Theory of Stochastic Processes
Vol. 29 (45), no. 1, 2025, pp. 118–128

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SOME ISSUES OF THE SECRETARY PROBLEM WITH RANDOM LIFETIMES OF SCANNED OBJECTS

We consider a variation of the secretary problem, in which n objects have random lifetimes after their scan. After scanning each object in turn, an observer either continues the process or chooses one of the scanned objects. In the latter case, the selection process ends, although the selected object may already disappear. We assume that its lifetime has a geometric distribution with parameter α. Unlike all previous formulations of the secretary problem, in which the scanning started with the first object, we investigate the optimal strategy of a belated observer who joined the selection process with a delay. In the final part of the article, we investigate the transient regime when $\alpha = \alpha(n) \to 1$ as $n \to \infty$. It turns out that the behavior of the threshold level $L = L(n)$ can be arbitrary, as long as $\varliminf_{n\to\infty} L/n \geq 1/e$.

1. Statement of the problem and preliminaries

P. R. Freeman [3] in his review article on the secretary problem indicates that the development of what has come to be known as the secretary problem began in the early 1960s. Now we see a new surge of interest in the best choice problem. The paper [8] provides an overview of publications on the issue under consideration as of 2020.

The classical statement of the problem of optimal choice of the best among $n \geq 2$ objects (the secretary problem) is given by the following conditions:

(1) All $n!$ permutations of n objects according to their quality are equally possible.
(2) The observer scans them in turn.
(3) Having scanned anything the observer takes a decision either to continue process or to stop and claim that the last scanned object is the best of all.
(4) The observer cannot go back and choose a previously scanned object.
(5) This decision is only based on information about the mutual arrangement of the scanned objects.

The decision may eventually be false, so the following problem arises: to determine the optimal strategy that provides the highest probability to correct decision.

Lindley [4] gave the first published solution of the standard problem. The optimal strategy is as follows.

Put $a_r = \frac{1}{r} + \frac{1}{r+1} + \ldots + \frac{1}{n-1}$. Let also r^* be such that $a_{r^*-1} \geq 1 > a_{r^*}$. Then the optimal policy is to reject the first $r^* - 1$ objects and then to accept the first object thereafter, which is better than all previous ones. Both r^*/n and the probability of winning using this policy tend to e^{-1}, as $n \to \infty$.

Most succeeding papers follow algebraic methods similar to Lindley's. Dynkin [1, 2] gave a completely different approach, which we are following. This approach allows us to consider the problem of the optimal strategy of a latecomer.

Condition 4 indicates that the observer cannot claim any previously missed object as the best one. Yang [9] and Petrucelli [5, 6] allowed the observer at any stage to go

2000 *Mathematics Subject Classification.* Primary 60G40.

Key words and phrases. secretary problem, optimal stopping, returning to previously scanned objects, transient regime, delayed observer.

back and try to accept a previously rejected object. If it is available, it is accepted, but otherwise, it remains unavailable ever after, and the observer must continue scanning new items.

The paper [10] considered another extension of the secretary problem. It changed conditions 3 and 4 by:

(6) After scanning an object, a counter of its lifetime starts. Having scanned any object, the observer takes a decision either to continue scanning or to stop and to claim an object already scanned as the best one among the totality. The observer can fail due to two reasons, the first one is the same as earlier, and the second consists in the observer's sluggishness and, consequently, in the possible disappearance of the chosen object.

We assume that lifetimes of different objects are independent and have common geometrical distribution with parameter α. We point out that, unlike [5, 6, 9], multiple returns are inadmissible, and the decision is final. The paper [10] considered such a case and established the structure of the support set of a two-dimensional Markov chain that specified the selection procedure. We continue the investigation of this model. We know only one work [7] that considered such a variant of the secretary problem. It used an algebraic method. Here, a system of relations between system performances was critical for the research.

