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CONSTRUCTING STOCHASTIC FLOWS OF KERNELS

In the paper we suggest a new construction of stochastic flows of kernels in a locally compact separable metric space M. Starting from a consistent sequence of Feller transtition function $(\mathsf{P}^{(n)} : n \geq 1)$ on M we prove existence of a stochastic flow of kernels $K = (K_{s,t} : -\infty < s \leq t < \infty)$ in M, such that distributions of n-point motions of K are determined by $\mathsf{P}^{(n)}$. Presented construction allows to find a single idempotent measurable presentation $\mathfrak{p}$ of distributions of all kernels $K_{s,t}$ from a flow, and to construct a flow that is invariant under $\mathfrak{p}$ and is jointly measurable in all arguments.

1. Introduction

Stochastic flows of kernels appear naturally as solutions to stochastic differential equations (SDE's) in the absence of strong uniqueness. Following fundamental works of Y. Le Jan and O. Raimond [8, 9], by a stochastic flow of kernels we understand a family $(K_{s,t} : -\infty < s \leq t < \infty)$ of random probability transition kernels on a locally compact separable metric space M that satisfy the evolutionary property $K_{r,s}K_{s,t} = K_{r,t}$, $K_{s,s}(x) = \delta_x$, $r \leq s \leq t$ (equalities must be understood in a proper sense that is explained below), have independent and homogeneous increments (if $t_1 \leq t_2 \leq \ldots \leq t_n$, then $K_{t_1,t_2}, \ldots, K_{t_{n-1},t_n}$ are independent; the distribution of $K_{s,t}$ depends only on $t-s$) and satisfy a variant of the Feller condition. Precise definition of a stochastic flow of kernels is given in Section 2.

One of the simplest examples of an SDE for which strong uniqueness fails is the Tanaka equation on $\mathbb{R}$

$$dX_t = \mathrm{sign}(X_t) d\mathrm{B}_t, \tag{1}$$

where $(\mathrm{B}_t : t \in \mathbb{R})$ is the standard Brownian motion on $\mathbb{R}$ [6, Ch. IV, §1]. Obviously, the solution X of (1) follows the trajectory of B when it is strictly positive, and follows the trajectory of $-\mathrm{B}$ when it is strictly negative. The reason for non-existence of a strong solution is that once the solution X reaches zero, it can randomly choose which excursion to follow: the excursion of B or the excursion of $-\mathrm{B}$. A natural extension of the Tanaka equation to kernels was suggested in [7] in the form

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}\left(f'\mathrm{sign}\right)(x) d\mathrm{B}(u) + \frac{1}{2}\int_s^t K_{s,u}(f'')(x)du, \ t \geq s, \tag{2}$$

where f is an arbitrary twice continuously differentiable function on $\mathbb{R}$ with compact support. If kernels $K_{s,t}$ are given by random mappings of $\varphi_{s,t} : \mathbb{R} \to \mathbb{R}$, i.e. $K_{s,t}(x) = \delta_{\varphi_{s,t}(x)}$, then the equation (2) is a consequence of (1) and the Itô formula. However, there are kernel solutions of (2) that are not given by random mappings. In [7] it was proved that all solutions of (2) are in one-to-one correspondence with probability measures m on $[0,1]$ with mean $\frac{1}{2}$, where m is the law of $K_{0,t}(0, [0,\infty))$. An amount of similar results for large classes of SDE's on manifolds and metric graphs were obtained in [2, 3, 4, 10, 11, 12]. Stochastic flows of kernels with Brownian n-point motions were studied in [5, 15, 17].

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In [8, 9] it was shown that to any sequence $(\mathsf{P}^{(n)} : n \geq 1)$ of consistent Feller transition functions (where $(\mathsf{P}^{(n)}_t : t \geq 0)$ is a Feller transition function on M^n) there corresponds a stochastic flow of kernels $(K_{s,t} : -\infty < s \leq t < \infty)$ such that for all $n \geq 1$, $t \geq 0$, $x \in M^n$

$$\mathsf{P}^{(n)}_t f(x) = \mathsf{E}\left[\int_{M^n} f(y)\left(\otimes_{i=1}^n K_{0,t}(x_i)\right)(dy)\right], \tag{3}$$

where f is an arbitrary continuous function on M^n that vanishes at infinity. Consistency of transition functions means that transition kernels $\mathsf{P}^{(n)}_t(x)$ behave properly under permutations of components of $x \in M^n$ and define transition kernels $\mathsf{P}^{(k)}_t(y)$ for all $k < n$ and $y \in M^k$. This result extends results of [1, 8, 9] on existence of stochastic flows of mappings. In [1] it was proved that to any sequence $(\mathsf{P}^{(n)} : n \geq 1)$ of consistent transition functions with additional property that $\mathsf{P}^{(2)}_t((x,x))$ is concentrated on a diagonal of M^2 (coalescing property) there corresponds a stochastic flow of random mappings $(\varphi_{s,t} : -\infty < s \leq t < \infty)$ of M such that for all $n \geq 1$, $t \geq 0$, $x \in M^n$

