mathbb{R}^d$*. See* [15, 16] *for details.*

We next introduce the moment and correlation measures used throughout the paper. Although we focus on first and second order, we define factorial moment measures for all orders since they enter naturally in correlation hierarchies.

1.2. **Moment and correlation measures.** Let X be a point process on E, viewed as a random counting measure. For a measurable set $B \subset E$, we werite $X(B)$ for the number of points of X in B. The *intensity measure* of X is the first moment measure

$$\alpha^{(1)}(B) := \mathbb{E}[X(B)], \qquad B \in \mathcal{B}(E).$$

To study interactions between points, we recall *factorial moment measures.* For $n \ge 1$, the nth factorial moment measure $\alpha^{(n)}$ is the measure on E^n defined on rectangles $B_1 \times \cdots \times B_n$ by

$$\alpha^{(n)}(B_1 \times \cdots \times B_n) := \mathbb{E}\left[\sum_{x_1,\dots,x_n \in X}^{\neq} \mathbf{1}_{B_1}(x_1) \cdots \mathbf{1}_{B_n}(x_n)\right],$$

where the sum is taken over n-tuples of distinct points of X. Equivalently, for bounded $B \subset E$,

$$\alpha^{(n)}(B^n) = \mathbb{E}\big[X(B)_{\downarrow n}\big],$$

where $k_{\downarrow n} := k(k-1)\cdots(k-n+1)$ is the falling factorial. We assume local finiteness of factorial moments, namely

$$\mathbb{E}\big[X(B)_{\downarrow n}\big] < \infty \quad \text{for every bounded } B \subset E \text{ and every } n \ge 1.$$

Under this assumption, $\alpha^{(n)}$ extends uniquely to a symmetric Radon measure on E^n; see [15, 64].

Whenever $\alpha^{(n)}$ is absolutely continuous with respect to a reference measure $m^{\otimes n}$ on E^n (for instance, m is Lebesgue measure on $\mathbb{R}^d$), we define the *nth-order product density* or *correlation function* $\rho^{(n)}$ by

$$\alpha^{(n)}(\mathrm{d}x_1 \cdots \mathrm{d}x_n) = \rho^{(n)}(x_1, \dots, x_n)\, m(\mathrm{d}x_1) \cdots m(\mathrm{d}x_n).$$

Informally, for a simple point process, one may think of $\rho^{(n)}(x_1, \dots, x_n)\, \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_n)$ as the probability that X has a point in each infinitesimal neighbourhood $\mathrm{d}x_1, \dots, \mathrm{d}x_n$, with no restriction elsewhere.

The function $\rho^{(1)}$ coincides with the intensity, while $\rho^{(2)}$ encodes pairwise interaction. In several classes of models, higher-order correlation functions $\rho^{(n)}$ are either known explicitly or can be described by hierarchical systems of equations [16, 36, 35, 17].

Factorial moment measures control expectations of sums over distinct tuples of points. In particular, if $\rho^{(1)}$ exists and $f \geq 0$ is measurable, then the Campbell formula reads

$$\mathbb{E}\left[\sum_{x\in X} f(x)\right] = \int_E f(x)\,\rho^{(1)}(x)\,\lambda(\mathrm{d}x),$$

and analogous formulas involving $\rho^{(2)}$ and $\alpha^{(2)}$ yield variance and covariance formulas.

Example 1.2 (Poisson point process: factorial moment measures and correlations)**.** *Continuing Example 1.1, let X be a Poisson point process on E with locally finite intensity measure Λ. Then, by definition,*

$$\alpha^{(1)}(B) = \mathbb{E}[X(B)] = \Lambda(B), \qquad B \in \mathcal{B}(E).$$

Moreover, for any $n \geq 1$ and measurable sets $B_1, \dots, B_n \subset E$, the nth factorial moment measure factorises:

$$\alpha^{(n)}(B_1 \times \cdots \times B_n) = \prod_{i=1}^n \Lambda(B_i).$$

Equivalently, for nonnegative measurable $f_1, \dots, f_n$ with bounded support,

$$\mathbb{E}\left[\sum_{x_1,\dots,x_n\in X}^{\neq} \prod_{i=1}^n f_i(x_i)\right] = \int_{E^n} \prod_{i=1}^n f_i(x_i)\,\Lambda(\mathrm{d}x_1)\cdots\Lambda(\mathrm{d}x_n).$$

Assume now that Λ is absolutely continuous with respect to the chosen reference measure λ on E, that is,

$$\Lambda(\mathrm{d}x) = \rho(x)\,\lambda(\mathrm{d}x).$$

Then $\alpha^{(n)}$ is absolutely continuous with respect to $\lambda^{\otimes n}$, and the n-point correlation functions are

$$\rho^{(n)}(x_1, \dots, x_n) = \prod_{i=1}^n \rho(x_i), \qquad n \geq 1.$$

In particular, in the homogeneous case on $\mathbb{R}^d$ with $\rho(x) \equiv \beta$,

$$\rho^{(n)}(x_1, \dots, x_n) = \beta^n.$$

See [15, 16, 64] *for these standard identities.*

In the stationary case on $E = \mathbb{R}^d$, the intensity is constant and the second-order structure is often summarized by a scalar *pair-correlation function* g, which reparametrizes $\rho^{(2)}$. We introduce g and its basic interpretations in the next subsection.

1.3. **Pair-correlation function** $g(r)$**.** We define the pair-correlation function as a normalization of the second factorial moment density. It measures the deviation of second-order joint intensities from the Poisson benchmark and is a standard summary in both probabilistic and statistical treatments of stationary point processes.

Assume that X is stationary and simple on $\mathbb{R}^d$ with constant intensity $\lambda > 0$, and that the second factorial moment measure $\alpha^{(2)}$ is absolutely continuous with respect to Lebesgue measure on $(\mathbb{R}^d)^2$, with density $\rho^{(2)}$. The *pair-correlation function* is defined by

$$g(x, y) := \frac{\rho^{(2)}(x, y)}{\lambda^2}, \qquad x \neq y.$$

Stationarity implies that $g(x, y) = g(x - y)$, and under isotropy one writes $g(x, y) = g(\|x - y\|) = g(r)$. Heuristically, for disjoint small Borel sets B_1, B_2 around x and y, $\alpha^{(2)}(B_1 \times B_2) \approx \lambda^2 g(x, y)\,|B_1|\,|B_2|$, so g compares second-order joint intensities to those of a Poisson process with the same λ.

The reference case is the homogeneous Poisson process, for which all points are independent and hence

$$g(x, y) \equiv 1.$$

Values $g(r) < 1$ at small r indicate inhibition (repulsion) at short distances, while $g(r) > 1$ indicates clustering.

Another interpretation can be made by comparing conditional and unconditional probabilities for small volume elements, using the heuristic for $\rho^{(2)}$ from Subsection 1.2. For small balls $B(x, \mathrm{d}x)$ and $B(y, \mathrm{d}y)$,

$$\mathbb{P}\big\{X(B(x, \mathrm{d}x)) \geq 1,\ X(B(y, \mathrm{d}y)) \geq 1\big\} \;\approx\; \lambda^2 g(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

while $\mathbb{P}\{X(B(y, \mathrm{d}y)) \geq 1\} \approx \lambda\, \mathrm{d}y$ and $\mathbb{P}\{X(B(x, \mathrm{d}x)) \geq 1\} \approx \lambda\, \mathrm{d}x$. Hence

$$\mathbb{P}\big\{X(B(x, \mathrm{d}x)) \geq 1 \mid X(B(y, \mathrm{d}y)) \geq 1\big\} \;\approx\; \lambda g(x, y)\, \mathrm{d}x.$$

A standard second-order calculation yields, for bounded $B \subset \mathbb{R}^d$ and under $\int_B \int_B |g(x-y) - 1|\, \mathrm{d}x\, \mathrm{d}y < \infty$,

$$\operatorname{Var} X(B) = \lambda |B| + \lambda^2 \int_B \int_B \big(g(x - y) - 1\big)\, \mathrm{d}x\, \mathrm{d}y,$$

so that $g(r) - 1$ quantifies the excess variance (over-dispersion or under-dispersion) relative to the Poisson case.

In spatial statistics, $g(r)$ is closely related to Ripley's K-function via

$$K(r) = \int_{\|x\| \leq r} g(x)\, \mathrm{d}x,$$

and empirical estimators of g and K are the basic tools for diagnosing repulsion and clustering in observed point patterns [73, 54, 3].

We next turn to point processes with explicit algebraic formulas for $\rho^{(n)}$, which provide tractable examples with characteristic second-order behaviour.

2. Algebraic Point Processes and Correlation Structures

In several important classes of point processes, the factorial moment densities admit closed algebraic formulas. We record the determinantal, permanental, and Pfaffian cases, since they provide explicit expressions for $\rho^{(n)}$ and hence for second-order quantities such as g. Such point processes arise naturally as mathematical models for physical systems.

2.1. **Determinantal point processes.** Determinantal point processes are characterized by determinantal formulas for their correlation functions. In particular, their second-order statistics satisfy $g \leq 1$ under standard assumptions on the kernel. This class of point processes represents a classical repulsive model for fermionic systems.

Let E be a locally compact Polish space with reference measure λ (e.g. Lebesgue measure on $\mathbb{R}^d$). A simple point process X on E is called a *determinantal point process* with kernel $K : E \times E \to \mathbb{C}$ if, for every $n \geq 1$, the n-point correlation function exists and satisfies

$$\rho^{(n)}(x_1, \ldots, x_n) = \det\big(K(x_i, x_j)\big)_{i,j=1}^{n}$$

with respect to $\lambda^{\otimes n}$; see [71, 80, 53, 6, 57]. A sufficient condition for existence is that K induces a self-adjoint, locally trace-class contraction on $L^2(E, \lambda)$ (that is, $0 \leq K \leq I$ in operator order). Under this hypothesis, there exists a unique DPP with kernel K; see [71, 80, 53].

Assume K is Hermitian, i.e. $K(y, x) = \overline{K(x, y)}$. Then

$$\rho^{(1)}(x) = K(x, x), \qquad \rho^{(2)}(x, y) = K(x, x)K(y, y) - |K(x, y)|^2.$$

In particular, for a stationary DPP on $\mathbb{R}^d$ with constant intensity $\lambda = \rho^{(1)}(x)$, the pair-correlation function

$$g(x,y) = \frac{\rho^{(2)}(x,y)}{\lambda^2} = 1 - \frac{|K(x,y)|^2}{K(x,x)K(y,y)}$$

satisfies $g(x,y) \le 1$, with strict inequality for $x \ne y$ whenever $K(x,y) \ne 0$. In particular, $g(x,y) \le 1$, with strict inequality for $x \ne y$ whenever $K(x,y) \ne 0$, so the joint intensity of nearby pairs is reduced relative to the Poisson case; see [53, 57, 6, 65] for examples and further discussion.

Examples. Examples from random matrix theory and integrable probability include:

- The eigenvalues of unitary invariant Hermitian ensembles (for example the GUE and more general $\beta = 2$ ensembles) form DPPs with correlation kernels expressed in terms of orthogonal polynomials; in the bulk limit one obtains the sine kernel, and at the edge the Airy kernel [57, 56, 53, 6].
- The complex Ginibre ensemble yields a DPP on $\mathbb{C}$ with kernel

$$K(z,w) = \frac{1}{\pi}\exp\bigl(-\tfrac{1}{2}(|z|^2 + |w|^2) + z\overline{w}\bigr),$$

  whose pair-correlation function $g(r)$ shows strong radial repulsion; see [53, 8].
- Many $1+1$-dimensional growth models and exclusion processes (TASEP, PNG model) have fixed-time distributions that are determinantal; this is the basis for exact fluctuation results in the KPZ universality class [7, 84, 56].

In spatial statistics, DPPs have been adopted as flexible models for repulsive point patterns, with likelihood-based and composite likelihood methods exploiting the explicit forms of $\rho^{(n)}$ and $g(r)$ [65, 73, 54]. From the perspective of second-order summaries, DPPs provide a tractable class of repulsive models with explicit $\rho^{(n)}$.

We next record the permanental and Cox constructions, which yield $g \ge 1$ under analogous hypotheses.

2.2. **Permanental and Cox processes.** Permanental point processes are defined by permanent formulas for $\rho^{(n)}$. Under Hermitian kernels, their second-order statistics satisfy $g \ge 1$, corresponding to clustering relative to the Poisson benchmark.

Let $K : E \times E \to \mathbb{C}$ be a non-negative definite kernel. A simple point process X on E is called a *permanental point process* with kernel K if, for every $n \ge 1$,

$$\rho^{(n)}(x_1,\dots,x_n) = \operatorname{perm}\bigl(K(x_i,x_j)\bigr)_{i,j=1}^n,$$

where

$$\operatorname{perm}(A) := \sum_{\sigma\in S_n}\prod_{i=1}^n A_{i,\sigma(i)}$$

is the permanent of the matrix A.Permanental processes are related to α-permanents; see [86, 79].

As in the determinantal case, the first two correlation functions can be written directly in terms of K. For a permanental point process with kernel K (so that $\rho^{(n)}(x_1,\dots,x_n) = \operatorname{perm}(K(x_i,x_j))_{i,j=1}^n$), one has

$$\rho^{(1)}(x) = K(x,x), \qquad \rho^{(2)}(x,y) = K(x,x)K(y,y) + K(x,y)K(y,x).$$

If K is Hermitian, that is $K(y,x) = \overline{K(x,y)}$, then

$$\rho^{(2)}(x,y) = K(x,x)K(y,y) + |K(x,y)|^2.$$

In particular, for a stationary permanental point process on $\mathbb{R}^d$ with constant intensity $\lambda = \rho^{(1)}(x)$, the pair-correlation function

$$g(x,y) = \frac{\rho^{(2)}(x,y)}{\lambda^2} = 1 + \frac{|K(x,y)|^2}{K(x,x)K(y,y)}$$

satisfies $g(x,y) \geq 1$, with strict inequality for $x \neq y$ whenever $K(x,y) \neq 0$. Hence $g(x,y) \geq 1$, with strict inequality for $x \neq y$ whenever $K(x,y) \neq 0$; see [79, 72].

A key structural feature is that many permanental processes can be represented as *Cox processes*. Recall that a Cox process is a Poisson point process with a random intensity measure: conditionally on a random field $\Lambda(x) \geq 0$, the process is Poisson with intensity Λ. In particular, if G is a centred *proper* complex Gaussian field on E with covariance kernel $K(x,y) = \mathbb{E}[G(x)\overline{G(y)}]$ and

$$\Lambda(x) = |G(x)|^2,$$

then, under suitable integrability conditions, the Cox process driven by Λ is a permanental point process with kernel K [79, 16]. This representation makes the clustering behaviour transparent: large values of $|G(x)|$ create areas where many points are likely to appear together.

While determinantal and permanental processes describe static repulsion or static clustering, Pfaffian point processes arise naturally from *dynamic* coalescence or annihilation. Their algebraic structure encodes merging histories rather than purely static geometry.

2.3. **Pfaffian point processes.** Pfaffian point processes are defined by Pfaffian formulas for $\rho^{(n)}$ associated with antisymmetric matrix kernels. They arise in several integrable models, including real symmetry classes in random matrix theory and one-dimensional coalescing or annihilating systems.