In all previous formulations of the secretary problem, the observer appeared at the very beginning of the scan. They did not consider the possibility of the observer being late. In our variant of the secretary problem, such an observer joins the scan and receives data on the number i of the last scanned object, and on the number j of the best object among all scanned. For i less than a threshold level, the observer must continue scanning. In the opposite case, the question arises: should the observer select the object j, although it could disappear, or should he (she) continue the scan? The first part of the article is devoted to the study of the optimal strategy under the specified conditions. We indicate the criterion for making a particular decision. In addition, we demonstrate the capabilities of the proposed method in obtaining already known results. The second part of the article considers the transient regime of the secretary problem with a geometric distribution of lifetimes, when $\alpha = \alpha(n)$ as $n \to \infty$. It turns out that the threshold level $L = L(n)$ can behave arbitrarily, as long as $\varliminf_{n\to\infty} L/n \geq 1/e$. The study makes use of Lemma 5.1 on the monotonicity of the sequence

$$k \to \left(1 - \sum_{j=n-k+1}^{n} \frac{1}{j-1}\right)^{\frac{1}{k}}.$$

This result was established in [10] because of other statements. Professor Marynych communicated another proof to the author. We give it in his edit.

2. The base process and notation

In this section, we give the main definitions, designations and conventions.

(1) The process $B(k) = (k, y(k))$, $1 \leq k \leq n$, is called the base process, if k is the number of the object just scanned, and $y(k)$ is the number of the object, the best among those scanned so far. It is a homogeneous Markov chain with absorbing states (n, j). Its one-step transitions from non-absorbing states are

$$p((i,j),(i+1,j)) = \frac{i}{i+1}, \quad p((i,j),(i+1,i+1)) = \frac{1}{i+1}.$$

(2) $S = \{(i,j) : 1 \leq j \leq i \leq n\}$ denotes the phase space of the base process.

(3) $D = \{(i,i) : 1 \le i \le n\}$ is the diagonal of S, $S_n = \{(n,1),(n,2),\dots,(n,n)\}$ is the last column of S, $\partial S = D \cup S_n$, and $\mathrm{Int}S = S\backslash\partial S$.
(4) $\mathbb{P}_{(i,j)}\{A\}$ and $\mathbb{E}_{(i,j)}\eta$ denote the conditional probability of the event A and conditional expectation of the random variable η under the condition $x(0) = (i,j)$.
(5) $\mathbf{P}g(i,j) = \mathbb{E}_{(i,j)}g(B(1))$ is the one-step shift operator of the function $g(\cdot,\cdot)$.
(6) $f(i,j) = \alpha^{i-j}\frac{i}{n}$ is the payoff function. It is the probability that the object j still exists at the moment of scanning the object i and is the best among all, provided that it is the best among initial i objects.
(7) The value of the game $v(i,j) = \sup \mathbb{E}_{(i,j)}f(B(\theta))$, where supremum is taken over the set of all Markov moments. It is known [8] that $v(\cdot) \ge 0$, $v(\cdot) = \max\{f(\cdot), \mathbf{P}v(\cdot)\}$, i.e. the value of the game is the minimal excessive majorant of the payoff function.
(8) $\Gamma = \{(i,j) : v(i,j) = f(i,j)\}$ is the support set.
(9) $\tau = \min\{k : B(k) \in \Gamma\}$ is the hitting time of the support set. It gives the optimal strategy.
(10) L denotes the threshold level of D.
(11) $m = n - L + 1$ is the number of states in $D \cap \Gamma$.
(12) r^* is the threshold level in the standard secretary problem.
(13) By default, from now on $(x,y) \in S$.
(14) We place the symbol $\square$ at the end of proofs.
(15) $\Rightarrow$ is the implication sign.

3. Optimal strategy of a delayed observer

Article [9] indicates the following properties of the support set:
(1) $(i,j) \in \Gamma \Rightarrow (i,j+1) \in \Gamma$.
(2) $(i,j) \in \Gamma \cap \mathrm{Int}S \Rightarrow (i-1,j) \in \Gamma$.
(3) $(i,j) \in \Gamma \Rightarrow (i+1,j+1) \in \Gamma$.

Definition 3.1. The threshold of the diagonal is a point (L,L) such that $(i,i) \notin \Gamma$ for any $i < L$, and $(i,i) \in \Gamma$ for any $i \ge L$. We shall call L the threshold level.

If $n \ge 2$, then $(1,1) \notin \Gamma$, $(n,n) \in \Gamma$, and property 3 implies the existence of such a level.

Definition 3.2. The threshold of the ith column is a point (i,R_i) such that $(i,j) \notin \Gamma$ for any $j < R_i$, and $(i,j) \in \Gamma$ for any $j \ge R_i$. We shall call R_i the threshold level of the ith column.