$$\mathsf{P}^{(n)}_t f(x) = \mathsf{E} f\left(\varphi_{0,t}(x_1), \ldots, \varphi_{0,t}(x_n)\right),$$

where f is an arbitrary continuous function on M^n that vanishes at infinity. In the construction of [1] the evolutionary property $\varphi_{s,t} \circ \varphi_{r,s} = \varphi_{r,t}$, $r \leq s \leq t$, holds without exceptions in r, s, t, ω, for any $t_1 \leq t_2 \leq \ldots \leq t_n$ mappings $\varphi_{t_1,t_2}, \ldots, \varphi_{t_{n-1},t_n}$ are independent, and the distribution of $\varphi_{s,t}$ depends only on $t-s$. However, in this construction the measurability of $\varphi_{s,t}(x)$ in any of the variables s, t or x is absent and only measurability in x can be achieved under rather strong restrictions on transition functions $(\mathsf{P}^{(n)} : n \geq 1)$. This limits the applicability of results of [1] in the context of equations like (2). To overcome the issue, in [8, 9] the Feller property of $\mathsf{P}^{(n)}$ is assumed and the definition of a stochastic flow is modified. Namely, a stochastic flow of mappings is a family $(\varphi_{s,t} : -\infty < s \leq t < \infty)$ of random elements in the space of measurable mappings of M (equipped with the cylindrical σ-field) that satisfies a variant of the Feller property and for which the evolutionary property is understood as follows:

for all $r \leq s \leq t$ and $x \in M$ with probability 1

$$\varphi_{r,t}(x) = \mathcal{J}_{t-s}\left(\varphi_{s,t}\right) \circ \varphi_{r,s}(x), \tag{4}$$

where $\mathcal{J}_{t-s}$ is a measurable presentation of the distribution of $\varphi_{s,t}$ in the space of measurable mappings of M. The usage of a measurable presentation $\mathcal{J}_{t-s}$ together with a variant of the Feller property for φ allows to settle a one-to-one correspondence between stochastic flows of mappings and coalescing sequences of consistent Feller transition functions. Similarly, the evolutionary property for stochastic flows of kernels in [8, 9] is understood as follows:

for all $r \leq s \leq t$ and $x \in M$ with probability 1

$$K_{r,t}(x) = K_{r,s}\mathfrak{p}_{t-s}\left(K_{s,t}\right)(x), \tag{5}$$

where $\mathfrak{p}_{t-s}$ is a measurable presentation of the distribution of $K_{s,t}$ in the space of kernels on M (see Section 2 for more details).

Presences of $\mathcal{J}_{t-s}$ in (4) and of $\mathfrak{p}_{t-s}$ in (5) do not look natural. However, they are necessary due to two reasons at least. Firstly, the convolution of kernels is in general a non-measurable operation and it is not clear how to define convolution of two independent random kernels in a measurable way. Secondly, the presence of $\mathfrak{p}_{t-s}$ in (5) allows to show that functions $\mathsf{P}^{(n)}_t(x)$ defined in (3) are actually transition functions. In [8, 9] a stochastic flow of mappings $(\varphi_{s,t} : -\infty < s \leq t < \infty)$ was constructed in such a way that equalities $\mathcal{J}_{t-s}(\varphi_{s,t}(\omega)) = \varphi_{s,t}(\omega)$ were satisfied without exceptions in s, t, ω. The same result for flows of kernels was absent. The reason is that in [8, 9] the flow of kernels is constructed from a certain stochastic flow of measure-valued mappings, and the procedure that produces the flow of kernels does not commute with measurable presentations of distributions of measure-valued mappings. In this paper we improve the approach suggested in [8, 9]. Starting from a consistent sequence of Feller transition functions we prove the existence of a single idempotent measurable presentation $\mathfrak{p}$ of corresponding distributions of kernels. Further, we construct a stochastic flow of kernels $(K_{s,t} : -\infty < s \le t < \infty)$ in such a way that equalities $K_{s,t}(\omega) = \mathfrak{p}\left(K_{s,t}(\omega)\right)$ are satisfied without exceptions in s, t, ω. Moreover, we achieve measurability of the mapping $(s, t, \omega) \mapsto K_{s,t}(\omega)$. Together with equalities $K_{s,t}(\omega) = \mathfrak{p}\left(K_{s,t}(\omega)\right)$ this implies the measurability of the mapping $(s, t, \omega, x) \mapsto K_{s,t}(\omega, x) = \mathfrak{p}(K_{s,t}(\omega))(x)$ and the evolutionary property in the usual form $K_{r,t}(x) = K_{r,s}K_{s,t}(x)$ a.s., where exceptional sets depend on $r \le s \le t$ and $x \in M$.