Let E be as before and let $K(x,y)$, $x, y \in E$, be a 2×2 matrix kernel

$$K(x,y) = \begin{pmatrix} K_{11}(x,y) & K_{12}(x,y) \\ K_{21}(x,y) & K_{22}(x,y) \end{pmatrix},$$

satisfying the antisymmetry condition $K(x,y) = -K(y,x)^{\mathsf{T}}$. A simple point process X on E is called a *Pfaffian point process* with kernel K if, for every $n \geq 1$,

$$\rho^{(n)}(x_1, \ldots, x_n) = \operatorname{Pf}\big(K(x_i, x_j)\big)_{i,j=1}^{n},$$

where Pf denotes the Pfaffian of the resulting $2n \times 2n$ antisymmetric block matrix. Recall that for an antisymmetric matrix A, the Pfaffian is characterised by $\operatorname{Pf}(A)^2 = \det(A)$ and can be written as a signed sum over pairings of indices; see, for example, [80, 8]. Under e trace-class and symmetry assumptions on K, such kernels arise naturally in random matrix models and interacting particle systems [80, 8, 85].

For $n = 1, 2$, the Pfaffian definition gives explicit formulas in terms of the entries of K. In particular,

$$\rho^{(1)}(x) = K_{12}(x,x),$$

and

$$\rho^{(2)}(x,y) = K_{12}(x,x)K_{12}(y,y) - K_{11}(x,y)K_{22}(x,y) + K_{12}(x,y)K_{21}(x,y).$$

Thus, writing $\lambda(x) = \rho^{(1)}(x)$,

$$g(x,y) = \frac{\rho^{(2)}(x,y)}{\lambda(x)\lambda(y)} = 1 + \frac{K_{12}(x,y)K_{21}(x,y) - K_{11}(x,y)K_{22}(x,y)}{K_{12}(x,x)\,K_{12}(y,y)}.$$

In some special representations the kernel is purely off-diagonal, $K_{11} = K_{22} = 0$ and $K_{21}(x,y) = -K_{12}(y,x)$; then this simplifies to

$$\rho^{(2)}(x,y) = \rho^{(1)}(x)\rho^{(1)}(y) - K_{12}(x,y)K_{12}(y,x),$$

which is formally analogous to the determinantal identity, with the 2×2 structure carried by the matrix kernel.

Examples: random matrices and coalescing systems. Two main families of examples illustrate the dual algebraic–dynamic nature of Pfaffian processes:

- In random matrix theory, eigenvalues of real and quaternionic ensembles (e.g. GOE, GSE, real Ginibre) form Pfaffian point processes on $\mathbb{R}$ or $\mathbb{C}$; their correlation functions are expressed as Pfaffians of antisymmetric block matrices built from a 2×2 kernel, encoding the underlying symmetry class and leading to explicit formulae for spacing distributions and $g(r)$ [80, 8, 6, 57].
- In interacting particle systems, one-dimensional coalescing and annihilating random walks, started from suitable initial conditions, have fixed-time configurations that are Pfaffian [85, 39]. Under diffusive rescaling these systems converge to coalescing Brownian motions and the Arratia flow, whose time-slices also possess Pfaffian structure [2, 37, 14]. In this context, the antisymmetric kernel encodes collision probabilities and the resulting short-range inhibition created by past coalescences. We return to these examples in Sections 4 and 5.

In coalescing or annihilating examples, the resulting $g(r)$ is typically < 1 for small r, reflecting inhibition of close pairs induced by the dynamics.

2.4. **Further algebraic families and quantum–optical origin.** Other algebraic correlation structures have been proposed, motivated in particular by quantum-optical and many-body models.

On the combinatorial side, immanants provide a representation-theoretic interpolation between determinants and permanents: for a matrix A and a class function χ on the symmetric group, the immanant $\mathrm{imm}_\chi(A)$ weights each permutation by $\chi(\sigma)$, with determinants and permanents corresponding to the sign and trivial characters. On finite state spaces one can define *immanantal point processes* by assigning probabilities proportional to $\mathrm{imm}_\chi(K_S)$ for principal submatrices K_S; these interpolate between determinantal and permanental statistics, although positivity constraints restrict the range of admissible characters [21].

For bosonic systems, a natural analogue of Pfaffian structure is given by *hafnians*. In the Gaussian (quasi-free) setting, normal-ordered $2n$-point correlation functions admit Wick-type expansions into sums over pairings, so that the relevant n-point detection statistics can be written as hafnians of matrices built from two-point covariances. The matrix function itself, viewed as the canonical pairing-sum for symmetric matrices, goes back to early work of Caianiello [13]. This connection becomes especially explicit in quantum optics: for multimode Gaussian states, the probability of observing a given photon-number pattern with number-resolving detectors is governed by hafnians of suitable submatrices (Gaussian boson sampling) [48]. For threshold (on-off) detectors, analogous click statistics are expressed in terms of a closely related matrix function, the *Torontonian* [77]. Interpreting these sampling rules as finite-window counting laws leads naturally to *hafnian point processes*, a point-process formulation of Gaussian boson sampling in which joint intensities inherit a hafnian structure from the underlying Gaussian field [55], and the perfect matching interpretation provides additional combinatorial intuition [9].

These constructions again express $\rho^{(n)}$ in terms of a two-point kernel via a matrix function (immanant, hafnian, Torontonian). We do not use them later, and therefore only record them briefly.

We next discuss interacting particle systems in which these algebraic correlation structures appear as fixed-time laws.

Part II. Dynamic Interacting Systems and Flows

3. Interacting Particle Systems Through Second-Order Structure

The remainder of the survey concerns interacting particle systems and stochastic flows whose time marginals can be viewed as point processes. We use second-order quantities (pair correlations, covariances, and factorial-moment hierarchies) as a common language for comparing models. Second-order structure detects inhibition and clustering effects in time, but questions such as invariant measures, scaling limits, and non-equilibrium behaviour typically require information beyond second order.

We begin with lattice models (exclusion, voter, contact), then briefly comment on continuum birth–death systems and the BBGKY hierarchy. These examples provide background for the coalescing and measure-dependent flows studied in Sections 5–6.

3.1. Lattice Interacting Particle Systems.

Lattice interacting particle systems are Markov processes on $\{0,1\}^{\mathbb{Z}^d}$ (or a finite spin space) with local transition mechanisms; see [66, 67, 29].

Configurations typically live in $\{0,1\}^{\mathbb{Z}^d}$ (or a finite spin space) and evolve according to a generator with local rates. Despite simple local update rules, correlations at positive times are typically non-trivial and encode inhibition or clustering. We recall the basic set-up and discuss three standard examples (exclusion, voter, contact), emphasizing their second-order behaviour.

General framework and particle interpretation. A configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ assigns an occupation variable $\eta(x)$ to each $x \in \mathbb{Z}^d$, with $\eta(x) = 1$ (occupied) and $\eta(x) = 0$ (vacant). Equivalently, η encodes the subset of occupied sites

$$\Xi(\eta) := \{x \in \mathbb{Z}^d : \eta(x) = 1\},$$

so one may view $(\Xi(\eta_t))_{t\geq 0}$ as a (simple) point process on the lattice. In the IPS literature the local variable $\eta(x)$ is also called a *spin* (borrowed from statistical mechanics); "spin-flip" means that a single site changes its value $0 \leftrightarrow 1$.

Let $\eta_t = (\eta_t(x))_{x\in\mathbb{Z}^d}$, $t \geq 0$, be a Markov process with state space $\{0,1\}^{\mathbb{Z}^d}$. For *spin-flip systems* the generator often has the form

$$Lf(\eta) = \sum_{x\in\mathbb{Z}^d} c(x,\eta)\big(f(\eta^x) - f(\eta)\big),$$

where η^x is obtained from η by flipping the value at x, and the rate $c(x,\eta)$ depends only on η in a finite neighbourhood of x. For *conservative particle systems* (such as exclusion), the elementary move is a *jump* of a particle from x to y (typically nearest neighbours), so the generator involves local exchanges $\eta \mapsto \eta^{x,y}$ rather than single-site flips; see [81, 22, 66].

For $t \geq 0$ and $x, y \in \mathbb{Z}^d$ we write the one- and two-point functions

$$\rho_t^{(1)}(x) := \mathbb{P}(\eta_t(x) = 1), \qquad \rho_t^{(2)}(x,y) := \mathbb{P}(\eta_t(x) = 1,\ \eta_t(y) = 1).$$

To quantify "repulsion vs. clustering" we compare $\rho_t^{(2)}$ to the product of marginals. Since point-process pair correlations are defined off the diagonal, we also restrict to distinct sites and set

$$g_t(x,y) := \frac{\rho_t^{(2)}(x,y)}{\rho_t^{(1)}(x)\rho_t^{(1)}(y)}, \qquad x \neq y,$$

whenever $\rho_t^{(1)}(x)\rho_t^{(1)}(y) > 0$. Thus $g_t(x,y) = 1$ corresponds to factorization at the level of two-point functions, while $g_t(x,y) < 1$ (resp. > 1) indicates suppression (resp. excess) of pairs. If the law of η_t is translation invariant, then $\rho_t^{(1)}(x) \equiv \rho_t$ and $g_t(x,y) = g_t(y-x)$.

Harris graphical construction (space–time diagram). Many lattice IPS admit a graphical construction in which independent Poisson marks in space–time determine the evolution pathwise. Concretely, one draws a random diagram on $\mathbb{Z}^d \times \mathbb{R}_+$:

- For each ordered edge $y \to x$ (or undirected edge $\{x, y\}$), place a Poisson process of *arrow events* at the prescribed rate (e.g. rate $p(y, x)$, or rate 1 for nearest-neighbour updates). An arrow $y \to x$ at time s means "site x consults site y at time s".
- For models with spontaneous flips at a site (e.g. recoveries in the contact process), place an independent Poisson process of *site marks* on each x (e.g. recovery marks at rate 1).

Almost surely, any bounded space–time window contains only finitely many marks, so one can update the configuration by scanning events in increasing time order. The rule at each mark depends on the model: for the voter model, an arrow $y \to x$ makes x copy the state of y; for the contact process, arrows transmit infection while recovery marks heal; for exclusion, one can use "stirring": a mark on $\{x, y\}$ exchanges the occupations at x and y, so the hard-core constraint is automatic. This yields a pathwise construction of $(\eta_t)_{t\geq 0}$ on a single probability space carrying only independent Poisson processes.

Two standard consequences are useful throughout. Using the same Poisson marks yields a canonical coupling of processes started from different initial configurations, which underpins attractiveness and comparison arguments. Moreover, backward tracing along the marks identifies ancestors and leads to standard dualities. In the voter model this produces coalescing ancestral random walks, which is the mechanism behind Theorem 3.1. See [49, 67, 68] for historical context.

Simple exclusion: hard-core repulsion. In the (symmetric) simple exclusion process each particle performs nearest-neighbour random walks subject to the rule that jumps onto occupied sites are suppressed. Formally, $\eta_t(x) = 1$ indicates a particle at x, and a particle at x attempts to jump to y at rate $p(x, y)$, but the jump is carried out only if $\eta_t(y) = 0$; see [67, 60] for precise definitions.

Exclusion enforces *hard-core repulsion* at the microscopic level: two particles can never occupy the same site, so $\rho_t^{(2)}(x, x) = 0$ for all t and x. At equilibrium in infinite volume, with Bernoulli product measure ν_ρ of density $\rho \in (0, 1)$, distinct sites are independent, so

$$\rho^{(1)}(x) = \rho, \qquad \rho^{(2)}(x, y) = \rho^2 \quad (x \neq y),$$

and away from the diagonal

$$g_{\mathrm{eq}}(x, y) = 1 \quad (x \neq y).$$

Here the hard-core constraint acts on the diagonal, while off-diagonal second-order behaviour depends on the ensemble and on whether the system is at equilibrium. In finite systems with a fixed number of particles (canonical ensembles), the constraint induces negative covariances between distinct sites; out of equilibrium one often observes short-range inhibition relative to product baselines.

From the second-order viewpoint, exclusion combines a hard-core diagonal constraint with ensemble-dependent off-diagonal correlations (product Bernoulli in the grand-canonical setting, and negative correlations in canonical finite-volume ensembles).

Voter model: neutral density and coalescing dual. Let $p(\cdot, \cdot)$ be a *jump kernel* on $\mathbb{Z}^d$, meaning that

$$p(x, y) \geq 0, \qquad \sum_{y \in \mathbb{Z}^d} p(x, y) = 1 \quad \text{for all } x \in \mathbb{Z}^d,$$

and typically $p(x, x) = 0$. (For concreteness one may assume p has finite range or is summable in y, so that the dynamics are local; see [49, 67].)

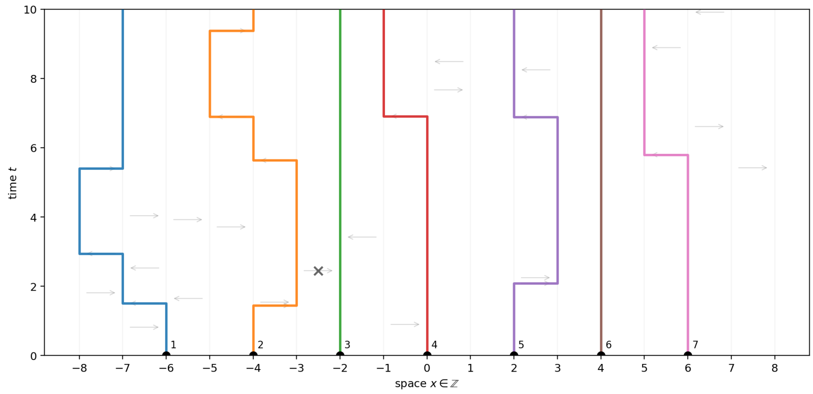

FIGURE 1. **Harris graphical construction for the one-dimensional symmetric simple exclusion process (attempted-jump representation).** Space is $x \in \mathbb{Z}$ and time is $t \in [0, T]$. Light gray arrows are the Poisson jump-attempt marks on oriented nearest-neighbour edges $x \to y$. Given these marks and the initial configuration, the evolution is deterministic: at an arrow time t on $x \to y$, a particle at x moves to y if $\eta_{t-}(x) = 1$ and $\eta_{t-}(y) = 0$, and the attempt is suppressed otherwise. Black dots at $t = 0$ mark the initial particle locations, and the coloured space–time paths trace labeled particle trajectories. The symbol $\times$ marks an attempted jump that was blocked because the target site was already occupied, illustrating the hard-core constraint (at most one particle per site).

The voter model $(\eta_t)_{t\geq 0}$ on $\{0,1\}^{\mathbb{Z}^d}$ is the Markov process in which each site x updates at rate 1 by choosing a site y with law $p(x,\cdot)$ and then copying its state. Its generator acts on cylinder functions f as

$$Lf(\eta) = \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} p(x,y)\,\big[f(\eta^{x\leftarrow y}) - f(\eta)\big],$$

where $\eta^{x\leftarrow y}$ agrees with η except that $\eta^{x\leftarrow y}(x) = \eta(y)$.

The voter model is obtained from the general Harris diagram above by using only arrow marks $y \to x$ (at rate $p(x,y)$), with the update rule that x copies y at each mark; see [49, 67].

Coalescing random walks. Fix the same kernel p. A system of *coalescing random walks* started from a finite set $A \subset \mathbb{Z}^d$ is a collection $(X^x_s)_{s\geq 0,\ x\in A}$ such that: (i) each X^x is a continuous-time random walk that waits an exponential time of mean 1 and then jumps from z to y with probability $p(z,y)$, until it meets another walk; and (ii) whenever two walks occupy the same site at the same time, they *coalesce* and move together thereafter.

In the graphical representation, the dual walks are obtained by following arrows backward in time: starting from a space–time point (x,t), move backward; whenever one encounters an arrow $y \to x$ pointing into the current site, jump to y. If two such backward paths meet, they coincide thereafter, which is exactly coalescence.