If $i < L$, then the threshold point of the column does not exist, because in the opposite case from property 1 it would follow that $(i,i) \in \Gamma$, which cannot be the case if $i < L$. If $i = n$, then the threshold point of the column does not exist because in this column all points are in Γ. If $i \in \{L, L+1, \dots, n-1\}$, then the threshold point exists due to property 1, $(i,1) \notin \Gamma$, $(i,L) \in \Gamma$ because of $(L,L) \in \Gamma$, existence of the threshold level L, and the condition $i \ge L$. The above, with appropriate edits, concerns the threshold of the row and the threshold of the parallel to the diagonal from property 3.

Lemma 3.1. *The sequence $(R_L, R_{L+1}, \dots, R_{n-1})$ is nondecreasing. Its neighboring elements differ by no more than one.*

Proof. If $L < k \le n-1$, then $(k,R_k) \in \Gamma$, and by property 2 $(k-1,R_k) \in \Gamma$. This implies $R_{k-1} \le R_k$. If $R_{k-1} < R_k - 1$, then from property 3 it follows that $(k, R_{k-1}+1) \in \Gamma$. In such a case $R_{k-1} + 1 \ge R_k$, which contradicts the assumption made. $\square$

The optimal strategy requires an immediate stop at the point of the support set. Otherwise, the scan must continue. A belated observer receives information that the scan result now is (i,j). In three cases, we already know the answer to the question: whether $(i,j) \notin \Gamma$ or $(i,j) \in \Gamma$:

(1) If $i < L$, then $(i,j) \notin \Gamma$, and the optimal strategy requires continuing scanning.
(2) If $i = j$, then $(i,i) \notin \Gamma$ at $i < L$, and one should continue scanning, and at $i \geq L$ the optimal strategy is to stop.
(3) If $i = n$, then $(n,j) \in \Gamma$.

Lemma 3.2. *If $R_{i+1} = (i+1, j+1)$, then* $\mathbf{P}v(i,j) = \frac{i}{n}\left(\sum\limits_{l=k}^{n-1} \frac{1}{l} + \frac{j}{n}\alpha^{n-j}\right)$.

Proof. The point $(i+1, j+1)$ is threshold, so

a) $(i+1, i+1) \in \Gamma$, $(k,k) \in \Gamma$ at $k \geq i+1$,
b) $(i+1, j) \notin \Gamma$.

Property 2 implies $(k,j) \notin \Gamma$ at $i+1 \leq k \leq n-1$. Further,

$$\mathbf{P}v(i,j) = \frac{i}{i+1}v(i+1,j) + \frac{1}{i+1}v(i+1,i+1).$$

Since $(i+1,j) \notin \Gamma$, we have $v(i+1,j) = \mathbf{P}v(i+1,j)$. Conversely, $(i+1,i+1) \in \Gamma$, and $v\,(i+1,i+1) = f(i+1,i+1) = \frac{i+1}{n}$. So, $\mathbf{P}v(i,j) = \frac{i}{i+1}\mathbf{P}v(i+1,j) + \frac{1}{n}$.

The obtained equality used only properties a) and b). This allows us to apply them to calculate $\mathbf{P}v(i+1,j), \ldots, \mathbf{P}v(n-2,j)$. As a result, we obtain a system of equations

$$\mathbf{P}v(k,j) = \frac{k}{k+1}\mathbf{P}v(k+1,j) + \frac{1}{n}, \quad k = i, i+1, \ldots, n-2. \tag{1}$$

Since the state (n,j) is absorbing,

$$\mathbf{P}v(n-1,j) = \frac{n-1}{n}v(n,j) + \frac{1}{n}v(n,n) = \frac{n-1}{n}\mathbf{P}v(n,j) + \frac{1}{n}.$$

Therefore, system (1) is valid at $k = i, i+1, \ldots, n-1$. Its solution admits the representation

$$\mathbf{P}v(k,j) = \frac{k}{n}\sum_{l=k}^{n-1}\frac{1}{l} + \frac{k}{n}\mathbf{P}v(n,j) \quad \text{at} \quad k = i, i+1, \ldots, n-1.$$

Again, the state (n,j) is absorbing, $\mathbf{P}v(n,j) = v(n,j) = f(n,j) = \alpha^{n-j}$, and eventually we get

$$\mathbf{P}v(k,j) = \frac{k}{n}\left(\sum_{l=k}^{n-1}\frac{1}{l} + \alpha^{n-j}\right) \quad \text{at} \quad k = i, i+1, \ldots, n-1. \tag{2}$$

We have established even stronger statement than in the lemma. □

Theorem 3.1. *If $R_{i+1} = (i+1, j+1)$, then $R_i = (i,j)$ if and only if*

$$\alpha^{i-j} \geq \sum_{l=i}^{n-1}\frac{1}{l} + \alpha^{n-j}. \tag{3}$$

Otherwise, $R_i = (i, j+1)$.