The paper is organized as follows. In Section 2 we give definitions of consistent sequences of Feller transition functions, Feller convolution semigroups in the space of kernels and stochastic flows of kernels on a locally compact separable metric space M. Also, we show that a Feller convolution semigroup on M defines a consistent sequence of Feller transition functions on M that determines finite-point motions with respect to the semigroup, and a stochastic flow of kernels in M defines a Feller convolution semigroup in the space of kernels on M that defines distributions of kernels in a flow. In Section 3 we prove that any consistent sequence of Feller transition functions on M defines a unique Feller convolution semigroup in the space of kernels on M with finite-point motions determined by the given sequence of transition functions. This result was obtained in [8, 9]. Our approach enables to construct a single idempotent measurable presentation $\mathfrak{p}$ of all distributions from a Feller convolution semigroup (Theorem 2.1). In Section 4 we prove that from any Feller convolution semigroup $(\nu_t : t \ge 0)$ in the space of kernels on M one can construct a stochastic flow of kernels $(K_{s,t} : -\infty < s \le t < \infty)$ in M, for which the distribution of each kernel $K_{s,t}$ coincides with ν_{t-s}, the mapping $(s, t, \omega) \mapsto K_{s,t}(\omega)$ is measurable and equalities $\mathfrak{p}(K_{s,t}(\omega)) = K_{s,t}(\omega)$ hold without exceptions in (s, t, ω) (Theorem 2.2). Auxiliary Propositions 4.3 and 4.4 about approximations of stochastic flows of kernels seem to be new and interesting on their own. Another interesting consequence of our approach is that constructions of Feller convolution semigroups and of stochastic flows of kernels are done using approximating procedures that are very similar in their nature, but differ in the domain of approximation: the approximation is in space for Feller convolution semigroups and is in time for stochastic flow of kernels.

Finally, we note that our definitions of stochastic flows of kernels and Feller convolution semigroups are slightly different from the ones given in [8, 9]. To show equivalence of definitions we give full proofs of several known statements from [8, 9].

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2. Definitions, Preliminaries and Main Results

Let (M, ρ) be a locally compact separable metric space equipped with the Borel σ-field $\mathcal{B}(M)$. Without loss of generality we assume that all ρ-bounded sets are relatively compact. In particular, (M, ρ) is a complete separable metric space. By $C(M)$ we denote the space of bounded continuous functions on M, and by $C_0(M)$ we denote the space of all continuous functions $f \in C(M)$ that vanish at infinity in the sense that for any $\varepsilon > 0$ there exists a compact $C \subset M$, such that $\sup_{x \in M \setminus C} |f(x)| \le \varepsilon$. With respect to the norm $\|f\| = \sup_M |f|$ the space $C_0(M)$ is a separable Banach space. $\mathcal{P}(M)$ denotes the space of all Borel probability measures on M.

Let $\hat{M}$ be the one-point compactification of M. The following construction will be useful in our considerations. Write M as a union $M = \bigcup_{j=1}^{\infty} L_j$ of compact sets L_j, such that L_j is contained in the interior of L_{j+1}. For each j fix a continuous function

$\zeta_j : \hat{M} \to [0,1]$, such that $\zeta_j|_{L_j} = 1$ and the support of ζ_j is contained in the interior of L_{j+1}. Sequences $(L_j : j \geq 1)$ and $(\zeta_j : j \geq 1)$ will be called exhaustive.

The space $\mathcal{P}(\hat{M})$ equipped with the topology of weak convergence is a compact metrizable space. Let $\hat{d}$ be the corresponding metric on $\mathcal{P}(\hat{M})$. The set $\mathcal{P}(M)$ is a G_δ subset in $\mathcal{P}(\hat{M})$, hence is a Polish space [13, Ch. II, Th. 6.5]. Denote by d a metric on $\mathcal{P}(M)$ that is compatible with the topology of weak convergence and turns $\mathcal{P}(M)$ into a complete separable metric space.

2.1. **Consistent sequences of Feller transition functions.** For $1 \leq k \leq n$ denote by $S_{k,n}$ the set of all injections $\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$. Any $\sigma \in S_{k,n}$ defines the mapping $\pi_\sigma : M^n \to M^k$, $\pi_\sigma x = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$.