Theorem 3.1 (Voter–coalescing duality [67, Ch. V])**.** *Let $(\eta_t)_{t\geq 0}$ be the voter model on $\mathbb{Z}^d$ with kernel p, constructed from a Harris diagram, and started from a* deterministic *initial configuration η_0. Fix $t \geq 0$. For each $x \in \mathbb{Z}^d$, let $(A^{x,t}_s)_{0\leq s\leq t}$ be the backward ancestral path defined by following arrows backward from (x,t) for time s (so $A^{x,t}_0 = x$,*

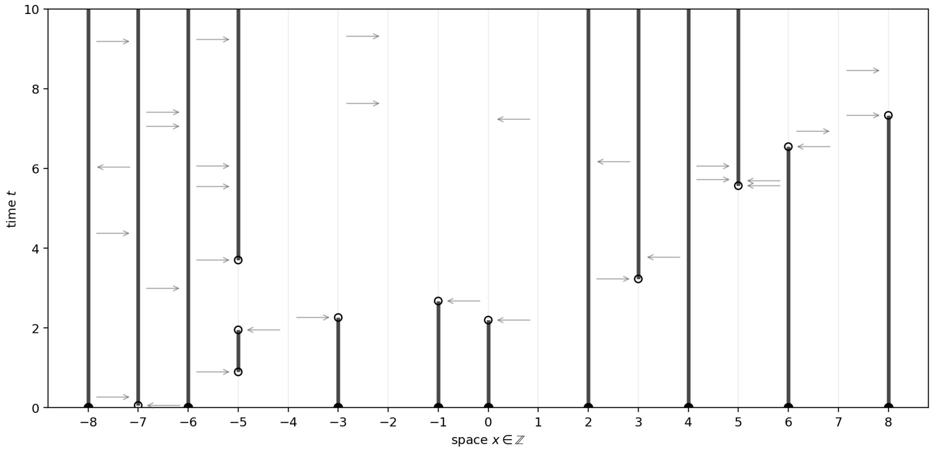

FIGURE 2. **Harris graphical construction for the one-dimensional voter model (illustration).** Space is $x \in \mathbb{Z}$ and time is $t \in [0, T]$. Gray arrows are Poisson copying marks $y \to x$ (in this figure only nearest-neighbour arrows are shown). At an arrow time t on $y \to x$, the update rule is $\eta_t(x) := \eta_{t-}(y)$, hence the sample path is a deterministic function of η_0 and the arrow marks. Black dots at $t = 0$ indicate sites with $\eta_0(x) = 1$. Thick black vertical segments show time intervals on which $\eta_t(x) = 1$. Circled times mark arrows that actually changed the value at the updated site.

and $A_s^{x,t}$ is the spatial location at physical time $t - s$). Then for every finite set $B \subset \mathbb{Z}^d$,

$$\mathbb{E}_{\eta_0}\Big[\prod_{x\in B}\eta_t(x)\Big] = \mathbb{E}\Big[\prod_{x\in B}\eta_0(A_t^{x,t})\Big],$$

where on the right the expectation is over the Harris diagram. Moreover, the family $(A_s^{x,t})_{x\in B}$ evolves as coalescing random walks with kernel p run backward in time.

Idea of proof. On a fixed Harris diagram, the state at (x, t) is exactly the state at time 0 of its ancestor obtained by tracing arrows backward, i.e. $\eta_t(x) = \eta_0(A_t^{x,t})$. For several sites, coalescence of backward paths corresponds to common ancestry. Taking products over $x \in B$ and then expectations yields the stated identity; see [49, 67].

As a first consequence, the one-point function is conserved under translation-invariant initial laws.

Proposition 3.2 (Conservation of density in the voter model [67, 29]). *Assume that the initial law of the voter model is translation invariant with density $\rho_0 = \mathbb{P}(\eta_0(0) = 1)$. Then for all $t \geq 0$ and $x \in \mathbb{Z}^d$,*

$$\rho_t^{(1)}(x) = \mathbb{P}(\eta_t(x) = 1) = \rho_0.$$

Idea of proof. Using the generator and translation invariance,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\eta_t(0)] = \mathbb{E}[L\eta(0)] = \sum_{y\in\mathbb{Z}^d} p(0,y)\,\big(\mathbb{E}[\eta_t(y)] - \mathbb{E}[\eta_t(0)]\big) = 0,$$

since translation invariance gives $\mathbb{E}[\eta_t(y)] = \mathbb{E}[\eta_t(0)]$ for all y. Alternatively, the case $|B| = 1$ in Theorem 3.1 gives $\mathbb{P}(\eta_t(0) = 1) = \sum_y \mathbb{P}(A_t^{0,t} = y)\,\mathbb{P}(\eta_0(y) = 1) = \rho_0$. □

Thus the voter model is *neutral* in the sense that it does not bias the overall density of 1's or 0's, even though it generates non-trivial spatial correlations. Using Theorem 3.1

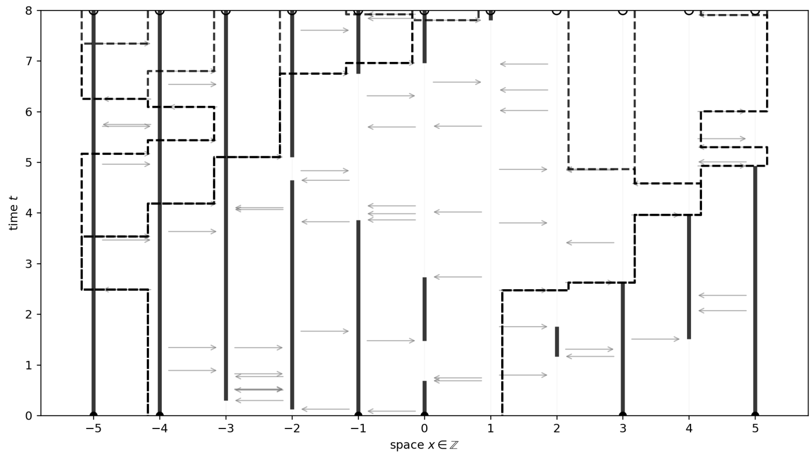

FIGURE 3. **Harris genealogy picture for the voter model and its coalescing dual.** Space is $x \in \mathbb{Z}$ (restricted here to $[-5, 5]$) and time is $t \in [0, T]$. Gray arrows are the Harris Poisson copying marks $y \to x$ (at an arrow time, site x updates by copying the state at y). The voter configuration $(\eta_t)_{0 \le t \le T}$ is shown by thick solid black vertical segments: a segment at site x indicates an interval of times on which $\eta_t(x) = 1$; black dots at $t = 0$ mark the initial ones. For each site x at time T, the dashed black path traces the *ancestral line* obtained by following arrows backward from (x, T) to time 0; these backward paths evolve as coalescing random walks. The small horizontal offset of dashed paths is for visual separation only.

with $|B| \ge 2$, correlation functions can be expressed in terms of coalescence probabilities; in dimensions $d \le 2$ this leads to clustering, while for $d \ge 3$ coexistence is possible, see [30].

Contact process: clustering and phase transitions. The contact process is a basic model for infection spread on $\mathbb{Z}^d$ [50, 67, 45]. Sites are either healthy (0) or infected (1); infected sites recover at rate 1, while healthy sites become infected at rate λ times the number of infected neighbours. For infection rates λ below a critical value λ_c the infection dies out, while for $\lambda > \lambda_c$ there exists a non-trivial invariant measure with sustained activity [50, 51, 30].

In the active phase, infections tend to occur in clusters: nearby sites have positively correlated occupation variables and

$$g_{\text{inv}}(x, y) = \frac{\mathbb{P}_{\text{inv}}(\eta(x) = 1, \eta(y) = 1)}{\mathbb{P}_{\text{inv}}(\eta(x) = 1)^2} > 1$$

for x, y close, where $\mathbb{P}_{\text{inv}}$ denotes the stationary law. Time-dependent two-point functions exhibit positive correlations at short spatial separations in the supercritical regime; see [45]. In this sense, the contact process provides a standard lattice example of clustering at the level of second-order structure.

Second-order comparison. These examples illustrate how g_t distinguishes inhibition and clustering mechanisms in lattice dynamics. Exclusion enforces a hard-core constraint and typically yields inhibition relative to product baselines; the contact process exhibits positive correlations in the active phase; the voter model preserves density under translation invariance and its correlations are governed by coalescence probabilities via the dual system.

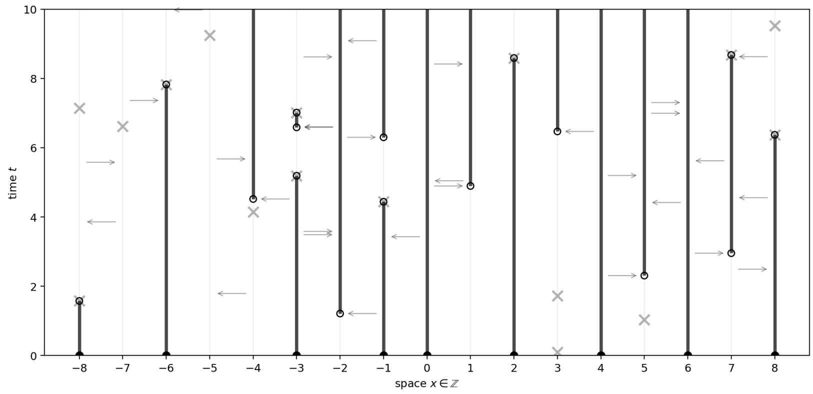

FIGURE 4. **Harris graphical construction for the one-dimensional contact process.** Space is $x \in \mathbb{Z}$ (restricted here to $[-8, 8]$) and time is $t \in [0, T]$. Gray arrows are Poisson infection-attempt marks on oriented nearest-neighbour edges $x \to y$, and gray "$\times$" marks are Poisson recovery events at sites. Given the marks and the initial configuration, the path $(\eta_t)_{t\in[0,T]}$ is obtained deterministically by applying the local rules in chronological order: at a recovery mark at (x, t) set $\eta_t(x) = 0$; at an infection arrow $x \to y$ at time t, set $\eta_t(y) = 1$ if $\eta_{t-}(x) = 1$ and $\eta_{t-}(y) = 0$. Black dots at $t = 0$ indicate the initially infected sites. Thick black vertical segments show time intervals on which $\eta_t(x) = 1$. Circled times mark events that actually changed the state (successful infections and effective recoveries).

We return to this second-order viewpoint for continuum systems and for coalescing/annihilating models, where it connects directly to the pair-correlation function $g_t(r)$ on $\mathbb{R}^d$.

3.2. **Continuum Birth–Death and Glauber Dynamics.** Continuum interacting particle systems evolve on the configuration space $\Gamma(\mathbb{R}^d)$ introduced in Subsection 1.1. A standard class is given by *spatial birth–death processes*, in which points are added and removed at configuration-dependent rates. This framework is closely connected with Gibbs measures via Papangelou conditional intensities and the Georgii–Nguyen–Zessin identity [40, 74, 41]. It also leads to evolution equations for factorial moment measures and for g_t; see [16, 64] (and the references therein).

Birth–death generators on configuration space. Let X_t be a Markov process on $\Gamma = \Gamma(\mathbb{R}^d)$ with generator L defined at least on local (cylinder) functions $F : \Gamma \to \mathbb{R}$ of the form

$$F(\gamma) = f\big(\langle\varphi_1, \gamma\rangle, \dots, \langle\varphi_k, \gamma\rangle\big), \qquad \langle\varphi, \gamma\rangle := \sum_{x\in\gamma} \varphi(x),$$

where $k \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^k)$, and $\varphi_i \in C_c^\infty(\mathbb{R}^d)$. A spatial birth–death generator has the standard form

$$(LF)(\gamma) = \int_{\mathbb{R}^d} \big[F(\gamma \cup \{x\}) - F(\gamma)\big]\, b(x, \gamma)\, \mathrm{d}x \;+\; \sum_{x\in\gamma} \big[F(\gamma \setminus \{x\}) - F(\gamma)\big]\, d(x, \gamma), \tag{1}$$

where $b(x, \gamma) \geq 0$ and $d(x, \gamma) \geq 0$ are the birth and death rates at location x in configuration γ; see [62, 34, 33] for constructions and analytic frameworks. Under locality and growth conditions ensuring non-explosion, one constructs a conservative Markov process solving the martingale problem for L. General existence and approximation results for spatial birth–death processes can be found in [38].

Gibbs point processes (finite-volume picture). A finite-volume Gibbs measure is obtained by weighting a reference Poisson process by an energy functional: configurations with lower interaction energy receive higher weight, and configurations with infinite energy have zero weight. In finite volume this yields an explicit density with respect to a Poisson process; see [40, 41, 20].

For simplicity, let S be a bounded state space; one may take $S = \Lambda \subset \mathbb{R}^d$ bounded Borel, or the flat torus $S = \mathbb{T}_L^d = (\mathbb{R}/L\mathbb{Z})^d$ to encode periodic boundary conditions. Write Γ_S for the space of (necessarily finite) point configurations in S.

Let $\Pi_{z,S}$ be the law of a homogeneous Poisson point process on S with intensity measure $z\,\mathrm{d}x$ ($z > 0$). Given a symmetric pair potential $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ (interpreted on the torus via periodic distances when $S = \mathbb{T}_L^d$), define the (grand-canonical) pair energy of a configuration $\gamma \in \Gamma_S$ by

$$H_S(\gamma) := \sum_{\{x,y\}\subset\gamma} \phi(x-y),$$

with the convention $H_S(\gamma) = +\infty$ if the sum diverges or any pair has $\phi(x-y) = +\infty$.

Definition 3.3 (Finite-volume (grand-canonical) Gibbs measure)**.** *Fix $z > 0$ and $\beta > 0$ (inverse temperature). The Gibbs point process on S associated with (z, β, ϕ) is the probability measure μ_S on Γ_S given by*

$$\mu_S(\mathrm{d}\gamma) := \frac{1}{Z_S} \exp\bigl(-\beta H_S(\gamma)\bigr)\, \Pi_{z,S}(\mathrm{d}\gamma), \qquad Z_S := \int_{\Gamma_S} \exp\bigl(-\beta H_S(\gamma)\bigr)\, \Pi_{z,S}(\mathrm{d}\gamma),$$

whenever $0 < Z_S < \infty$.

Remark 3.4 (What changes in infinite volume)**.** *On $S = \mathbb{R}^d$, one typically cannot write a* global *density with respect to a Poisson process on $\mathbb{R}^d$. Instead, infinite-volume Gibbs measures are defined via consistent finite-volume conditional distributions (with boundary conditions) or as thermodynamic limits; see* [40, 41, 20]*. What we use from the Gibbs formalism is the associated Papangelou conditional intensity (when it exists), which in the pair potential case takes the familiar form*

$$\lambda(x \mid \gamma) = z \exp\Bigl(-\beta \sum_{y\in\gamma} \phi(x-y)\Bigr),$$

interpreted as the energy increment of inserting a point at x.

Papangelou intensity and the GNZ identity. Let μ be a probability measure on Γ with finite local first moments. Define the *reduced Campbell measure* $C_\mu^!$ on $\mathbb{R}^d \times \Gamma$ by

$$C_\mu^!(A \times B) := \int_\Gamma \sum_{x\in\gamma} \mathbf{1}_A(x)\, \mathbf{1}_B(\gamma \setminus \{x\})\, \mu(\mathrm{d}\gamma).$$

If $C_\mu^! \ll \mathrm{d}x \otimes \mu$, its Radon–Nikodym derivative $\lambda(x \mid \gamma)$ is called the *Papangelou conditional intensity* of μ; equivalently, λ is characterized by the GNZ identity [40, 74, 20]:

$$\int_\Gamma \sum_{x\in\gamma} h(x, \gamma\setminus\{x\})\, \mu(\mathrm{d}\gamma) = \int_\Gamma \int_{\mathbb{R}^d} h(x,\gamma)\, \lambda(x \mid \gamma)\, \mathrm{d}x\, \mu(\mathrm{d}\gamma), \tag{2}$$

for all measurable $h \geq 0$.