Proof. By Lemma (3.1), R_i must be either $(i,j+1)$ or (i,j). If it is $(i,j+1)$, and not (i,j), then by the threshold property $(i,j) \notin \Gamma$ and $f(i,j) < v(i,j)$. In this situation, $v(i,j) = \max\{f(i,j), \mathbf{P}v(i,j)\} = \mathbf{P}v(i,j)$, and $f(i,j) < \mathbf{P}v(i,j)$. So, $(i,j) \notin \Gamma$ if and only if $f(i,j) < \mathbf{P}v(i,j)$. $\mathbf{P}v(i,j)$ is indicated in (2), $f(i,j) = \frac{i}{n}\alpha^{i-j}$, from which follows the statement of the theorem (more precisely, inverse to contrary, which is equivalent). □

To summarize, let us formulate an optimal rule for an observer who, at the moment of his arrival, finds the process in the state (i,j):

(1) If $i < L$, he should not stop (too few observations have passed).

(2) If $i = n$, he has to choose the jth object best (he arrived too late and missed all the opportunities to make a choice).
(3) If $L \le i \le n-1$, he should check the inequality $j \ge R_i$. If it is right, he should stop and choose the jth object. Otherwise, he should continue scanning. To use this rule, he should know the threshold levels $R_L, R_{L+1}, \dots, R_{n-1}$.

Remark 3.1. Note that $R_L = L$. The reverse procedure for finding $R_{L+1}, R_{L+2}, \dots, R_{n-1}$ has no cycles in the cycle. It has 2 cycles, the first for finding R_{n-1}, and the second for finding $R_{n-2}, R_{n-3}, \dots, R_{L+1}$.

Remark 3.2. If in the specified procedure one gets $R_k = k$, then $R_s = s$ for all $s \le k$, if any. This follows from the threshold property of all R_s and Lemma 3.1.

Remark 3.3. The threshold level L is determined by the condition

$$\sum_{j=L}^{n} \frac{1}{j-1} + \alpha^{n-L+1} > 1 \ge \sum_{j=L+1}^{n} \frac{1}{j-1} + \alpha^{n-L}.$$

We establish it in the next section.

4. Embedded chain

In this section, we only consider the case $B(1) = (1,1)$, $\tau = \min\{i : B(i) \in \Gamma\}$, and demonstrate efficiency of the approach proposed.

Lemma 4.1. *If $B(1) = (1,1)$ and $\tau = \min\{i : B(i) \in \Gamma\}$, then any point $(i,j) \in \Gamma \cap \mathrm{Int}S$ is unattainable for the base process. If $(i,i) \in \Gamma$ at $i \le n-1$, then (n,i) is also unattainable.*

Proof. The base process arrives at the given point (i,j) only along the trajectory, the final part of which consists of points $(j,j), (j+1,j), \dots, (i-1,j), (i,j)$. Since $(i,j) \in \Gamma \cap \mathrm{Int}S \Rightarrow (i-1,j) \in \Gamma$, all the specified points belong to the support set, and the selected optimal strategy would stop the base process no later than at the point (j,j), making all later ones unattainable. The second statement of lemma is true for the same reason. □

Lemma 4.1 allows us to reduce the problem to scanning the values of the base process in $\partial S = D \cup \Gamma_n$. Such successive observations generate an embedded Markov chain $Q(\cdot)$ with the phase space ∂S, the states from Γ_n are absorbing. Its one-step transitions from non-absorbing states have the form

a) $(i,i) \to (j,j)$ with probability $\frac{i}{j(j-1)}$, $1 \le i < j \le n$, and
b) $(i,i) \to (n,i)$ with probability $\frac{1}{n}$.