Assume that for each $n \in \mathbb{N}$ a Feller transition function $\mathsf{P}^{(n)}$ on M^n is defined.

Definition 2.1. [8, Def. 1.1] A sequence $(\mathsf{P}^{(n)} : n \in \mathbb{N})$ is called a consistent sequence of Feller transition functions on M, if

for all $1 \leq k \leq n$, $\sigma \in S_{k,n}$, $x \in M^n$ and $t \geq 0$

$$\mathsf{P}_t^{(n)}(x) \circ \pi_\sigma^{-1} = \mathsf{P}_t^{(k)}(\pi_\sigma x). \tag{6}$$

The following Lemma contains one useful property of Feller transition functions.

Lemma 2.1. *Let $(\mathsf{P}_t : t \geq 0)$ be a Feller transition function on a locally compact separable metric space M. Then for any compact $C \subset M$, $T \geq 0$ and $\varepsilon > 0$, there exists a compact $L \subset M$, such that*

$$\inf_{x \in C, t \in [0,T]} \mathsf{P}_t(x, L) \geq 1 - \varepsilon.$$

Proof. Feller property implies that the map $(t, x) \mapsto \mathsf{P}_t(x) \in \mathcal{P}(M)$ is continuous. In particular, the set $\{\mathsf{P}_t(x) : t \in [0,T], x \in C\}$ is compact in $\mathcal{P}(M)$. The result follows from Prokhorov's theorem [13, Th. 6.7, Ch. II].

□

2.2. **Feller convolution semigroups in the space of kernels.** A kernel on M is a measurable mapping $K : M \to \mathcal{P}(M)$. By E we denote the set of all kernels on M. For $K_1, K_2 \in E$ denote by K_1K_2 a kernel

$$K_1K_2(x) = \int_M K_2(y) K_1(x, dy).$$

For $\mu \in \mathcal{P}(M)$ we denote by μK a probability measure $\mu K(B) = \int_M K(x,B)\mu(dx)$, and for a bounded measurable function $f : M \to \mathbb{R}$ we denote by Kf a measurable function $Kf(x) = \int_M f(y)K(x, dy)$.

The set E is equipped with the cylindrical σ-field $\mathcal{E}$, i.e. the smallest σ-field on E with respect to which all mappings $K \mapsto K(x)$, $x \in M$, are $\mathcal{E}/\mathcal{B}(\mathcal{P}(M))$-measurable.

Definition 2.2. [8, Def. 1.2, Def. 2.1] A probability measure ν on $(E, \mathcal{E})$ is called *regular*, if there exists a mapping $\mathfrak{p} : E \to E$, such that the mapping $E \times M \ni (K, x) \mapsto \mathfrak{p}(K)(x) \in \mathcal{P}(M)$ is measurable, and for all $x \in M$, $\mathfrak{p}(K)(x) = K(x)$ ν-a.s.

The mapping $\mathfrak{p}$ is called a measurable presentation of a regular measure ν. Let ν_1, ν_2 be regular probability measures on $(E, \mathcal{E})$, and let $\mathfrak{p}$ be a measurable presentation of ν_2. Then the mapping $(K_1, K_2) \mapsto K_1\mathfrak{p}(K_2)$ is $\mathcal{E}^{\otimes 2}/\mathcal{E}$-measurable and its distribution with respect to the product measure $\nu_1 \otimes \nu_2$ is independent from the choice of $\mathfrak{p}$. The latter distribution is denoted by $\nu_1 * \nu_2$ and is called a convolution of ν_1 and ν_2 [8].

Definition 2.3. [8, Def. 1.4, Def. 1.5] A family $(\nu_t : t \geq 0)$ of regular probability measures on $(E, \mathcal{E})$ is called a Feller convolution semigroup in the space of kernels on M, if

(1) for all $t, s \geq 0$, $\nu_t * \nu_s = \nu_{t+s}$;
(2) for any $f \in C_0(M)$ and any $\varepsilon > 0$,

$$\lim_{t \to 0+} \sup_{x \in M} \nu_t \{K : |Kf(x) - f(x)| \geq \varepsilon\} = 0;$$

(3) for any $f \in C_0(M)$, $t \geq 0$, $x \in M$ and $\varepsilon > 0$,

$$\lim_{y \to x} \nu_t \{K : |Kf(y) - Kf(x)| \geq \varepsilon\} = 0, \ \lim_{y \to \infty} \nu_t \{K : |Kf(y)| \geq \varepsilon\} = 0.$$

To each Feller convolution semigroup in the space of kernels $(\nu_t : t \geq 0)$ one can associate a consistent sequence of Feller transition functions $(\mathsf{P}^{(n)} : n \geq 1)$ as follows: for all $n \geq 1$, $x \in M^n$, $B \in \mathcal{B}(M^n)$, $t \geq 0$,

$$\mathsf{P}_t^{(n)}(x, B) = \int_E \left(\otimes_{i=1}^n K(x_i)\right)(B) \nu_t(dK). \tag{7}$$

Proposition 2.1. $(\mathsf{P}^{(n)} : n \geq 1)$ *is a consistent sequence of Feller transition functions on* M.