Remark 3.5 (Intuition)**.** *The function $\lambda(\cdot \mid \gamma)$ is a conditional* rate *(intensity), not a probability density. Heuristically, for a small Borel set B and writing $\varkappa = \gamma|_{B^c}$,*

$$\mathbb{E}_\mu[\gamma(B) \mid \gamma|_{B^c} = \varkappa] \approx \int_B \lambda(x \mid \varkappa)\, \mathrm{d}x,$$

i.e. the expected number of points in B given the outside configuration is approximately the integral of the local insertion rate. The GNZ identity (2) *is the precise characterization.*

For Gibbs measures with pair potential ϕ, the Papangelou intensity takes the canonical form (with the convention $\exp(-\infty) = 0$)

$$\lambda(x \mid \gamma) = z \exp\Big(-\beta \sum_{y\in\gamma} \phi(x-y)\Big),$$

under standard assumptions ensuring the energy increment is well-defined; see [40, 41, 20].For hard-core or purely repulsive interactions, one expects short-range inhibition at equilibrium, e.g. $g(r) < 1$ for small r (and $g(r) = 0$ inside the hard-core radius, when present). In low-activity/high-temperature uniqueness regimes, this behaviour can often be justified quantitatively, whereas attractive regimes are more delicate and may exhibit phase coexistence [41, 20, 73, 4].

Glauber dynamics for Gibbs measures. Given a Gibbs measure μ with Papangelou intensity λ, a canonical reversible birth–death dynamics is the *(continuum) Glauber dynamics* with generator (1) and rates

$$b(x, \gamma) = \lambda(x \mid \gamma), \qquad d(x, \gamma) \equiv 1.$$

The invariance and reversibility of μ for this dynamics follow from the GNZ identity (the birth and death parts satisfy detailed balance under μ); see, e.g., [62, 34, 20]. Construction and well-posedness of such dynamics (under suitable conditions on ϕ and z) can be achieved by analytic semigroup or correlation-function methods [62, 34, 33].

Theorem 3.6 (Invariance and reversibility of Glauber dynamics)**.** *Assume that μ is a Gibbs measure on Γ with Papangelou intensity $\lambda(\cdot \mid \cdot)$, and that the birth–death dynamics with rates $b(x, \gamma) = \lambda(x \mid \gamma)$ and $d(x, \gamma) \equiv 1$ is well-posed as a conservative Markov process on Γ. Then μ is invariant for this dynamics, and the dynamics is reversible with respect to μ.*

Concrete sufficient conditions for well-posedness, invariance, and (in uniqueness regimes) exponential relaxation are given in [62, 34, 33, 69, 5].

Formally, the family of correlation functions $\{\rho_t^{(n)}\}_{n\geq1}$ satisfies an infinite hierarchy of evolution equations (cf. Subsection 3.3). In this hierarchy, the birth mechanism depends on the interaction through $\lambda(x \mid \gamma)$, while the death term contributes a linear decay.
Examples. The following cases indicate how the choice of rates affects g_t.

- *Immigration–death (constant rates).* If $b(x, \gamma) \equiv z$ and $d(x, \gamma) \equiv 1$, then the process is an immigration–death dynamics. It has a unique invariant law, namely the homogeneous Poisson process of intensity z [16, 64].Consequently, at stationarity one has $g(r) \equiv 1$. For general initial laws, g_t need not equal 1 at finite times; under mild assumptions one has $g_t \to 1$ as $t \to \infty$; see [16].
- *Repulsive Gibbsian models.* For hard-core or purely repulsive interactions, equilibrium typically exhibits short-range inhibition, e.g. $g(r) < 1$ for small r (and $g(r) = 0$ inside a hard-core radius). In a uniqueness/high-temperature regime, Glauber dynamics converges to the Gibbs equilibrium; in particular, $g_t(r) \to g_\mu(r)$ as $t \to \infty$, and $g_t(r)$ develops the corresponding short-range inhibition. [41, 20, 73, 4].
- *Cox-type clustering from random environments.* If births occur in a (static) random environment, e.g. $b(x, \gamma) = \Lambda(x)$ for a random field $\Lambda \geq 0$ and $d \equiv 1$, then conditional on Λ the invariant law is Poisson with intensity Λ, hence the unconditional invariant law is a Cox process. If Λ is stationary with $\rho := \mathbb{E}\Lambda(0) \in$

$(0,\infty)$ and $\mathbb{E}[\Lambda(0)\Lambda(h)] < \infty$, then the pair-correlation function satisfies

$$g(h) = 1 + \frac{\mathrm{Cov}(\Lambda(0), \Lambda(h))}{\rho^2},$$

so nonnegative covariance at scale h implies $g(h) \geq 1$ at that scale [16, 73].

The dependence of $b(x,\gamma)$ on γ governs whether the evolution favours inhibition or clustering at short range, and the BBGKY-type hierarchy describes the resulting dynamics of $\{\rho_t^{(n)}\}_{n\geq 1}$.

3.3. **Correlation Function Hierarchies.** For spatial birth–death dynamics, the family $(\rho_t^{(n)})_{n\geq 1}$ formally evolves according to an infinite system of coupled equations linking neighbouring correlation orders. We call this system a BBGKY-type correlation hierarchy to emphasize the inter-order coupling $\rho_t^{(n)} \leftrightarrow \rho_t^{(n\pm 1)}$. Here the hierarchy is obtained directly from the configuration-space generator, without invoking a kinetic scaling limit. Correlation hierarchies provide one analytic route to continuum birth–death dynamics and their correlation structure; see [36, 34, 35].

Let $(X_t)_{t\geq 0}$ be a Markov process on $\Gamma(\mathbb{R}^d)$ with (formal) birth–death generator L as in (1). As test functionals, fix $n \geq 1$ and consider the polynomial (factorial) observables

$$F_f(\gamma) := \sum_{(x_1,\dots,x_n)\in\gamma^n_{\neq}} f(x_1,\dots,x_n),$$

where $f : (\mathbb{R}^d)^n \to \mathbb{R}$ is symmetric, bounded, measurable, and compactly supported, and $\gamma^n_{\neq}$ denotes ordered n-tuples of pairwise distinct points of γ. Assuming that the n-point correlation function $\rho_t^{(n)}$ exists, the defining relation yields

$$\mathbb{E}\big[F_f(X_t)\big] = \int_{(\mathbb{R}^d)^n} f(x_1,\dots,x_n)\, \rho_t^{(n)}(x_1,\dots,x_n)\, dx_1\cdots dx_n.$$

Formally differentiating in t and invoking the Kolmogorov forward identity

$$\partial_t \mathbb{E}[F_f(X_t)] = \mathbb{E}[(LF_f)(X_t)]$$

(whenever $F_f \in \mathrm{Dom}(L)$ and differentiation may be interchanged with expectation), one obtains an evolution equation for $\rho_t^{(n)}$ whose right-hand side couples to adjacent orders.

At the formal level, for general local rates $b(x,\gamma)$ and $d(x,\gamma)$ the nth equation takes the schematic form

$$\partial_t \rho_t^{(n)} = A^{(n)}\rho_t^{(n)} \;+\; B^{(n)}\big(\rho_t^{(n+1)}\big) \;+\; D^{(n)}\big(\rho_t^{(n-1)}\big), \tag{3}$$

where: (i) $A^{(n)}$ collects terms depending only on the n tagged points (for instance, configuration-independent death contributions), (ii) $B^{(n)}$ arises from configuration dependence of b and/or d, which, after rewriting in correlation form, introduces one additional point and hence $\rho_t^{(n+1)}$, and (iii) $D^{(n)}$ corresponds to configuration-independent immigration, which couples the nth equation to $\rho_t^{(n-1)}$.

In concrete models, (3) is obtained by applying L to the observables F_f and identifying the resulting expressions with the correlation functions; see [36, 34].

Example: immigration–death hierarchy. Consider the spatial immigration–death process with constant rates $b(x,\gamma) \equiv z$ and $d(x,\gamma) \equiv 1$.

Proposition 3.7 (Hierarchy for immigration–death)**.** *Assume the correlation functions $(\rho_t^{(n)})_{n\geq 0}$ exist (with the convention $\rho_t^{(0)} \equiv 1$). Then for each $n \geq 1$,*

$$\partial_t \rho_t^{(n)}(x_1,\dots,x_n) = -n\, \rho_t^{(n)}(x_1,\dots,x_n) \;+\; z\sum_{i=1}^{n} \rho_t^{(n-1)}(x_1,\dots,\widehat{x}_i,\dots,x_n), \tag{4}$$

where $\widehat{x_i}$ means that x_i is omitted. In particular, the stationary solution of (4) is given by $\rho^{(n)}(x_1,\dots,x_n) \equiv z^n$. Moreover, under standard hypotheses ensuring well-posedness and convergence of correlations (e.g. sufficiently integrable initial factorial moments so that the hierarchy is valid for all n), one has $\rho_t^{(1)}(x) \to z$ and $\rho_t^{(n)}(x_1,\dots,x_n) \to z^n$ as $t \to \infty$.

Sketch of proof. Apply L to $F_f(\gamma) = \sum_{\gamma^n_{\neq}} f$: each of the n tagged points is removed at rate 1 (yielding the term $-n\rho_t^{(n)}$), while immigration at rate $z\,dx$ creates a new point which can play the role of any one of the n tagged positions (yielding the sum over i and $\rho_t^{(n-1)}$). The resulting identity follows by inserting F_f into the generator formula and matching terms against the defining relation for $\rho_t^{(n)}$; see [15, 64].

For general nonnegative local rates $b(x,\gamma)$ and $d(x,\gamma)$, the explicit form of (3) depends on the chosen interaction mechanism. For continuum Glauber dynamics with $d \equiv 1$ and $b(x,\gamma)$ given by a Papangelou intensity (as in Subsection 3.2), the configuration dependence of b induces the $(n+1)$-order term in (3), hence coupling to $\rho_t^{(n+1)}$. The resulting explicit hierarchy (written in terms of the dual operator on correlation functions) is developed in [34, 33].

Because the hierarchy is infinite, one often introduces a closure to obtain a finite system for selected low orders. For instance, the mean-field ansatz $\rho_t^{(2)}(x,y) \approx \rho_t^{(1)}(x)\rho_t^{(1)}(y)$ is, in general, a heuristic replacement; absent a regime implying propagation of chaos, it should be treated as an additional modelling assumption [82]. More refined closures (e.g. Kirkwood-type superposition) replace $\rho_t^{(3)}$ by combinations of $\rho_t^{(2)}$ and arise, for example, in spatial ecological modelling [30].

Since g_t is determined by $\rho_t^{(2)}$, second-order information is governed by the $n = 2$ equation of the hierarchy. In general, the $n = 2$ equation is not closed, as it involves $\rho_t^{(3)}$; analysis therefore either exploits integrable structures (e.g. determinantal/Pfaffian models) or supplies a closure justified by an additional approximation scheme; cf. [30, 41, 34].

The hierarchy makes explicit how local birth–death mechanisms enter the evolution of the correlation functions. We return to $g_t(r)$ as a second-order summary statistic distinguishing inhibition, aggregation, and coalescence signatures.

4. Coalescing systems and Pfaffian kernels

The algebraic point-process structures from Section 2 also arise in certain *stochastic particle systems*. We focus on *coalescing* and *annihilating* systems: particles evolve by random motion, and upon collision the interacting pair is replaced either by a single particle (coalescence) or by the empty configuration (annihilation). In one dimension, the fixed-time configuration at any $t > 0$ admits a *Pfaffian point-process* description; this is developed from the work of Arratia [1, 2], with further analysis by Bramson–Griffeath [11, 10] and a systematic Pfaffian formulation by Tribe–Zaboronski [85].

The Pfaffian description yields explicit formulas for $\rho_t^{(n)}$ and, in particular, for $\rho_t^{(1)}$, $\rho_t^{(2)}$, and the associated pair-correlation ratio. These formulas quantify short-range inhibition relative to Poisson and permit the evaluation of collision-related probabilities (e.g. meeting and survival events). Moreover, dualities between coalescing and annihilating dynamics enter at the level of the kernel and translate into identities for correlation functions. This section therefore, connects the algebraic viewpoint of Section 2 with the measure-dependent stochastic flows discussed later in Section 5.

We begin with coalescing random walks on $\mathbb{Z}$ and their diffusive scaling limits, and then pass to coalescing Brownian motions and the Arratia flow. We emphasize fixed-time second-order quantities (pair correlations and mixing), semigroup descriptions of

m-point motions, and discrete-to-continuum limits. The Pfaffian fixed-time structure needed later is collected in Subsection 4.5.

Coalescing and annihilating random-walk systems are standard examples in the theory of interacting particle systems [81, 22, 46, 66]. Although the configuration-valued process is Markovian, the motion of a tagged particle is not autonomous because collisions induce pathwise dependence.

Annihilating random walks were introduced and studied in [31, 44]. On $\mathbb{Z}$, particles perform independent random walks and annihilate upon collision (in discrete time one may equivalently formulate annihilation via parity at each site after the move). Questions raised in this line of work include almost-sure hitting and recurrence properties under dense initial conditions [31, 44].

Coalescing random walks appear naturally as duals of the voter model [52, 66]. In CRW, colliding particles merge and subsequently evolve as a single random walk [44]. Arratia constructed the continuum counterpart, coalescing Brownian motions on $\mathbb{R}$, and the associated Arratia flow [1, 85].

Historical research has focused on the transition from infinite configurations to equilibrium and the resulting spatial patterns:

- Asymptotic density. A basic question concerns the decay of the occupation probability p_t (or particle density) as $t \to \infty$. In transient dimensions $d \geq 3$ one obtains $p_t \asymp t^{-1}$ (with model-dependent constants), whereas in $d = 1, 2$ recurrent collisions produce slower decay and logarithmic corrections in $d = 2$ [10, 2, 70].
- Coming down from infinity. Under dense entrance laws (e.g. full occupancy in lattice systems or maximal entrance laws in continuum limits), the configuration becomes locally finite for each $t > 0$ [23, 78].
- Clustering versus dispersion. Via voter-model duality, one relates coalescence to clustering phenomena for spins, while the particle system itself becomes increasingly sparse at large times in low dimensions [11, 83].

4.1. **Coalescing random walks on $\mathbb{Z}$.** Coalescing random walks on $\mathbb{Z}$ are defined by independent nearest-neighbour random-walk motion up to collision, with instantaneous merging of any particles occupying the same site. In one dimension the recurrence of the walk produces (i) a density scale of order $n^{-1/2}$, (ii) negative correlations at fixed times (hence suppression of close pairs), and (iii) a non-trivial diffusive scaling limit to coalescing Brownian motions (Arratia flow). We recall the construction and the coalescing–annihilating relations used later in Pfaffian formulas.

To exclude synchronous "crossing without meeting" in discrete time, we impose the parity convention and start from the even sublattice. That is, we take the initial configuration $\eta_0 \subset 2\mathbb{Z}$. Then every particle position has the same parity as the time: at time n all particles lie in $2\mathbb{Z} + n$, so two trajectories cannot swap positions in one step without actually meeting.

We define the dynamics from a single i.i.d. space–time arrow field on the lattice

$$\mathcal{L} := \{(n, x) \in \mathbb{N}_0 \times \mathbb{Z} : x + n \text{ is even}\}.$$

Let

$$\{\xi(n, x)\}_{(n,x)\in\mathcal{L}}, \qquad \xi(n, x) \in \{-1, +1\}, \qquad \mathbb{P}(\xi(n, x) = \pm 1) = \tfrac{1}{2},$$

be independent over $(n, x) \in \mathcal{L}$. Given a starting site $x \in 2\mathbb{Z}$, define the (forward) path $(X^x_n)_{n\geq 0}$ recursively by

$$X^x_0 = x, \qquad X^x_{n+1} = X^x_n + \xi(n, X^x_n), \qquad n \geq 0,$$

where $\xi(n, X^x_n)$ is well-defined because $(n, X^x_n) \in \mathcal{L}$ for all n.