The values of the payoff function, the threshold level L, and the game value on the set ∂S remain unchanged. The support set of the chain $Q(\cdot)$ is $\Delta = \Gamma \cap \partial S$. In addition, $\tau = \min\{i : B(i) \in \Gamma\} = \min\{i : Q(i) \in \Delta\}$.

Let $\mathbf{Q}$ denote the shift operator generated by the chain $Q(\cdot)$.

The level L is determined by the threshold conditions

$$f((L,L)) \ge \mathbf{Q}v((L,L)), \quad f((L-1,L-1)) < \mathbf{Q}v((L-1,L-1)).$$

One can write them as

$$\sum_{j=L}^{n} \frac{1}{j-1} + \alpha^{n-L+1} > 1 \ge \sum_{j=L+1}^{n} \frac{1}{j-1} + \alpha^{n-L}. \tag{4}$$

The probability of success when using the optimal strategy is

$$\mathbb{E}_{(1,1)}v(Q(\tau)) = \sum_{j=L}^{n} \frac{L-1}{j(j-1)} v(j,j) + \sum_{j=1}^{L-1} \frac{1}{n}\alpha^{n-j} =$$

$$\frac{L-1}{n}\sum_{j=L}^{n}\frac{1}{j-1} + \frac{1}{n}\frac{\alpha^{n-L+1}-\alpha^{n}}{1-\alpha}.$$

Up to notation, these formulas are given in [10].

Remark 4.1. Condition (4) can be rewritten as

$$\left(1-\sum_{j=L}^{n}\frac{1}{j-1}\right)^{\frac{1}{n-L+1}} < \alpha \le \left(1-\sum_{j=L+1}^{n}\frac{1}{j-1}\right)^{\frac{1}{n-L}}. \tag{5}$$

We got this inequality due to the existence of the threshold level L. A direct proof of the increase of the sequence $k \mapsto \left(1-\sum_{j=k}^{n}\frac{1}{j-1}\right)^{\frac{1}{n-k+1}}$, provided that $1-\sum_{j=L}^{n}\frac{1}{j-1} > 0$, exists for arbitrary constant k and is given separately.

Remark 4.2. These results entail $L/n \to e^{-1}$ as $n \to \infty$. The probability of success has the same limit.

5. Transient regime

In this section, we consider the transient regime, in which α and m depend on n, $\alpha = \alpha(n)$, $m = m(n)$, and $\alpha(n) \to 1$ as $n \to \infty$. We investigate three cases:

a) $m(n)$ is a constant;
b) $m^2(n) = o(n)$, $m(n) \to \infty$ as $n \to \infty$;
c) $m(n) = Kn + o(n)$, where K is a constant, $1-\frac{1}{e} < K < 1$.

Let us put $\alpha_0 = 1$, $\alpha_k = \left(1-\sum_{j=n-k+1}^{n}\frac{1}{j-1}\right)^{\frac{1}{k}}$ for $k \ge 1$, provided that

$$1-\sum_{j=n-k+1}^{n}\frac{1}{j-1} > 0. \tag{6}$$

Lemma 5.1. *Subject to requirement* (6) $\alpha_k > \alpha_{k+1}$.

Proof. Let us check that the sequence $k \to -\ln\alpha_k$ is increasing. Given the Taylor series expansion

$$-\ln\alpha_k = \sum_{j\ge1}\frac{1}{j}\frac{1}{k}\left(\frac{1}{n-1}+\ldots+\frac{1}{n-k}\right)^j$$

for $k \ge 1$, it is sufficient to check that the sequences $k \to \left(\frac{1}{n-1}+\ldots+\frac{1}{n-k}\right)$ and $k \to \frac{1}{k}\left(\frac{1}{n-1}+\ldots+\frac{1}{n-k}\right)$ are increasing. Then each term in the expansion increases as the product of increasing sequences. The fact that the first specified sequence is increasing is obvious. The monotonicity of the second follows from the elementary fact applied to the sequence $k \to c_k = \frac{1}{n-k}$: if the sequence $k \to c_k$ is increasing, then the sequence $k \to \frac{1}{k}(c_1+\ldots+c_k)$ is also increasing. $\square$

By Lemma 5.1, we can define the intervals

$$I_1 = (\alpha_1, \alpha_0], \quad I_2 = (\alpha_2, \alpha_1], \quad \ldots \quad I_m = (\alpha_m, \alpha_{m-1}].$$