Proof. Let $\mathfrak{p}_t$ be a measurable presentation of ν_t. Measurability of $\mathsf{P}_t^{(n)}(x, B)$ in x and the Chapman-Kolmogorov equation for $\mathsf{P}^{(n)}$ follow from the representation

$$\mathsf{P}_t^{(n)}(x, B) = \int_E \left(\otimes_{i=1}^n \mathfrak{p}_t(K)(x_i)\right)(B) \nu_t(dK)$$

and the convolution semigroup property of ν.

We verify consistency. Let $\sigma \in S_{k,n}$. Then

$$\begin{aligned} \mathsf{P}_t^{(n)}\left(x, \pi_\sigma^{-1}(B)\right) &= \int_E \left(\otimes_{j=1}^n K(x_j)\right)\left(\pi_\sigma^{-1}(B)\right) \nu_t(dK) \\ &= \int_E \left(\otimes_{j=1}^k K(x_{\sigma(j)})\right)(B) \nu_t(dK) = \mathsf{P}_t^{(k)}\left(\pi_\sigma x, B\right). \end{aligned}$$

It remains to verify the Feller property of $\mathsf{P}^{(n)}$. By the Stone-Weierstrass theorem, it is enough to consider functions $f \in C_0(M^n)$ of the form $f(x) = \prod_{j=1}^n g_j(x_j)$, $g_j \in C_0(M)$. Then

$$\begin{aligned} |\mathsf{P}_t^{(n)} f(x) - \mathsf{P}_t^{(n)} f(y)| &= \left| \int_E \prod_{j=1}^n K g_j(x_j) \nu_t(dK) - \int_E \prod_{j=1}^n K g_j(y_j) \nu_t(dK) \right| \\ &\leq \sum_{k=1}^n \left| \int_E \left(\prod_{j=1}^k K g_j(x_j) \prod_{j=k+1}^n K g_j(y_j) - \prod_{j=1}^{k-1} K g_j(x_j) \prod_{j=k}^n K g_j(y_j) \right) \nu_t(dK) \right| \\ &\leq \sum_{k=1}^n \left| \int_E \prod_{j=1}^{k-1} K g_j(x_j) \prod_{j=k+1}^n K g_j(y_j) \times \left(K g_k(x_k) - K g_k(y_k)\right) \nu_t(dK) \right| \\ &\leq 2n \prod_{j=1}^n \|g_j\| \times \sup_{1 \leq k \leq n} \nu_t\{K : \ |K g_k(x_k) - K g_k(y_k)| \geq \varepsilon\} + n\varepsilon \prod_{j=1}^n (\|g_j\| + 1) \\ &\to n\varepsilon \prod_{j=1}^n (\|g_j\| + 1), \ y \to x. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\mathsf{P}_t^{(n)} f$ is continuous on M^n.

For any $\varepsilon > 0$ there exists a compact $L \subset M$, such that

$$\sup_{1 \leq k \leq n} \sup_{y \notin L} \nu_t\{K : |K g_k(y)| \geq \varepsilon\} \leq \varepsilon.$$

If $x \notin L^n$ with, say, $x_k \notin L$, then

$$|\mathsf{P}_t^{(n)} f(x)| = \left| \int_E \prod_{j=1}^n K g_j(x_j) \nu_t(dK) \right|$$
$$\leq \prod_{j=1}^n \|g_j\| \times \nu_t\{K : |Kg_k(x_k)| \geq \varepsilon\} + \varepsilon \prod_{j=1}^n (\|g_j\| + 1) \leq 2\varepsilon \prod_{j=1}^n (\|g_j\| + 1).$$

It follows that $\lim_{x\to\infty} \mathsf{P}_t^{(n)} f(x) = 0$. So, $\mathsf{P}_t^{(n)}(C_0(M^n)) \subset C_0(M^n)$.