If two paths meet at some space-time point, they subsequently follow the same future arrows and therefore coincide thereafter; this encodes the coalescence rule.

For a (finite or infinite) initial configuration $\eta_0 \subset 2\mathbb{Z}$, the coalescing system at time n is the random subset

$$Y_n := \{X_n^x : \ x \in \eta_0\} \subset \mathbb{Z},$$

where multiplicities are ignored because merged paths are identical. We view Y_n as a simple point configuration on $\mathbb{Z}$ (supported on $2\mathbb{Z} + n$).

For translation-invariant dense initial laws, the model quickly becomes sparse at large times: coalescences occur repeatedly in and this forces a $n^{-1/2}$ density scale and a reduction in the frequency of close pairs.

Theorem 4.1 (Density decay and negative correlations for coalescing random walks)**.** *Let $(Y_n)_{n\geq 0}$ be the discrete-time nearest-neighbour coalescing system on $\mathbb{Z}$ started from full occupancy ($\eta_0 = \mathbb{Z}$). Then:*

(1) [11] *There exists $c > 0$ such that, as $n \to \infty$,*

$$\mathbb{P}(0 \in Y_n) \sim c\, n^{-1/2}.$$

For the unit-step simple symmetric walk , the constant is $c = 1/\sqrt{\pi}$. Consequently, for any interval $I \subset \mathbb{Z}$ of length L,

$$\mathbb{E}\big[|Y_n \cap I|\big] = L\,\mathbb{P}(0 \in Y_n) \asymp L\, n^{-1/2} \qquad \text{as } n \to \infty.$$

(2) [2] *For every pair of disjoint finite sets $A, B \subset \mathbb{Z}$ and every $n \geq 0$,*

$$\mathbb{P}(A \cup B \subset Y_n) \leq \mathbb{P}(A \subset Y_n)\,\mathbb{P}(B \subset Y_n).$$

In particular, for distinct $x, y \in \mathbb{Z}$,

$$\mathbb{P}(x, y \in Y_n) \leq \mathbb{P}(x \in Y_n)\,\mathbb{P}(y \in Y_n),$$

so the pair-correlation ratio

$$g_n(x, y) := \frac{\mathbb{P}(x, y \in Y_n)}{\mathbb{P}(x \in Y_n)\,\mathbb{P}(y \in Y_n)}$$

satisfies $g_n(x, y) \leq 1$ for all $n \geq 0$ and all $x \neq y$ (when the denominator is nonzero).

Discussion. The exponent $1/2$ in part (1) is the one-dimensional signature of diffusive motion combined with frequent merging: many potential neighbours eventually become the same particle, so the configuration thins out at the $n^{-1/2}$ rate. Part (2), going back to Arratia [2], says that occupancy events at different sites are negatively correlated. Part (2) implies negative association of occupancy events and, in particular, a pointwise bound $g_n(x, y) \leq 1$ for $x \neq y$ whenever the denominator is nonzero. Interpreted at second order, this expresses suppression of near-neighbour pairs relative to an independent (Poisson/Bernoulli) reference at matched intensity.

From the point-process viewpoint, Y_n is a random subset of $\mathbb{Z}$ whose correlation functions can be studied using duality with annihilating systems and, in one dimension, non-intersecting path representations of Karlin–McGregor type [59] and the Pfaffian analysis of [85]. Under diffusive rescaling, the discrete system converges to coalescing Brownian motions (and, under dense entrance laws, to the Arratia flow), and at fixed times the limiting correlation functions admit Pfaffian representations; see Subsection 4.5.

Using the same arrow field $\{\xi(n, x)\}$, one defines an *annihilating* nearest-neighbour system $(A_n)_{n\geq 0}$ as follows. Start from $A_0 \subset \mathbb{Z}$, move each particle from x at time n to $x + \xi(n, x)$ at time $n + 1$, and then *annihilate in pairs* at every site (equivalently: after the move, keep exactly one particle at a site if and only if an odd number of particles arrived there). This produces a simple configuration $A_n \subset \mathbb{Z}$ for each n.

Two relations between (A_n) and (Y_n) will be used repeatedly. *(i) Thinning relation.* Let $\Theta(\eta)$ denote a 1/2-thinning of a configuration η, obtained by keeping each particle independently with probability 1/2. A fixed-time identity, stated and used systematically in [85] (and appearing already in Arratia's scaling-limit work), is that thinning intertwines coalescence and annihilation:

$$\big(A_n \text{ started from } \Theta(\eta_0)\big) \overset{d}{=} \Theta\big(Y_n \text{ started from } \eta_0\big), \qquad n \geq 0.$$

This is an equality in distribution at each fixed time n: thinning may be applied either at time 0 (followed by annihilation dynamics) or at time n (after coalescence dynamics), with identical resulting laws.

(ii) Parity/interval observables. For integers $a < b$ and a configuration $\eta \subset \mathbb{Z}$, write $N_\eta([a,b)) := |\eta \cap [a,b)|$ and define the spin variable

$$\Sigma(\eta; a, b) := (-1)^{N_\eta([a,b))}.$$

In the Pfaffian approach of [85], one combines:

- duality identities that convert certain coalescing *empty-interval* events into events for a related annihilating system started from the interval endpoints, and
- the thinning relation above, which connects annihilating parity quantities to coalescing occupation/emptiness quantities.

As a result, many multi-point probabilities for the coalescing configuration (and hence its correlation functions and pair-correlation ratios g_n) can be reduced to finite-dimensional computations for these parity/interval observables. This reduction leads to explicit Pfaffian formulas for fixed-time laws as developed in [85].

4.2. **Coalescing Brownian motions and Arratia flow.** Coalescing Brownian motions (CBM) arise as diffusive limits of coalescing random walks. The Arratia flow constructed in [1] is a stochastic flow of maps on $\mathbb{R}$ in which trajectories coalesce upon meeting. We summarize the scaling construction and two analytic viewpoints used later: (i) discrete-time Gaussian approximations and (ii) semigroups for m-point motions.

Starting from coalescing random walks on $\mathbb{Z}$, one can define for each $n \in \mathbb{N}$ a rescaled process by

$$X_t^{(n)}(x) := \frac{1}{\sqrt{n}}\, X_{nt}(x\sqrt{n}),$$

appropriately interpolated in space and time. Arratia [1] constructs a limiting stochastic flow $x(u,t)$ on $\mathbb{R}$ such that for each finite collection $u_1 < \cdots < u_m$ the joint law of

$$X_t^{(n)}(u_1), \ldots, X_t^{(n)}(u_m)$$

converges to that of coalescing Brownian motions $(x(u_1,t), \ldots, x(u_m,t))$ started from $(u_1, \ldots, u_m)$. The mapping $(u,t) \mapsto x(u,t)$ is almost surely non-decreasing in u for each fixed t and continuous in t for each fixed u; the coalescing property says that $x(u_i,t) = x(u_j,t)$ for all $t \geq \tau_{ij}$ once two trajectories have met.

At any fixed time $t > 0$ the set of images

$$X_t := \{x(u,t) : u \in \mathbb{R}\}$$

is almost surely a locally finite point configuration in $\mathbb{R}$ whose law encodes the entire collision history up to time t. The second-order structure of X_t — intensity, pair correlations, and Palm measures — reflects both the diffusive spreading of Brownian motion and the recurrent coalescence in one dimension.

A useful comparison object for the Arratia flow is a system of non-intersecting Brownian motions. For m independent Brownian motions started from strictly ordered positions $u_1 < \cdots < u_m$, the event that no collisions occur up to time t has probability described by the Karlin–McGregor determinant [59], and the joint density of the non-intersecting system at time t is given by

$$\det\big[p_t(u_i, y_j)\big]_{i,j=1}^m,$$

where p_t is the one-dimensional Brownian transition density.

Although the Arratia flow allows coalescence rather than forbidding collisions, the Karlin–McGregor structure remains an important analytic ingredient: it appears in Green functions and boundary value problems for the m-point semigroups and in the derivation of Pfaffian kernels for fixed-time distributions.

4.3. **Discrete-time Gaussian approximation and disordering.** The Arratia flow has two characteristic geometric properties: (i) particles coalesce upon meeting, and (ii) the flow is order-preserving: if $u_1 < u_2$ then $x(u_1, t) \le x(u_2, t)$ for all $t \ge 0$ almost surely. Equivalently, at any fixed time $t > 0$ the two-point law of $(x(u_1, t), x(u_2, t))$ is supported on the half-space $\{x_1 \le x_2\}$. The following discrete-time approximation, due to [75, 42], illustrates how this property emerges from a Gaussian scheme and provides quantitative control of "disordering" events.

For each $n \ge 1$ consider a sequence of independent stationary Gaussian processes $\{\xi_k^n : k = 1, \dots, n\}$ on $\mathbb{R}$, with zero mean and covariance function Γ_n. Define random maps $\{x_k^n : k = 0, \dots, n\}$ on $\mathbb{R}$ by the recursion

$$x_0^n(u) = u, \qquad x_{k+1}^n(u) = x_k^n(u) + \frac{1}{\sqrt{n}} \xi_{k+1}^n\big(x_k^n(u)\big), \quad k = 0, \dots, n-1. \tag{5}$$

Let $x_n(u, \cdot)$ be the piecewise linear interpolation of $x_k^n(u)$ on the interval $[0, 1]$ with mesh size $1/n$.

Theorem 4.2 ([75]). *Assume that*

$$\Gamma_n(0) = 1, \qquad C_n := \sup_{x \ne 0} \frac{2 - 2\Gamma_n(x)}{x^2} < \infty,$$

and that for each $\delta > 0$, $\sup_{\mathbb{R}\setminus[-\delta,\delta]} |\Gamma_n(x)| \to 0$ as $n \to \infty$. Then for every $m \in \mathbb{N}$ and ordered $u_1 < \cdots < u_m$ the finite-dimensional distributions of

$$\{x_n(u_1, \cdot), \dots, x_n(u_m, \cdot)\}$$

converge to those of the m-point motions of the Arratia flow as $n \to \infty$.

Fix $u_1 < u_2$ and consider the two-point motion

$$y_n(t) = x_n(u_2, t) - x_n(u_1, t), \qquad t \in [0, 1].$$

For the limiting Arratia flow one has $y(t) \ge 0$ for all $t \ge 0$ almost surely. In the discrete scheme (5), however, the order of the two particles may temporarily be reversed. To measure this, [42] introduces the functional

$$\Phi_n := \int_0^1 \mathbf{1}_{\{y_n(s)<0\}} \, \mathrm{d}s,$$

which represents the total amount of time in $[0, 1]$ during which the particles are in the "wrong" order. Thus $\Phi_n > 0$ is the event that a disordering occurs at some time, and Φ_n quantifies its duration.

The process y_n is equidistributed with the solution of an Euler-type scheme for the one-dimensional SDE

$$\mathrm{d}\tilde{y}_n(t) = B_n(\tilde{y}_n(t)) \, \mathrm{d}w(t), \qquad \tilde{y}_n(0) = u_2 - u_1 > 0,$$

where w is a standard Brownian motion and $B_n(u) = \sqrt{2 - 2\Gamma_n(u)}$.

Theorem 4.3 ([42]). *Under the assumptions of Theorem 4.2 the probability of any disordering satisfies*

$$\mathbb{P}\{\Phi_n > 0\} \le n\, F\left(\sqrt{\frac{n}{C_n}}\right),$$

where $F(x) = \int_x^\infty (2\pi)^{-1/2} e^{-u^2/2}\,du$ *is the standard Gaussian tail. Under additional monotonicity and lower-boundedness assumptions on* B_n*, for each fixed* $\varepsilon > 0$ *there exist* $K > 0$ *and a sequence* $A_n \to A_\infty > 0$ *such that*

$$\mathbb{P}\{\Phi_n > \varepsilon\} \;\geq\; F\left(K\sqrt{\frac{n}{C_n}}\right) A_n.$$

In particular,

$$\limsup_{n\to\infty} \frac{2C_n}{n} \log \mathbb{P}\{\Phi_n > 0\} \leq -1, \qquad \liminf_{n\to\infty} \frac{2C_n}{n} \log \mathbb{P}\{\Phi_n > \varepsilon\} \geq -K^2.$$

The functional Φ_n is a purely two-point quantity: it integrates over time the indicator that the pair $(x_n(u_1,t), x_n(u_2,t))$ lies in the region $\{x_1 > x_2\}$. The bounds above show that, under the assumptions on Γ_n, the law of the two-point motion concentrates exponentially fast (in the scale n/C_n) on the ordered half-space $\{x_1 \leq x_2\}$. In the limit $n \to \infty$ the two-point distribution is supported entirely on $\{x_1 \leq x_2\}$, as in the Arratia flow. Thus Theorems 4.2–4.3 give a quantitative pre-limit control on the support of the two-point correlation measure: the "forbidden" region $\{x_1 > x_2\}$ carries vanishing mass, with explicit exponential rate.

4.4. **Semigroups of m-point motions and binary forests.** A complementary analytic description of the Arratia flow is given by the m-point motion semigroups. This viewpoint makes the genealogy of collisions explicit and provides a natural bridge to determinantal and Pfaffian structures.

Let $x(u,t)$ be the Arratia flow on $\mathbb{R}$ and, for $u = (u_1, \dots, u_m)$ with $u_1 < \dots < u_m$, let

$$X(u,t) = (x(u_1,t), \dots, x(u_m,t))$$

denote the m-point motion started from u. The family of m-point motions can be constructed as the coalescing version of the product Brownian semigroup, yielding a compatible family of Feller semigroups $(Q_{m,t})_{m\geq 1}$ on the ordered simplex

$$\Delta_m := \{u \in \mathbb{R}^m : u_1 \leq \dots \leq u_m\}.$$

For $f \in C_0(\Delta_m)$ we write

$$Q_{m,t} f(u) := \mathbb{E}\big[f(X(u,t))\big], \qquad u \in \Delta_m.$$

The object of interest is the generator A_m of $(Q_{m,t})$ and the way coalescence is encoded in its domain and boundary conditions.

The main structural result of [43] is the identification of a natural core D_m for A_m on which the generator acts as the free Laplacian, with all coalescence encoded in the boundary behaviour of the semigroup rather than in explicit drift terms.