Corollary 5.1. *For any natural-valued function $K(n)$, with $r^* \le K(n) \le n$, there exists a sequence $\alpha(n)$ such that $L = L(n, \alpha(n)) = K(n)$.*

Proof. Condition (5) gives

$$L = k \Leftrightarrow \left(1 - \sum_{j=k}^{n} \frac{1}{j-1}\right)^{\frac{1}{n-k+1}} < \alpha \le \left(1 - \sum_{j=k+1}^{n} \frac{1}{j-1}\right)^{\frac{1}{n-k}} \Leftrightarrow \alpha \in I_k,$$

and it suffices to take $k = K(n)$, as $\alpha(n)$ take any point of the interval I_k. □

The interval $I_m = (\alpha_m, \alpha_{m-1}]$ with $m = n - L + 1$ being the number of states in $D \cap \Gamma$ is of particular interest. Now we pass to investigating the asymptotic behavior of α_m and $\alpha_{m-1} - \alpha_m$, as $n \to \infty$.

Theorem 5.1. *The following statements are true:*

a) *for m being a constant, not less than 2,*

$$\alpha_m = 1 - \frac{1}{n-1} - \frac{m-1}{n^2} + O(n^{-3})$$

and

$$\alpha_{m-1} - \alpha_m = n^{-2} + O(n^{-3});$$

b) *for $m = m(n)$, $m^2 = o(n)$ and $m \to \infty$ as $n \to \infty$,*

$$\alpha_m = 1 - \frac{1}{n-1} - \frac{m}{(n-1)^2} + O\left(\frac{m}{n^3}\right)$$

and

$$\alpha_{m-1} - \alpha_m = n^{-2} + O(mn^{-3});$$

c) *for $m \sim Kn$, as $n \to \infty$, and $0 < K < 1 - 1/e$,*

$$\alpha_m = 1 + \frac{\ln(1 + \ln(1-K))}{Kn} + o\left(\frac{1}{n}\right).$$

Proof. Let us begin with the statement a). Inasmuch as $\alpha_1 = 1 - \frac{1}{n-1}$, its two equalities are equivalent, and it suffices to prove the second one. Since $m = n - L + 1$, one can write (5) as

$$\alpha_m = \left(1 - \sum_{j=n+1-m}^{n} \frac{1}{j-1}\right)^{\frac{1}{m}} < \alpha \le \left(1 - \sum_{j=n+2-m}^{n} \frac{1}{j-1}\right)^{\frac{1}{m-1}} = \alpha_{m-1}.$$

Let us denote

$$x = \sum_{j=n+2-m}^{n} \frac{1}{j-1}, \quad p = \frac{1}{m-1},$$

$$y = \sum_{j=n+1-m}^{n} \frac{1}{j-1}, \quad q = \frac{1}{m}.$$

Then

$$\alpha_{m-1} - \alpha_m = (1-x)^p - (1-y)^q.$$

From the Taylor series expansion

$$(1-z)^s = 1 - sz - \frac{s(1-s)}{2!}z^2 - \frac{s(1-s)(2-s)}{3!}z^3 - \dots,$$

it follows that for $0 < s < 1$ and $0 \le z \le 1 - \varepsilon$, $\varepsilon > 0$

$$(1-z)^s = 1 - sz - \frac{s(1-s)}{2!}z^2 + O(sz^3). \tag{7}$$

Here $O(sz^3) = sz^3 O(1)$, and $O(1)$ is a function bounded on its domain.
We have

$$\alpha_{m-1} - \alpha_m = (1-x)^p - (1-y)^q = \\ -px + qy - \frac{p(1-p)}{2}x^2 + \frac{q(1-q)}{2}y^2 + O\left(px^3 \vee qy^3\right). \tag{8}$$

Further,

$$\begin{aligned} -px + qy &= -\frac{1}{m-1}\sum_{j=n+2-m}^{n}\frac{1}{j-1} + \frac{1}{m}\sum_{n+1-m}^{n}\frac{1}{j-1} \\ &= \frac{1}{(m-1)m}\left(\frac{m}{n-m} - \sum_{j=n+1-m}^{n}\frac{1}{j-1}\right) \\ &= \frac{1}{(m-1)m}\sum_{j=n+1-m}^{n}\left(\frac{1}{n-m} - \frac{1}{j-1}\right) \\ &= |j = k+n+1-m| \\ &= \frac{1}{(m-1)m}\sum_{k-0}^{m-1}\frac{k}{(n-m)(k+n-m)}. \end{aligned} \tag{9}$$