Further,

$$|\mathsf{P}_t^{(n)} f(x) - f(x)| = \left| \int_E \left(\prod_{j=1}^n K g_j(x_j) - \prod_{j=1}^n g_j(x_j) \right) \nu_t(dK) \right|$$
$$\leq \sum_{k=1}^n \left| \int_E \left(\prod_{j=1}^k K g_j(x_j) \prod_{j=k+1}^n g_j(x_j) - \prod_{j=1}^{k-1} K g_j(x_j) \prod_{j=k}^n g_j(x_j) \right) \nu_t(dK) \right|$$
$$\leq \sum_{k=1}^n \left| \int_E \prod_{j=1}^{k-1} K g_j(x_j) \prod_{j=k+1}^n g_j(x_j) \times (K g_k(x_k) - g_k(x_k)) \, \nu_t(dK) \right|$$
$$\leq 2n \prod_{j=1}^n \|g_j\| \times \sup_{1\leq k\leq n} \sup_{y\in M} \nu_t\{K : \ |Kg_k(y) - g_k(y)| \geq \varepsilon\} + n\varepsilon \prod_{j=1}^n (\|g_j\| + 1).$$

It follows that

$$\sup_{x\in M^n} |\mathsf{P}_t^{(n)} f(x) - f(x)| \leq 2n \prod_{j=1}^n \|g_j\| \times \sup_{1\leq k\leq n} \sup_{y\in M} \nu_t\{K : \ |Kg_k(y) - g_k(y)| \geq \varepsilon\}$$
$$+ n\varepsilon \prod_{j=1}^n (\|g_j\| + 1) \to n\varepsilon \prod_{j=1}^n (\|g_j\| + 1), \ t \to 0.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $(\mathsf{P}_t^{(n)} : t \geq 0)$ is strongly continuous at $t = 0$. □

The sequence $(\mathsf{P}^{(n)} : n \geq 1)$ completely determines the semigroup $(\nu_t : t \geq 0)$. To show this we introduce an algebra $\mathbb{A}_n(M)$ of continuous functions on $\mathcal{P}(M)^n$, that consists of all functions $g : \mathcal{P}(M)^n \to \mathbb{R}$ of the form

$$g(\mu_1, \ldots, \mu_n) = \int_{M^N} f(y) \, (\mu_{i_1} \otimes \ldots \otimes \mu_{i_N}) \, (dy), \tag{8}$$

where $f \in C_0(M^N)$, $(i_1, \ldots, i_N) \in \{1, \ldots, n\}^N$.

Lemma 2.2. *A probability measure Π on $\mathcal{P}(M)^n$ is completely determined by integrals of the form*

$$\int_{\mathcal{P}(M)^n} g(\mu_1, \ldots, \mu_n) \Pi(d\mu), \tag{9}$$

where $g \in \mathbb{A}_n(M)$.

Proof. Let M be compact. Then $\mathcal{P}(M)^n$ is a compact metric space and $\mathbb{A}_n(M)$ is dense in $C(\mathcal{P}(M)^n)$ by the Stone-Weierstrass theorem. Hence, integrals of the form (9) with $g \in \mathbb{A}_n(M)$ define integrals of the form (9) with $g \in C(\mathcal{P}(M)^n)$. In this case the result is proved.

In general case, consider the one-point compactification $\hat{M}$ of M. Π can be viewed as a probability measure on $\mathcal{P}(\hat{M})^n$. It is completely determined by integrals of the form (9) with $g \in \mathbb{A}_n(\hat{M})$. Consider $g \in \mathbb{A}_n(\hat{M})$ of the form

$$g(\mu_1, \ldots, \mu_n) = \int_{\hat{M}^N} f(y)(\mu_{i_1} \otimes \ldots \otimes \mu_{i_N})(dy),$$

where $f \in C(\hat{M}^N)$, $(i_1, \ldots, i_N) \in \{1, \ldots, n\}^N$. Let $g_j \in \mathbb{A}_n(M)$ be defined as

$$g_j(\mu_1, \ldots, \mu_n) = \int_{M^N} f(y)\zeta_j^{\otimes N}(y)(\mu_{i_1} \otimes \ldots \otimes \mu_{i_N})(dy),$$

where $(\zeta_j : j \geq 1)$ is the exhaustive sequence introduced in the beginning of Section 2. The result follows, since

$$\int_{\mathcal{P}(M)^n} g(\mu_1, \ldots, \mu_n)\Pi(d\mu) = \int_{\mathcal{P}(M)^n} \left(\int_{M^N} f(y)(\mu_{i_1} \otimes \ldots \otimes \mu_{i_N})(dy) \right) \Pi(d\mu)$$
$$= \lim_{j \to \infty} \int_{\mathcal{P}(M)^n} g_j(\mu_1, \ldots, \mu_n)\Pi(d\mu).$$

□

Lemma 2.3. *The sequence* $(\mathsf{P}^{(n)} : n \geq 1)$ *completely determines the Feller convolution semigroup* $(\nu_t : t \geq 0)$.