Let

$$C_0^2(\Delta_m) := \Big\{ f \in C^2(\Delta_m) : f(x) \to 0,\ \partial^2_{x_i x_j} f(x) \to 0 \text{ as } \|x\| \to \infty \Big\},$$

and define

$$D_m := \Big\{ f \in C_0^2(\Delta_m) : \partial^2_{x_i x_j} f(x)\, \mathbf{1}_{\{x_i = x_j\}}(x) = 0 \text{ for all } i \neq j \Big\}.$$

Theorem 4.4 ([43]). *The set* D_m *is dense and invariant under* $Q_{m,t}$*, hence a core for the generator* A_m*. Moreover, for all* $f \in D_m$*,*

$$A_m f(u) = \frac{1}{2} \Delta f(u), \qquad u \in \Delta_m,$$

where Δ *is the usual Laplacian on* $\mathbb{R}^m$ *restricted to* Δ_m*. In particular, the singular coalescing interaction does not appear as an explicit term in* A_m*; it is entirely encoded in the domain of* A_m *and in the boundary conditions at coalescence hyperplanes.*

The behaviour of $Q_{m,t}$ on the boundary $\partial\Delta_m$ reflects the fact that, once two coordinates have coalesced, the system continues as an $(m-1)$-point motion with merged starting points. For $1 \le i \le m-1$ define

$$S_i^m := \{u \in \partial\Delta_m : u_i = u_{i+1}\},$$

and the projection and embedding maps

$$\pi_i(u_1, \dots, u_m) := (u_1, \dots, u_i, u_{i+2}, \dots, u_m) \in \Delta_{m-1},$$

$$\pi_i^{-1}(v_1, \dots, v_{m-1}) := (v_1, \dots, v_i, v_i, v_{i+1}, \dots, v_{m-1}) \in \Delta_m.$$

Then on the boundary pieces S_i^m one has the compatibility condition

$$Q_{m,t} f(u) = Q_{m-1,t}\Big(f \circ \pi_i^{-1}\Big)(\pi_i u), \qquad u \in S_i^m.$$

Thus $Q_{m,t} f$ solves the heat equation in the interior,

$$\partial_t Q_{m,t} f(u) = \frac{1}{2}\Delta Q_{m,t} f(u), \qquad u \in \mathring{\Delta}_m,$$

with initial condition $Q_{m,0} f = f$ and boundary condition obtained by recursively gluing in $(m-1)$-point semigroups along the collision faces. This yields a system of coupled boundary value problems for $Q_{k,t}$, $k = 1, \dots, m$, in which coalescence appears as a reduction in dimension at the boundary rather than as an interior interaction term.

To solve this system explicitly, [43] introduces a combinatorial description in terms of binary forests. Each forest encodes a hierarchy of pairwise coalescences: its leaves represent the m initial particles, its internal vertices record collision events, and its roots correspond to the surviving lineages. Time parameters are attached to the levels of the forest and space variables to all vertices.

Analytically, the interior dynamics are governed by the Karlin–McGregor determinantal Green function

$$\mathcal{G}_m(u, y, t, s) = \det\big[p_{t-s}(u_i, y_j)\big]_{i,j=1}^m,$$

where p_t is the one-dimensional Brownian transition density. The boundary conditions give rise to normal derivatives of $\mathcal{G}_m$ along the collision faces; these can be written as finite sums of products of heat kernels with algebraic prefactors. Each such term corresponds to a binary merging of two neighbouring coordinates and naturally matches a binary branching in the genealogical forest.

Combining these ingredients, $Q_{m,t} f(u)$ admits an expansion of the form

$$Q_{m,t} f(u) = \int_{\Delta_m} f(y)\, \mathcal{G}_m(u, y, t, 0)\, \mathrm{d}y \;+\; \sum_{T \in \mathcal{T}^m} (-1)^{\varepsilon(T)} I_T(f, u, t),$$

where $\mathcal{T}^m$ is a finite set of binary forests on m leaves, $\varepsilon(T)$ is the parity of edge intersections in T, and each term I_T is an explicit multiple integral over intermediate times and positions, with integrand given by a product of heat kernels (one per edge of T) and f composed with the successive coalescence maps associated with T.

From the perspective of this survey, this representation shows how multi-point structure in the Arratia flow is built from Brownian kernels and coalescence events: the interior evolution is diffusive (Laplacian), while all singular interaction is pushed to the boundary via coalescence conditions. This will match the Pfaffian picture in Subsection 4.5, where the same genealogy is encoded algebraically in explicit kernels for correlation functions.

4.5. **Pfaffian structure and second-order limit theorems.** We now return to the Pfaffian algebraic structure of coalescing systems and its implications for second-order quantities. In one dimension, coalescing random walks and coalescing Brownian motions (and hence the Arratia flow) have fixed-time laws that are Pfaffian point processes with explicitly computable antisymmetric kernels. This structure controls pair correlations, inhibition zones, and mixing, and it suggests central limit behaviour for linear statistics of the Arratia point process.

Throughout this subsection we restrict to one-dimensional systems: coalescing random walks on $\mathbb{Z}$ and coalescing Brownian motions on $\mathbb{R}$. The Pfaffian structure is known rigorously for a class of translation-invariant initial configurations, including full occupancy, Bernoulli product measures, and certain Poisson entrance laws; see [85, 39] and the references therein.

Fix $t > 0$ and consider a coalescing system on $\mathbb{Z}$ or $\mathbb{R}$ started from a translation-invariant initial configuration. Let $\rho_t^{(n)}(x_1, \dots, x_n)$ denote the n-point correlation function of the configuration at time t (when defined).

For one-dimensional coalescing and annihilating random walks on $\mathbb{Z}$ started from dense initial states, Tribe and Zaboronski [85] show that, at each fixed $t > 0$, the configuration is a Pfaffian point process. More precisely, there exists an antisymmetric 2×2 matrix-valued kernel

$$K_t(x, y) = \begin{pmatrix} K_t^{11}(x, y) & K_t^{12}(x, y) \\ K_t^{21}(x, y) & K_t^{22}(x, y) \end{pmatrix}, \qquad x, y \in \mathbb{Z},$$

such that for every $n \geq 1$ and distinct $x_1, \dots, x_n$,

$$\rho_t^{(n)}(x_1, \dots, x_n) = \mathrm{Pf}\big[K_t(x_i, x_j)\big]_{1 \leq i,j \leq n}.$$

The entries of K_t are expressed in terms of the heat kernel and its spatial derivatives. The proof combines the duality between coalescing and annihilating systems with Karlin–McGregor-type determinantal identities for non-intersecting paths [59], together with parity arguments that turn determinants into Pfaffians.

Passing to the diffusive scaling of Subsection 4.2, coalescing random walks converge to coalescing Brownian motions and, in the dense initial condition regime, to the Arratia flow [1, 2, 37, 14]. At each fixed $t > 0$ the image set

$$X_t := \{x(u, t) : u \in \mathbb{R}\}$$

is almost surely a locally finite subset of $\mathbb{R}$. For suitable translation-invariant initial laws, X_t again has a Pfaffian structure with correlation functions

$$\rho_t^{(n)}(x_1, \dots, x_n) = \mathrm{Pf}\big[K_t(x_i, x_j)\big]_{1 \leq i,j \leq n}, \qquad x_1 < \dots < x_n,$$

where K_t is now built from the one-dimensional Brownian transition density and its spatial derivatives; see [85] for precise formulas.

Theorem 4.5 (Pfaffian fixed-time laws for coalescing/annihilating systems, [85])**.** *Let $(\eta_t)_{t \geq 0}$ be a one-dimensional system of coalescing or annihilating random walks on $\mathbb{Z}$, or coalescing Brownian motions on $\mathbb{R}$, started from a translation-invariant dense initial law (full occupancy, product Bernoulli, or maximal entrance law). Then for every $t > 0$ the random configuration η_t is a Pfaffian point process: there exists an antisymmetric 2×2 kernel $K_t(x, y)$ such that*

$$\rho_t^{(n)}(x_1, \dots, x_n) = \mathrm{Pf}\big[K_t(x_i, x_j)\big]_{1 \leq i,j \leq n},$$

for all $n \geq 1$ and distinct $x_1, \dots, x_n$. The kernel K_t is expressible in terms of the corresponding (discrete or continuous) heat kernel and its spatial derivatives.

Two-point correlations, inhibition and mixing. From the kernel K_t one can extract the first and second correlation functions at time $t > 0$. For translation-invariant initial conditions, the one-point function is spatially homogeneous,

$$\rho_t^{(1)}(x) \equiv \rho_t,$$

with ρ_t decaying as $t^{-1/2}$ in dimension one, in agreement with the Bramson–Griffeath density asymptotics [12, 11, 2].

Proposition 4.6 (Two-point correlations and inhibition for Pfaffian coalescing systems, [85, 2])**.** *Assume the setting of Theorem 4.5 and that the initial law is translation invariant with finite, non-zero intensity. Let $\rho_t^{(1)}$ and $\rho_t^{(2)}$ denote the first and second correlation functions of η_t at time $t > 0$. Then:*

(1) *There exists a continuous, even function $H_t : \mathbb{R} \to [0, \infty)$ (respectively $H_t : \mathbb{Z} \to [0, \infty)$) such that, for all $x \neq y$,*

$$\rho_t^{(2)}(x, y) = \rho_t^2 - H_t(y - x).$$

In particular, the pair-correlation function

$$g_t(x, y) := \frac{\rho_t^{(2)}(x, y)}{\rho_t^{(1)}(x)\rho_t^{(1)}(y)} = 1 - \frac{H_t(y - x)}{\rho_t^2}$$

satisfies $g_t(x, y) \le 1$ for all $x \neq y$, with strict inhibition $g_t(x, y) < 1$ when $|x - y|$ is bounded and $\rho_t > 0$.

(2) *As $|y - x| \to \infty$ one has $H_t(y - x) \to 0$, and hence $\rho_t^{(2)}(x, y) \to \rho_t^2$ and $g_t(x, y) \to 1$. Thus the system is mixing in space at fixed time $t > 0$.*

4.6. **Gaussian structure in coalescing stochastic flows.** In [27] the authors study the Arratia flow. Despite the singular merging of paths, many natural linear and nonlinear functionals of the flow have *Gaussian* structure, and their covariance can be expressed explicitly in terms of the interaction kernel.

At each fixed time $t > 0$, the image set $x(\mathbb{R}, t)$ is almost surely countable and locally finite. One can therefore represent the random set of cluster positions by a point measure

$$N_t = \sum_{i \in \mathbb{Z}} \delta_{u_i}, \qquad \{u_i\}_{i \in \mathbb{Z}} = x(\mathbb{R}, t),$$

and study linear and higher-order functionals of N_t.

For suitable f one considers linear statistics of the form

$$\frac{1}{\sqrt{n}} \int_0^n f(u)\, N_t(\mathrm{d}u),$$

and proves a central limit theorem as $n \to \infty$. The limits define a generalized centered Gaussian element ζ on $L^2([0, 1])$ whose covariance operator can be written as $1 + \widetilde{G}_t$, where $\widetilde{G}_t$ is an integral operator built from the symmetrized second-order density of N_t:

$$\mathrm{Cov}\big(\zeta_f, \zeta_g\big) = (f, (1 + \widetilde{G}_t)g)_{L^2([0,1])}.$$

Thus the entire Gaussian limit field ζ is determined by the one-point intensity and the two-point correlation function of the Arratia point measure.

Higher-order statistics of N_t are encoded by *multiple integrals* with respect to the factorial powers $N_t^{(k)}$. For a symmetric kernel $f \in L^2([0, 1]^k)$ one can think of

$$\int f(x_1, \ldots, x_k)\, N_t^{(k)}(\mathrm{d}x_1 \cdots \mathrm{d}x_k)$$

as a sum of f over k-tuples of *distinct* clusters. These functionals capture k-point statistics of the coalescing system and are the natural analogue of multiple Wiener–Itô integrals for this non-Gaussian point process.

A central result of [27] is that, after suitable normalization and periodization, these multiple integrals have limits that can be described purely in terms of the Gaussian field ζ and the second-order kernel G_t.

Theorem 4.7 (Gaussian chaos limit for multiple integrals of the Arratia point measure [27, Thm 5.1])**.** *Let $t > 0$, let N_t be the point measure of clusters of the Arratia flow at time t, and let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric, 1-periodic in each coordinate, with $f|_{[0,1]^k} \in L^2([0,1]^k)$ and*

$$\int_0^1 f(x_1, \dots, x_k)\, \mathrm{d}x_j = 0, \qquad j = 1, \dots, k.$$

Denote by $N_t^{(k)}$ the kth factorial power of N_t, and let $I_{k,n}(f)$ be the rescaled k-fold multiple integral

$$I_{k,n}(f) := \frac{1}{n^{k/2}} \int_{[0,n]^k} f(x_1, \dots, x_k)\, N_t^{(k)}(\mathrm{d}x_1 \cdots \mathrm{d}x_k).$$

Then, as $n \to \infty$, $I_{k,n}(f)$ converges in distribution to a finite polynomial in the Gaussian element ζ whose coefficients are given by iterated integrals of the kernel G_t. In particular, if f is off-diagonal (vanishes whenever $x_i = x_j$ for some $i \neq j$), the limit lies in the kth Wiener chaos of ζ and can be written as

$$I_{k,n}(f) \implies A_f(\zeta, \dots, \zeta),$$

where A_f is the k-linear Hilbert–Schmidt form associated with f.

Informally, the theorem says that the appropriately rescaled k-point statistics of the Arratia point measure behave, in the large interval limit, like homogeneous polynomials in a Gaussian field whose covariance encodes only second-order information about N_t. The extra terms that appear in the general statement for non–off-diagonal f correspond to partial pairings of arguments and give lower-order chaos components weighted by iterated integrals of G_t.

As a result, even for the non-Gaussian, coalescing Arratia flow, high-order functionals of the point measure N_t have scaling limits that are fully determined by:

- a Gaussian generalized element ζ;
- its covariance operator $1 + \widetilde{G}_t$, built from the two-point density of N_t.

In other words, the *Gaussian chaos* limits for multiple integrals are governed entirely by the second-order geometry of the underlying coalescing system. This provides a second-order counterpart to the Pfaffian fixed-time description of Section 4: the Pfaffian kernel captures the exact finite-dimensional distributions of N_t, while the Gaussian chaos limit shows that, under suitable rescaling, the large-scale fluctuations of k-point statistics are universal and controlled by the same second-order kernel G_t.

The coalescing systems in this section involve interactions determined only by pairwise collisions; their fixed-time laws are Pfaffian and their second-order structure is controlled by inhibition and mixing. In the next section we turn to more general measure-dependent flows, where drift and noise depend on the entire empirical distribution and second-order quantities evolve under nonlinear feedback from μ_t.

5. Measure-Dependent Flows and Second-Order Geometry

The coalescing systems of Section 4 are driven by local binary interactions: particles merge upon collision, and the Pfaffian structure encodes the resulting fixed-time correlation functions. We next consider *measure-dependent* dynamics, in which the drift and noise acting on each particle depend on the empirical distribution of the full system.

This setting includes McKean–Vlasov diffusions, stochastic flows of maps with coefficients depending on the push-forward measure, and Harris-type flows driven by spatially correlated noise.

At each time t, the empirical measure

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)}$$

is a random probability measure (equivalently, a normalized point measure). If one wishes to use point-process terminology, it is often more natural to consider the associated counting measure $N_t^N := \sum_{i=1}^N \delta_{X_i^N(t)}$. In the classical propagation-of-chaos regime, the limit μ_t is deterministic.

The aim of this section is not to survey the full mean-field theory, but to highlight a few paradigmatic examples where second-order statistics can be described in a relatively explicit way and to connect them with the flow-based perspective of Dorogovtsev and co-authors [24, 25, 27, 26].

5.1. **McKean–Vlasov interactions.** McKean–Vlasov models describe mean-field interaction through coefficients depending on the empirical measure: each particle solves an SDE coupled to μ_t^N. In the large-N limit this yields a nonlinear evolution equation for μ_t; under additional assumptions, the centered $\sqrt{N}$-fluctuations converge to Gaussian fields whose covariance is characterized by linearized dynamics [82, 18, 19, 63, 47, 76].

Let $(W_i)_{1\le i\le N}$ be i.i.d. Brownian motions and consider

$$\mathrm{d}X_i^N(t) = b\big(X_i^N(t),\mu_t^N\big)\,\mathrm{d}t + \sigma\big(X_i^N(t),\mu_t^N\big)\,\mathrm{d}W_i(t), \qquad \mu_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_j^N(t)},$$

on a state space $E\subset\mathbb{R}^d$, where b and σ are assumed to be sufficiently regular (for instance Lipschitz in x and Lipschitz in the measure argument for a Wasserstein-type metric). Under such conditions, $\{X_i^N\}_{i=1}^N$ defines a Markov process on E^N whose interaction enters only through μ_t^N; see [32, 82].