Each of the terms of the last sum in (9) has the form $kn^{-2} + O(n^{-3})$, and

$$-px + qy = \frac{1}{m(m-1)}\sum_{k=0}^{m-1} kn^{-2} + O(n^{-3}) = \frac{1}{2}n^{-2} + O(n^{-3}).$$

Furthermore,

$$-\frac{p(1-p)}{2}x^2 + \frac{q(1-q)}{2}y^2 = -\frac{p}{2}x^2 + \frac{q}{2}y^2 + \frac{p^2x^2 - q^2y^2}{2}.$$

Simple estimates

$$\frac{m-1}{n-1} \le x \le \frac{m-1}{n+1-m}, \quad \frac{m}{n-1} \le y \le \frac{m}{n-m} \tag{10}$$

give

$$\begin{aligned} x^2 &= \left(\frac{m-1}{n}\right)^2 + O(n^{-3}), \\ y^2 &= \left(\frac{m}{n}\right)^2 + O(n^{-3}), \\ -\frac{p}{2}x^2 + \frac{q}{2}y^2 &= -\frac{1}{2(m-1)}\left(\frac{m-1}{n}\right)^2 + \frac{1}{2m}\left(\frac{m}{n}\right)^2 + O(n^{-3}) = \frac{1}{2n^2} + O(n^{-3}), \\ -\frac{p^2x^2 - q^2y^2}{2} &= -\frac{(px-qy)(px+qy)}{2} = -\frac{1}{2}\frac{1}{2n^2}O(n^{-1}) = O(n^{-3}), \\ x^3 &= O(n^{-3}), \\ y^3 &= O(n^{-3}). \end{aligned}$$

Combining these results, we get $\alpha_{m-1} - \alpha_m = n^{-2} + O(n^{-3})$, if $m \ge 2$.

Now let us turn to the case b). Instead of (8), we will use a more accurate formula

$$\alpha_{m-1}-\alpha_m=-px+qy-\frac{p(1-p)}{2}x^2 \\ +\frac{q(1-q)}{2}y^2-\frac{p(1-p)(2-p)}{3!}x^3+\frac{q(1-q)(2-q)}{3!}y^3+O\left(px^4\vee qy^4\right).$$

We will evaluate terms of the same order in turn.

Let us designate

$$r_k(n;m)=\frac{1}{(n-m)(k+n-m)}-\frac{1}{n^2}=\frac{m(2n-m)-k(n-m)}{(n-m)(n-m+k)n^2}.$$

Then $0<r_k(n;m)<r(n,m)=\frac{m(2n-m)}{(n-m)^2n^2}=O\left(\frac{m}{n^3}\right)$ and formula (9) gives

$$-px+qy=\frac{1}{(m-1)m}\sum_{k-0}^{m-1}\left(\frac{k}{n^2}+\frac{k}{(n-m)(k+n-m)}-\frac{k}{n^2}\right)=\frac{1}{2n^2}+O\left(\frac{m}{n^3}\right).$$

Estimates (10) allow us to evaluate

$$x=\frac{m-1}{n-m+1}+O\left(\frac{m}{n^2}\right),$$
$$y=\frac{m}{n-m}+O\left(\frac{m}{n^2}\right),$$
$$px^2=\frac{m-1}{(n-m+1)^2}+O\left(\frac{m}{n^3}\right),$$
$$qy^2=\frac{m}{(n-m)^2}+O\left(\frac{m}{n^3}\right),$$
$$p^2x^2-q^2y^2=(px-qy)(px+qy)=O\left(\frac{m}{n^3}\right).$$

Remark also that $\frac{m-1}{(n-m+1)^2}-\frac{m-1}{(n-m)^2}=O\left(\frac{m}{n^3}\right)$. This gives $-\frac{p}{2}x^2+\frac{q}{2}y^2=\frac{1}{2n^2}+O\left(\frac{m}{n^3}\right)$, and $-\frac{p(1-p)}{2}x^2+\frac{q(1-q)}{2}y^2=\frac{1}{2n^2}+O\left(\frac{m}{n^3}\right)$.