Proof. The probability measure ν_t is completely determined by distributions of $\mathcal{P}(M)^n$-valued random elements $(K(x_1), \ldots, K(x_n))$, where $x \in M^n$, $n \geq 1$. Hence, ν_t is completely determined by integrals of the form

$$\int_E \left(\int_{M^N} f(y) \left(K(x_{i_1}) \otimes \ldots \otimes K(x_{i_N}) \right) (dy) \right) \nu_t(dK), \tag{10}$$

where $f \in C_0(M^N)$, $(i_1, \ldots, i_N) \in \{1, \ldots, n\}^N$. It remains to note that (10) is equal to $\mathsf{P}_t^{(N)} f(x_{i_1}, \ldots, x_{i_N})$.

□

In [8] it was proved that to any consistent sequence of Feller transition functions $(\mathsf{P}^{(n)} : n \geq 1)$ on M there corresponds a unique Feller convolution semigroup $(\nu_t : t \geq 0)$ on E, such that (7) holds. Theorem 2.1 gives a strengthed version of this result. The main difference is that we find one idempotent measurable presentation $\mathfrak{p}$ of all measures ν_t.

Theorem 2.1. *Let* $(\mathsf{P}^{(n)} : n \geq 1)$ *be a consistent sequence of Feller transition functions on* M*. There exists a unique Feller convolution semigroup* $(\nu_t : t \geq 0)$ *that satisfies* (7)*. Moreover, there exists a mapping* $\mathfrak{p} : E \to E$ *which is a measurable presentation of every measure* ν_t, $t \geq 0$, *and satisfies the relation* $\mathfrak{p} \circ \mathfrak{p} = \mathfrak{p}$.

2.3. **Stochastic flows of kernels.**

Definition 2.4. [8, Def. 2.3], [9, Def. 7] A stochastic flow of kernels in M is a family $K = (K_{s,t} : -\infty < s \leq t < \infty)$ of random elements in $(E, \mathcal{E})$ that are defined on a common probability space $(\Omega, \mathcal{A}, \mathsf{P})$ and satisfy the following properties:

(1) the law of $K_{s,t}$ is regular and coincides with the law of $K_{0,t-s}$;
(2) for all $r \leq s \leq t$, $x \in M$ and any measurable presentation $\mathfrak{p}_{t-s}$ of the law of $K_{s,t}$,

$$K_{r,t}(x) = K_{r,s}\mathfrak{p}_{t-s}(K_{s,t})(x) \quad \mathsf{P} - \text{a.s.};$$

(3) if $t_1 \leq \ldots \leq t_n$, then $K_{t_1,t_2}, \ldots, K_{t_{n-1},t_n}$ are mutually independent;

(4) for any $f \in C_0(M)$ and $\varepsilon > 0$,

$$\lim_{t\to 0+} \sup_{x\in M} \mathsf{P}\left\{|K_{0,t}f(x) - f(x)| \geq \varepsilon\right\} = 0;$$

(5) for any $f \in C_0(M)$, $x \in M$, $t \geq 0$ and $\varepsilon > 0$,

$$\lim_{y\to x} \mathsf{P}\{|K_{0,t}f(y) - K_{0,t}f(x)| \geq \varepsilon\} = 0,\ \lim_{y\to\infty} \mathsf{P}\{|K_{0,t}f(y)| \geq \varepsilon\} = 0.$$

Let ν_t denote the law of $K_{0,t}$. Clearly, $(\nu_t : t \geq 0)$ is a Feller convolution semigroup in the space of kernels on M. The converse result is also true: if $(\nu_t : t \geq 0)$ is a Feller convolution semigroup in the space of kernels on M, then there exists a stochastic flow of kernels $K = (K_{s,t} : -\infty < s \leq t < \infty)$ in M, such that for all $s \leq t$ the law of $K_{s,t}$ coincides with ν_{t-s} [8, Th 2.1]. We prove that such stochastic flow can be always constructed as a measurable function from (s,t,ω) that satisfies relations $\mathfrak{p}_{t-s}(K_{s,t}(\omega)) = K_{s,t}(\omega)$ without exceptions in (s,t,ω).

Theorem 2.2. *Let $(\nu_t : t \geq 0)$ be a Feller convolution semigroup in the space of kernels on M. There exist a common idempotent measurable presentation $\mathfrak{p}$ of measures ν_t (Theorem 2.1) and a stochastic flow of kernels K in M, such that*

(1) *For all $s \leq t$ the law of $K_{s,t}$ coincides with ν_{t-s};*
(2) *The mapping $(s,t,\omega) \mapsto K_{s,t}(\omega)$ is jointly measurable;*
(3) *$K_{s,s}(\omega)(x) = \delta_x$ for all $s \in \mathbb{R}$, $x \in M$, $\omega \in \Omega$;*
(4) *$\mathfrak{p}(K_{s,t}(\omega)) = K_{s,t}(\omega)$ for all $s \leq t$ and $\omega \in \Omega$.*

3. Proof of Theorem 2.1

3.1. **Probability measures $\Pi_t^{(n)}(x)$.** Out of the sequence $(\mathsf{P}^{(n)} : n \geq 1)$ we construct for any $n \geq 1$, $x \in M^n$ and $t \geq 0$ a probability measure $\Pi_t^{(n)}(x)$ on $\mathcal{P}(M)^n$ which will be the distribution of $K \mapsto (K(x_1), \ldots, K(x_n))$ under ν_t.