Propagation of chaos asserts that, as $N\to\infty$, μ_t^N converges (in an appropriate sense) to a deterministic curve $(\mu_t)_{t\ge 0}$ identified as the law of the nonlinear (McKean–Vlasov) SDE

$$\mathrm{d}X(t) = b\big(X(t),\mu_t\big)\,\mathrm{d}t + \sigma\big(X(t),\mu_t\big)\,\mathrm{d}W(t), \qquad \mu_t = \mathrm{Law}(X(t)),$$

and equivalently as a weak solution of the nonlinear Fokker–Planck equation

$$\partial_t\mu_t = -\nabla\cdot\big(b(\cdot,\mu_t)\mu_t\big) + \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}\big(a_{ij}(\cdot,\mu_t)\,\mu_t\big), \qquad a=\sigma\sigma^\top,$$

with initial condition μ_0 given by the law of $X_i^N(0)$ [82, 63]. In particular, limiting finite-dimensional distributions are determined by μ_t, so that at the level of first moments the system behaves as if particles were independent.

The interacting nature of the system becomes visible at the second-order level through fluctuations of μ_t^N. Define the fluctuation field

$$\mathcal{Y}_t^N(\varphi) := \sqrt{N}\Big(\langle\varphi,\mu_t^N\rangle - \langle\varphi,\mu_t\rangle\Big), \qquad \varphi\in C_c^\infty(E).$$

Under additional regularity and non-degeneracy assumptions, one can show that $\{\mathcal{Y}_t^N\}_{t\ge 0}$ converges in law to a centered Gaussian process in a suitable negative Sobolev space. One convenient description is as a generalized Ornstein–Uhlenbeck process obtained from the linearization of the McKean–Vlasov dynamics; see [18, 19, 47]. In particular, for test functions φ, ψ,

$$\mathrm{Cov}\big(\mathcal{Y}_t(\varphi), \mathcal{Y}_s(\psi)\big) = \iint \varphi(x)\, \psi(y)\, C_{t,s}(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

where the kernel $C_{t,s}$ can be characterized as the solution of a linear (two-time) evolution associated with the linearized McKean–Vlasov operator along $(\mu_r)_{r\geq 0}$. The family $C_{t,s}$ plays the role of a time-dependent second-order correlation kernel for the fluctuation field.

Recent works address, among other topics, quantitative control of equilibrium fluctuations [47], parameter estimation from empirical covariances [76], and interactions depending on point-process functionals [61].

McKean–Vlasov systems provide a benchmark regime in which properly rescaled fluctuations are Gaussian in the limit, so second-order statistics determine the limiting field. This contrasts with Sections 2 and 4, where fixed-time laws are encoded by algebraic kernels; here the relevant second-order object is instead the solution of the linearized covariance evolution for $C_{t,s}$.

5.2. **Dorogovtsev measure-valued flows.** Dorogovtsev introduced and developed a class of *measure-valued Markov dynamics* in which the randomness enters through an underlying stochastic flow of maps and the measure evolves purely by *transport* (push-forward). Concretely, one starts with a random mapping $x(u, t)$, $u \in \mathbb{R}^d$, and induced measure-valued process by

$$\begin{cases} dx(u,t) = a\big(x(u,t), \mu_t, t\big)\, dt + \displaystyle\sum_{k=1}^{\infty} b_k\big(x(u,t), \mu_t, t\big)\, dw_k(t), \\ x(u,0) = u, \qquad u \in \mathbb{R}^d, \\ \qquad \mu_t = \mu_0 \circ X(\cdot, t)^{-1}, \end{cases} \tag{6}$$

where $w_k,\ k \geq 1$ are independent starnard Brownian motions and $a : \mathbb{R}^d \times \mathcal{M}_2 \to \mathbb{R}^d$, $b : \mathbb{R}^d \times \mathcal{M}_2 \to \mathbb{R}^{d\times d}$ are jointly Lipschitz.

This representation excludes "diffusive" creation of mass at new locations: the mass present at time t is exactly the image of the initial mass under the random map $x(\cdot, t)$. The central mathematical point is that one can build Markov dynamics directly at the level of $(\mu_t, x(\cdot, t))$, without passing through a finite-N empirical measure limit.

Dorogovtsev formulates an axiomatic notion of an *evolutionary* measure-valued process (also called an evolutionary Markov process) in a general metric state space (X, ρ) with $\mathcal{M}$ denoting the space of Borel probability measures on X. An evolutionary process consists of a pair (μ_t, f_t) where $f_t(\omega, \cdot) : X \to X$ is a random map and μ_t is a random probability measure such that, for every $t \geq 0$,

$$\mu_t = \mu_0 \circ f_t^{-1} \qquad (\mathrm{mod}\ \mathbb{P}), \tag{7}$$

together with measurability and a Markov property for the joint process $(\mu_t, f_t(u_1), \ldots, f_t(u_n))$ for every finite collection $u_1, \ldots, u_n \in X$. A convenient way to encode such models is through the *finite-particle transition probabilities* ("particle kernels")

$$Q(\mu, t, u_1, \ldots, u_n)(\Gamma)\,,$$

interpreted as the probability that particles started at $u_1, \ldots, u_n$ are in $\Gamma \subset X^n$ at time t under the ambient configuration μ. Dorogovtsev provides consistency and continuity conditions ensuring that such a family of kernels induces an evolutionary Markov process; see [24].

When the coefficient of the equation 6 are sufficiently smooth in space, the flow inherits Kunita-type regularity: $x(\cdot,t)$ becomes a C^2-diffeomorphism and the inverse flow is an Itô process in t, [24].

Even though for each fixed $\varepsilon > 0$ the Dorogovtsev flow $x^\varepsilon(\cdot,t)$ is a smooth (indeed C^2) diffeomorphism and hence cannot itself coalesce, letting the spatial correlation scale $\varepsilon \downarrow 0$ yields a weak limit in which distinct trajectories behave as independent Brownian motions until meeting and then move together, i.e. the coalescing Brownian (Arratia-type) motion. Let W be an $\mathbb{R}$-valued Wiener sheet on $\mathbb{R} \times [0,1]$ and let $\varphi \in C_0^\infty(\mathbb{R})$ be nonnegative, spherically symmetric, $\int \varphi = 1$. Define φ_ε as in [28] and consider the SDE

$$dx^\varepsilon(u,t) = \int_{\mathbb{R}} \varphi_\varepsilon\big(x^\varepsilon(u,t) - q\big)\, W(dq,dt), \qquad x^\varepsilon(u,0) = u. \tag{8}$$

Dorogovtsev [28] notes two key properties: (i) x^ε is a flow of homeomorphisms, and (ii) for each fixed u, $t \mapsto x^\varepsilon(u,t)$ is a Wiener process with $\langle x^\varepsilon(u,\cdot)\rangle_t = t$. Since in $d = 1$ the flow is order-preserving, and $\varepsilon \downarrow 0$ the limiting particle system corresponds to coalescing Brownian motions [28].

6. Outlook and open problems

The preceding sections present a second-order viewpoint on static algebraic point processes, classical IPS, coalescing systems, and measure-dependent flows. We conclude by formulating several concrete questions suggested by this perspective. In each case the theme is the same: to what extent can models be classified, approximated, or constructed on the basis of their second-order geometry?

6.1. Pfaffian models with controlled attraction.

Most probabilistic Pfaffian models appearing in this survey arise either from one-dimensional coalescing or annihilating systems. In these examples one observes strong short-range inhibition: the pair-correlation function satisfies $g(r) < 1$ for small $r > 0$, reflecting either collision of particles or eigenvalue repulsion. At intermediate scales, however, the behaviour of g is much less understood from a systematic, model-independent viewpoint.

The following questions ask whether Pfaffian structure is compatible with controlled attraction at some scales and how this is constrained by the antisymmetric kernel.

Problem 6.1. *Does there exist a translation-invariant Pfaffian point process on $\mathbb{R}$ whose pair-correlation function satisfies*

$$g(r) < 1 \text{ for all sufficiently small } r > 0, \qquad g(r_0) > 1 \text{ for some } r_0 > 0?$$

Can one construct such a process in a genuinely probabilistic way; for instance, as the fixed-time law of a spatially homogeneous coalescing or annihilating system with modified local rules (biased annihilation, longer-range branching, multi-species interactions), or as a scaling limit of a reaction–diffusion IPS?

Problem 6.2. *Let K be the antisymmetric matrix kernel of a stationary Pfaffian point process on $\mathbb{R}$, and let g be its pair-correlation function. Identify structural conditions on K that force $g(r) \le 1$ for all r (or at least on a neighbourhood of 0). Can such conditions be expressed in spectral terms, for example as sign or boundedness constraints on the Fourier transform of K, or on the spectrum of an associated operator?*

Conversely, can one formulate spectral criteria that allow $g(r) > 1$ on some interval, while still ensuring that the Pfaffian field exists and has non-negative correlations in the sense of [58]?

In determinantal settings, analogous spectral conditions on the kernel operator (eigenvalues in $[0,1]$ for the associated integral operator) are known to impose repulsiveness, in particular $g(r) \le 1$ in translation-invariant models. For Pfaffian fields, general existence criteria in terms of spectra of quaternionic kernels are available [58], but their implications for the fine shape of $g(r)$ appear to be largely unexplored. A clearer understanding here would indicate how much short- or medium-range "attraction" is compatible with Pfaffian structure.

6.2. **Classification of IPS by second-order structure.** In equilibrium statistical mechanics and spatial statistics it is standard to describe point configurations as *repulsive*, *Poisson-like*, or *clustered* according to the shape of the pair-correlation function $g(r)$ and related summaries (e.g. the structure factor or Ripley's K-function). Determinantal processes and hard-core Gibbs fields provide canonical repulsive examples ($g(r) < 1$ near 0), while Cox and cluster processes exhibit $g(r) > 1$ over a range of scales. For interacting particle systems, however, most information of this kind is available only model by model; there is no systematic "second-order taxonomy" at the level of IPS dynamics.

The following questions ask for such a taxonomy and for an identifiability theory based on second-order data.

Problem 6.3. *Develop a classification scheme for stationary IPS on $\mathbb{Z}^d$ (or $\mathbb{R}^d$) in which models are grouped according to the short- and long-range behaviour of their pair-correlation functions and structure factors. For example, one might distinguish:*

$$\textit{repulsive} \;\Rightarrow\; g(r) < 1 \textit{ near } 0, \quad \textit{Poisson-like} \;\Rightarrow\; g(r) \approx 1,$$

$$\textit{clustering} \;\Rightarrow\; g(r) > 1, \quad \textit{inhibition} \;\Rightarrow\; g(r) \ll 1,$$

supplemented by the small-k behaviour of the structure factor (hyperuniformity, long-range order, etc.).

Can one define second-order invariants (e.g. decay exponents for $g(r)-1$, small-k exponents for the structure factor, mixing rates of covariances) that are robust under natural rescalings (diffusive or hydrodynamic limits), and that place classical IPS (exclusion, contact, voter, Glauber dynamics for Ising-type models) into well-defined second-order universality classes?

6.3. **Multi-species and signed interactions.** Many natural systems involve several particle types, possibly with both attractive and repulsive components. From the second-order viewpoint, this leads naturally to *matrix-valued* correlation structure: for a multi-type point process one may consider $g_{ij}(r)$ describing interactions within and between species i and j. Such objects are well established, for example, in multivariate Cox and log-Gaussian Cox models, where the second-order structure is encoded by a matrix of cross-covariance functions. By contrast, systematic treatments of multi-type determinantal, permanental, or Pfaffian models remain comparatively scarce.

The following questions ask for a more explicit second-order theory in this setting.

Problem 6.4. *Develop a second-order framework for multi-type point processes in which the correlation structure is described by a matrix-valued pair-correlation function $g_{ij}(r)$ (or equivalently by a matrix of cross-covariance kernels). Under what conditions can such systems be represented by block-structured determinantal, permanental, or Pfaffian kernels, or by hybrid combinations of these (for example, determinantal within a species and Cox-type across species)?*

Problem 6.5. *For multi-species IPS with both attractive and repulsive components, investigate the possibility of "effective" one-species descriptions at the second-order level. When do the cross-correlations $g_{ij}(r)$ decay sufficiently fast (in space or in time) that the marginal of a given species behaves, on large scales, like a determinantal or Cox process with some effective kernel or random intensity? Can this be made precise in terms of convergence of pair-correlation functions or spectral densities, and can one quantify the error of such effective descriptions?*

Taken together, these questions fit into the broader program of understanding how far second-order geometry can carry the classification and construction of stochastic systems, and where genuinely higher-order features must enter. Even partial progress would clarify the role of matrix-valued correlation structure as a bridge between algebraic models, classical IPS, and nonlinear stochastic flows.

References

1. Richard Arratia, *Coalescing Brownian motions on the line*, Ph.D. thesis, University of Wisconsin–Madison, 1979.
2. Richard Arratia, *Limiting point processes for rescalings of coalescing and annihilating random walks on* $\mathbb{Z}^d$, Ann. Probab. **9** (1981), no. 6, 909–936. https://doi.org/10.1007/BF00531428
3. Adrian Baddeley, Ege Rubak, and Rolf Turner, *Fast approximation of the intensity of Gibbs point processes by the Poisson saddlepoint*, Electronic Journal of Statistics **6** (2012), 1155–1169. https://doi.org/10.1214/12-EJS707
4. Adrian Baddeley, Ege Rubak, and Rolf Turner, *Spatial point patterns: Methodology and applications with R*, Chapman and Hall/CRC, 2015. https://doi.org/10.1201/b19708
5. Lorenzo Bertini, Nicoletta Cancrini, and Filippo Cesi, *The spectral gap for a Glauber-type dynamics in a continuous gas*, Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques **38** (2002), no. 1, 91–108. https://doi.org/10.1016/S0246-0203(01)01085-8
6. Alexei Borodin, *Determinantal point processes*, The Oxford Handbook of Random Matrix Theory, Oxford University Press, 2011. https://doi.org/10.1093/oxfordhb/9780199534073.013.0012
7. Alexei Borodin, Patrik L. Ferrari, Michael Prähofer, and Tomohiro Sasamoto, *Fluctuation properties of TASEP with periodic initial configuration*, Journal of Statistical Physics **129** (2007), 1055–1080. https://doi.org/10.1007/s10955-007-9383-0
8. Alexei Borodin and Christopher D. Sinclair, *The Ginibre ensemble of real random matrices and its scaling limits*, Communications in Mathematical Physics **291** (2009), no. 1, 177–224. https://doi.org/10.1007/s00220-009-0793-0
9. Kamil Brádler, Pierre-Luc Dallaire-Demers, Patrick Rebentrost, Daiqin Su, and Christian Weedbrook, *Gaussian boson sampling for perfect matchings of arbitrary graphs*, Physical Review A **98** (2018), no. 3, 032310. https://doi.org/10.1103/PhysRevA.98.032310
10. Maury Bramson and David Griffeath, *Asymptotics for interacting particle systems on* $\mathbb{Z}^d$, Z. Wahrsch. Verw. Gebiete **53** (1980), no. 2, 183–196.
11. Maury Bramson and David Griffeath, *Clustering and dispersion rates for some interacting particle systems on* $\mathbb{Z}$, Annals of Probability **8** (1980), no. 2, 183–213. https://doi.org/10.1214/aop/1176994771
12. Maury Bramson and David Griffeath, *On the Williams–Bjerknes tumour growth model: II*, Annals of Probability **8** (1980), no. 6, 952–970. https://doi.org/10.1214/aop/1176994673
13. E. R. Caianiello, *On quantum field theory I*, Il Nuovo Cimento **10** (1953), 1634–1652.
14. David Coupier, Kumarjit Saha, Anish Sarkar, and Viet Chi Tran, *Collision times of random walks and applications to the Brownian web*, 2020, pp. 267–293. https://doi.org/10.1142/9789811206092_0007
15. Daryl J. Daley and David Vere-Jones, *An introduction to the theory of point processes: Volume I: Elementary theory and methods*, 2nd ed., Probability and Its Applications, Springer, New York, 2003. https://doi.org/10.1007/b97277
16. Daryl J. Daley and David Vere-Jones, *An introduction to the theory of point processes: Volume II: General theory and structure*, 2nd ed., Probability and Its Applications, Springer, New York, 2008. https://doi.org/10.1007/978-0-387-49835-5
17. Laurent Decreusefond and Ian Flint, *Functional marked point processes and Poisson point processes*, Comptes Rendus Mathematique **352** (2014), no. 4, 357–361. https://doi.org/10.1016/j.crma.2013.11.016
18. Pierre Del Moral and Alice Guionnet, *Large deviations for interacting particle systems: Applications to nonlinear filtering problems*, Stochastic Processes and their Applications **78** (1998), no. 1, 69–95. https://doi.org/10.1016/S0304-4149(98)00057-X
19. Pierre Del Moral and Laurent Miclo, *Branching and interacting particle systems: Approximations of Feynman–Kac formulae with applications to nonlinear filtering*, Séminaire de Probabilités XXXIV (J. Azéma, M. Émery, M. Ledoux, and M. Yor, eds.), Lecture Notes in Mathematics, vol. 1729, Springer, Berlin, 2007, pp. 1–145.