In evaluating

$$\frac{p(1-p)(2-p)}{3!}x^3-\frac{q(1-q)(2-q)}{3!}y^3,$$

we make use of the elementary identity $AB-CD=(A-C)B+(B-D)C$ with $A=x^3$, $B=\frac{p(1-p)(2-p)}{3!}$, $C=y^3$, $D=\frac{q(1-q)(2-q)}{3!}$. It gives

$$\frac{p(1-p)(2-p)}{3!}x^3-\frac{q(1-q)(2-q)}{3!}y^3=O\left(\frac{m}{n^3}\right).$$

Next, both px^4 and qy^4 are $\frac{1}{m}O\left(\frac{m^4}{n^4}\right)=o\left(\frac{m}{n^3}\right)$, so

$$\alpha_{m-1}-\alpha_m=n^{-2}+O(mn^{-3}).$$

To proceed to evaluating α_m, let us consider harmonic numbers $H_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, $n\geq 1$. The Euler-Maclaurin formula, applied to the function $f(x)=x^{-1}$, results in the equality

$$H_n=\ln n+\gamma+\frac{1}{2n}+\sum_{k=1}^{r}\frac{B_{2k}}{2kn^{2k}}-\theta_{r,n}\frac{B_{2r+2}}{(2r+2)n^{2r+2}},$$

where $0<\theta_{r,n}<1$, γ is the Euler-Mascheroni constant, and B_i are Bernoulli numbers [11]. We will limit ourselves to $r=1$:

$$H_n=\ln n+\gamma+\frac{1}{2n}+\frac{1}{12n^2}+O\left(\frac{1}{n^4}\right). \tag{11}$$

We have $\alpha_m = (1 - H_{n-1} + H_{n-m-1})^{\frac{1}{m}}$. In the case under consideration $H_{n-1} - H_{n-m-1} \underset{n\to\infty}{\to} 0$. This makes it possible to use equality (7) with $s = \frac{1}{m}$ and $z = H_{n-1} - H_{n-m-1}$, i.e. $\alpha_m = (1-z)^s = 1 - sz - \frac{s(1-s)}{2!}z^2 + O(sz^3)$. In accordance with (11),

$$z = -\ln\left(1 - \frac{m}{n-1}\right) - \frac{1}{2}\frac{m}{(n-1)(n-1-m)} + O\left(\frac{m}{n^3}\right)$$
$$= \frac{m}{n-1} + \frac{1}{2}\frac{m^2}{(n-1)^2} + \frac{1}{3}\frac{m^3}{(n-1)^3}(1+o(1)) - \frac{1}{2}\frac{m}{(n-1)(n-m-1)} + O\left(\frac{m}{n^3}\right).$$

Since $m^2 = o(n)$, $z = \frac{m}{n-1} + \frac{1}{2}\frac{m^2}{(n-1)^2} - \frac{1}{2}\frac{m}{(n-1)^2} + O\left(\frac{m}{n^3}\right)$. The major term in $-\frac{1}{2}z^2$ to be taken into account is $-\frac{1}{2}\frac{m^2}{(n-1)^2}$. Eventually we get

$$\alpha_m = 1 - \frac{1}{n-1} - \frac{m}{(n-1)^2} + O\left(\frac{m}{n^3}\right).$$

Now we turn to the case c). If $n \to \infty$, then $\frac{m}{n} = K + o(1)$, and it follows from (11) that $1 - H_{n-1} + H_{n-m-1} = 1 + \ln(1-K) + o(1)$. For brevity, let us denote $C = 1 + \ln(1-K)$. We have $\alpha_m = (C + o(1))^{\frac{1}{m}}$. Since $C > 0$,

$$\ln \alpha_m = \frac{1}{m}\ln(C + o(1)) = \frac{1}{m}\ln C + o\left(\frac{1}{m}\right)$$

and

$$\alpha_m = \exp\{\ln \alpha_m\} = \exp\left\{\frac{\ln C}{m} + o\left(\frac{1}{m}\right)\right\} = 1 + \frac{\ln C}{m} + o\left(\frac{1}{m}\right) = 1 + \frac{\ln C}{Kn} + o\left(\frac{1}{n}\right).$$

$\square$

Acknowledgement

The author is grateful to the reviewer for pointing out inaccuracies in the original version of the paper. This prompted the author to improve the presentation in some other places.

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