20. David Dereudre, *Introduction to the theory of Gibbs point processes*, Stochastic Geometry: Modern Research Frontiers, Lecture Notes in Mathematics, vol. 2237, Springer, Cham, 2019, pp. 181–229. https://doi.org/10.1007/978-3-030-13547-8_5
21. Persi Diaconis and Steven N. Evans, *Immanants and finite point processes*, Journal of Combinatorial Theory, Series A **91** (2000), no. 1-2, 305–321. https://doi.org/10.1006/jcta.2000.3097
22. R. L. Dobrushin, *Markov processes with a large number of locally interacting components: Existence of a limit process and its ergodicity*, Problemy Peredachi Informatsii **7** (1971), no. 2, 70–87, English translation available; original in Russian.
23. Peter Donnelly, Steven N. Evans, Klaus Fleischmann, Thomas G. Kurtz, and Xiaowen Zhou, *Continuum-sites stepping-stone models, coalescing exchangeable partitions and random trees*, The Annals of Probability **28** (2000), no. 3, 1063–1110.
24. A. A. Dorogovtsev, *Stochastic flows with interaction and measure-valued processes*, Int. J. Math. Math. Sci. **2003** (2003), no. 63, 3963–3977.
25. A. A. Dorogovtsev, *Measure-valued Markov processes and stochastic flows on abstract spaces*, Stoch. Stoch. Rep. **76** (2004), no. 5, 395–407.
26. A. A. Dorogovtsev, K. Hlyniana, and S. Liu, *Ergodic theorem for the differential equations with interaction*, arXiv preprint arXiv:2506.13347 (2025).
27. A. A. Dorogovtsev and K. V. Hlyniana, *Gaussian structure in coalescing stochastic flows*, Stoch. Dyn. **23** (2023), no. 06.
28. Andrey A. Dorogovtsev, *Measure-valued processes and stochastic flows*, De Gruyter, Berlin, Boston, 2024. https://doi.org/doi:10.1515/9783110986518
29. Richard Durrett, *An introduction to infinite particle systems*, Stochastic Processes and their Applications **11** (1981), no. 2, 109–150.
30. Rick Durrett and Simon A. Levin, *The importance of being discrete (and spatial)*, Theoretical Population Biology **46** (1994), no. 3, 363–394. https://doi.org/10.1006/tpbi.1994.1032
31. Paul Erdős and Philip Ney, *Some problems on random intervals and annihilating particles*, Ann. Probability **2** (1974), 828–839.
32. Stewart N. Ethier and Thomas G. Kurtz, *Markov processes: Characterization and convergence*, Wiley, 1986. https://doi.org/10.1002/9780470316658
33. D. Finkelshtein, Y. Kondratiev, and Y. Kozitsky, *Glauber dynamics in continuum: A constructive approach to evolution of states*, Discrete Contin. Dyn. Syst. **33** (2013), no. 4, 1431–1450.
34. Dmitri Finkelshtein, Yuri Kondratiev, and Oleksandr Kutoviy, *Semigroup approach to birth-and-death stochastic dynamics in continuum*, Journal of Functional Analysis **262** (2011), no. 3, 1274–1308. https://doi.org/10.1016/j.jfa.2011.11.005
35. Dmitri Finkelshtein, Yuri Kondratiev, and Oleksandr Kutoviy, *Statistical dynamics of continuous systems: Perturbative and approximative approaches*, Arab Journal of Mathematics **4** (2015), 255–300. https://doi.org/10.1007/s40065-015-0126-0
36. Dmitri L. Finkelshtein, Yuri G. Kondratiev, and Maria João Oliveira, *Markov evolutions and hierarchical equations in the continuum. I: one-component systems*, Journal of Evolution Equations **9** (2009), 197–233. https://doi.org/10.1007/s00028-009-0007-9
37. Luiz Renato G. Fontes, Monica Isopi, Charles M. Newman, and Krishnamurthi Ravishankar, *The Brownian web: Characterization and convergence*, Annals of Probability **32** (2004), no. 4, 2857–2883. https://doi.org/10.1214/009117904000000557
38. Nadia L. Garcia and Thomas G. Kurtz, *Spatial birth and death processes as solutions of stochastic equations*, ALEA. Latin American Journal of Probability and Mathematical Statistics **1** (2006), 281–303.
39. Barnaby Garrod, Mihail Poplavskyi, Roger P. Tribe, and Oleg V. Zaboronski, *Examples of interacting particle systems on $\mathbb{Z}$ as Pfaffian point processes: Annihilating and coalescing random walks*, Annales Henri Poincaré **18** (2018), 3635–3662. https://doi.org/10.1007/s00023-018-0719-x
40. Hans-Otto Georgii, *Canonical and grand canonical Gibbs states for continuum systems*, Communications in Mathematical Physics **48** (1976), no. 1, 31–51. https://doi.org/10.1007/BF01609410
41. Hans-Otto Georgii, *Gibbs measures and phase transitions*, De Gruyter, Berlin, New York, 2011. https://doi.org/doi:10.1515/9783110250329
42. E. V. Glinyanaya, *Disordering asymptotics in the discrete approximation of an Arratia flow*, Theory of Stochastic Processes **18(34)** (2012), no. 2, 8–14.
43. E. V. Glinyanaya, *Semigroups of m-point motions of the Arratia flow, and binary forests*, Theory of Stochastic Processes **19** (2014), no. 2, 31–41.
44. David Griffeath, *Annihilating and coalescing random walks on $\mathbb{Z}^d$*, Z. Wahrsch. Verw. Gebiete **46** (1978), no. 1, 55–65. https://doi.org/10.1007/BF00535688

45. David Griffeath, *The basic contact processes*, Stochastic Processes and their Applications **11** (1981), no. 2, 151–185. https://doi.org/10.1016/0304-4149(81)90024-8
46. David Griffeath, *Frank Spitzer's pioneering work on interacting particle systems*, The Annals of Probability **21** (1993), no. 2, 608–621.
47. Chenlin Gu, Jean-Christophe Mourrat, and Maximilian Nitzschner, *Quantitative equilibrium fluctuations for interacting particle systems*, arXiv preprint arXiv:2401.10080 (2024). https://doi.org/10.48550/arXiv.2401.10080
48. Craig S. Hamilton, Regina Kruse, Linda Sansoni, Sonja Barkhofen, Christine Silberhorn, and Igor Jex, *Gaussian boson sampling*, Physical Review Letters **119** (2017), no. 17, 170501. https://doi.org/10.1103/PhysRevLett.119.170501
49. T. E. Harris, *Additive set-valued Markov processes and graphical methods*, Annals of Probability **6** (1978), no. 3, 355–378.
50. Theodore E. Harris, *Contact interactions on a lattice*, Annals of Probability **2** (1974), no. 6, 969–988. https://doi.org/10.1214/aop/1176996493
51. Richard Holley and Thomas M. Liggett, *The survival of contact processes*, Annals of Probability **6** (1978), no. 2, 198–206. https://doi.org/10.1214/aop/1176995567
52. Richard A. Holley and Thomas M. Liggett, *Ergodic theorems for weakly interacting infinite systems and the voter model*, Annals of Probability **3** (1975), no. 4, 643–663.
53. J. Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág, *Zeros of Gaussian analytic functions and determinantal point processes*, University Lecture Series, vol. 51, American Mathematical Society, 2009. https://doi.org/10.1090/ulect/051
54. Janine Illian, Aila Penttinen, Helga Stoyan, and Dietrich Stoyan, *Statistical analysis and modelling of spatial point patterns*, John Wiley & Sons, 2008.
55. Soran Jahangiri, Juan Miguel Arrazola, Nicolás Quesada, and Nathan Killoran, *Point processes with Gaussian boson sampling*, Physical Review E **101** (2020), no. 2, 022134. https://doi.org/10.1103/PhysRevE.101.022134
56. K. Johansson, *Shape fluctuations and random matrices*, Comm. Math. Phys. **209** (2000), no. 2, 437–476.
57. Kurt Johansson, *Course 1 - random matrices and determinantal processes*, Mathematical Statistical Physics (Anton Bovier, François Dunlop, Aernout van Enter, Frank den Hollander, and Jean Dalibard, eds.), Les Houches, vol. 83, Elsevier, 2006, pp. 1–56. https://doi.org/https://doi.org/10.1016/S0924-8099(06)80038-7
58. Vladislav Kargin, *On Pfaffian random point fields*, Journal of Statistical Physics **154** (2012), no. 3, 681–704.
59. Samuel Karlin and James McGregor, *Coincidence probabilities*, Pacific Journal of Mathematics **9** (1959), no. 4, 1141–1164.
60. Claude Kipnis and Claudio Landim, *Scaling limits of interacting particle systems*, Springer, 1999. https://doi.org/10.1007/978-3-662-03995-6
61. Nikolaos Kolliopoulos, Martin Larsson, and Zeyu Zhang, *Propagation of chaos for point processes induced by particle systems with mean-field drift interaction*, Journal of Theoretical Probability **38** (2025), no. 27. https://doi.org/10.1007/s10959-024-01397-3
62. Yuri Kondratiev and Eugene Lytvynov, *Glauber dynamics of continuous particle systems*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **41** (2005), no. 4, 685–702. https://doi.org/10.1016/j.anihpb.2004.06.001
63. Claudio Landim, *Hydrodynamic limits of interacting particle systems*, School and Conference on Probability Theory, ICTP Lecture Notes, no. 17, Abdus Salam International Centre for Theoretical Physics, Trieste, 2004, pp. 57–100.
64. Günter Last and Mathew D. Penrose, *Lectures on the Poisson process*, Institute of Mathematical Statistics Textbooks, Cambridge University Press, 2017. https://doi.org/10.1017/9781316104477
65. Frédéric Lavancier, Jesper Møller, and Ege Rubak, *Determinantal point process models and statistical inference*, Journal of the Royal Statistical Society: Series B (Statistical Methodology) **77** (2015), no. 4, 853–877.
66. Thomas M. Liggett, *Interacting particle systems*, Springer, 1985.
67. Thomas M. Liggett, *Stochastic interacting systems: Contact, voter and exclusion processes*, Grundlehren der Mathematischen Wissenschaften, vol. 324, Springer, Berlin, 1999. https://doi.org/10.1007/978-3-662-03990-8
68. Thomas M. Liggett, *T. E. Harris' contributions to interacting particle systems and percolation*, Annals of Probability **39** (2011), no. 2, 318–331. https://doi.org/10.1214/10-AOP584
69. Sheng Lin Lu and Horng-Tzer Yau, *Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics*, Communications in Mathematical Physics **156** (1993), 399–433. https://doi.org/10.1007/BF02098489

70. Jamie Lukins, Roger Tribe, and Oleg Zaboronski, *Multi-point correlations for two dimensional coalescing random walks*, Journal of Applied Probability **55** (2018), no. 4, 1158–1185. https://doi.org/10.1017/jpr.2018.77
71. Odile Macchi, *The coincidence approach to stochastic point processes*, Advances in Applied Probability **7** (1975), no. 1, 83–122. https://doi.org/10.2307/1425855
72. Peter McCullagh and Jesper Møller, *The permanental process*, Advances in Applied Probability **38** (2006), no. 4, 873–888. https://doi.org/10.1017/S0001867800001361
73. Jesper Møller and Rasmus Plenge Waagepetersen, *Statistical inference and simulation for spatial point processes*, Chapman & Hall/CRC, 2003. https://doi.org/10.1201/9780203496930
74. X. X. Nguyen and H. Zessin, *Integral and differential characterizations of Gibbs processes*, Math. Nachr. **88** (1979), 105–115.
75. I. I. Nishchenko, *Discrete time approximation of coalescing stochastic flows on the real line*, Theory of Stochastic Processes **17(33)** (2011), no. 1, 70–78.
76. Grigorios A. Pavliotis and Andrea Zanoni, *A method of moments estimator for interacting particle systems and their mean-field limit*, SIAM/ASA Journal on Uncertainty Quantification **12** (2024), no. 2, 262–288. https://doi.org/10.1137/22M153848X
77. Nicolás Quesada, Juan Miguel Arrazola, and Nathan Killoran, *Gaussian boson sampling using threshold detectors*, Physical Review A **98** (2018), no. 6, 062322. https://doi.org/10.1103/PhysRevA.98.062322
78. Emmanuel Schertzer, Rongfeng Sun, and Jan M. Swart, *The Brownian web, the Brownian net, and their universality*, Advances in Disordered Systems, Random Processes and Some Applications (2016), 270–368.
79. Tomoyuki Shirai and Yoichiro Takahashi, *Random point fields associated with certain fredholm determinants I: Fermion, Poisson and boson point processes*, Journal of Functional Analysis **205** (2003), no. 2, 414–463. https://doi.org/10.1016/S0022-1236(02)00099-3
80. Alexander Soshnikov, *Determinantal random point fields*, Russian Mathematical Surveys **55** (2000), no. 5, 923–975. https://doi.org/10.1070/RM2000v055n05ABEH000321
81. Frank Spitzer, *Interaction of Markov processes*, Advances in Mathematics **5** (1970), no. 2, 246–290. https://doi.org/10.1016/0001-8708(70)90034-4
82. A.-S. Sznitman, *Topics in propagation of chaos*, Lecture Notes in Math., vol. 1464, Springer, 1991.
83. Bálint Tóth and Wendelin Werner, *The true self-repelling motion*, Probability Theory and Related Fields **111** (1998), no. 3, 375–452. https://doi.org/10.1007/s004400050170
84. Craig A. Tracy and Harold Widom, *Asymptotics in ASEP with step initial condition*, Communications in Mathematical Physics **290** (2009), no. 1, 129–154. https://doi.org/10.1007/s00220-009-0761-8
85. Roger Tribe and Oleg Zaboronski, *Pfaffian formulae for one-dimensional coalescing and annihilating systems*, Electronic Journal of Probability **16** (2011), 2080–2103. https://doi.org/10.1214/EJP.v16-945
86. David Vere-Jones, *A generalization of permanents and determinants*, Linear Algebra and its Applications **111** (1988), 119–124. https://doi.org/https://doi.org/10.1016/0024-3795(88)90053-5

School of Mathematics, Jilin University, Changchun, China; Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine
E-mail address: glinkate@gmail.com