Recall the dense algebra $\mathbb{A}_n(\hat{M})$ in the space of continuous functions on $\mathcal{P}(\hat{M})^n$. Let $g \in \mathbb{A}_n(\hat{M})$ be of the form (8) with $f \in C(\hat{M}^N)$, $(i_1, \ldots, i_N) \in \{1, \ldots, n\}^N$. For $x \in M^n$ and $t \geq 0$ define

$$\Pi_t^{(n)}(x)g = \int_{M^N} f(y)\mathsf{P}_t^{(N)}((x_{i_1}, \ldots, x_{i_N}), dy).$$

Lemma 3.1. *$\Pi_t^{(n)}(x)$ is a correctly defined linear non-negative functional on $\mathbb{A}_n(\hat{M})$, such that $\Pi_t^{(n)}(x)1 = 1$.*

Proof. Let us check correctness of the definition of $\Pi_t^{(n)}(x)$. Assume that $g \in \mathbb{A}_n(\hat{M})$ has two representations: for all $(\mu_1, \ldots, \mu_n) \in \mathcal{P}(\hat{M})^n$

$$\begin{aligned} g(\mu_1, \ldots, \mu_n) &= \int_{\hat{M}^N} f(y)\,(\mu_{i_1} \otimes \ldots \otimes \mu_{i_N})\,(dy) \\ &= \int_{\hat{M}^R} v(y)\,(\mu_{j_1} \otimes \ldots \otimes \mu_{j_R})\,(dy), \end{aligned}$$

where $f \in C(\hat{M}^N)$, $(i_1, \ldots, i_N) \in \{1, \ldots, n\}^N$, $v \in C(\hat{M}^R)$, $(j_1, \ldots, j_R) \in \{1, \ldots, n\}^R$. Consider injections $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N+R\}$, $\delta : \{1, \ldots, R\} \to \{1, \ldots, N+R\}$, defined by

$$\sigma(i) = i,\ 1 \leq i \leq N,\ \delta(j) = N + j,\ 1 \leq j \leq R.$$

Then

$$\begin{aligned} g(\mu_1, \ldots, \mu_n) &= \int_{\hat{M}^{N+R}} f \circ \pi_\sigma(y)\,(\mu_{i_1} \otimes \ldots \otimes \mu_{i_N} \otimes \mu_{j_1} \otimes \ldots \otimes \mu_{j_R})\,(dy) \\ &= \int_{\hat{M}^{N+R}} v \circ \pi_\delta(y)\,(\mu_{i_1} \otimes \ldots \otimes \mu_{i_N} \otimes \mu_{j_1} \otimes \ldots \otimes \mu_{j_R})\,(dy). \end{aligned}$$

By consistency,

$$\int_{M^{N+R}} f \circ \pi_\sigma(y) \mathsf{P}_t^{(N+R)}((x_{i_1}, \ldots, x_{i_N}, x_{j_1}, \ldots, x_{j_R}), dy) = \int_{M^N} f(y) \mathsf{P}_t^{(N)}((x_{i_1}, \ldots, x_{i_N}), dy),$$

$$\int_{M^L} v \circ \pi_\delta(y) \mathsf{P}_t^{(N+R)}((x_{i_1}, \ldots, x_{i_N}, x_{j_1}, \ldots, x_{j_R}), dy) = \int_{M^R} v(y) \mathsf{P}_t^{(R)}((x_{j_1}, \ldots, x_{j_R}), dy).$$

So, it is enough to consider the case $(i_1, \ldots, i_N) = (j_1, \ldots, j_R)$. Further, it is enough to prove that the equality

$$\int_{\hat{M}^N} f(y) \left(\mu_{i_1} \otimes \ldots \otimes \mu_{i_N}\right)(dy) = 0, \ (\mu_1, \ldots, \mu_n) \in \mathcal{P}(\hat{M})^n, \tag{11}$$

implies

$$\int_{M^N} f(y) \mathsf{P}_t^{(N)}((x_{i_1}, \ldots, x_{i_N}), dy) = 0.$$

Assume that (11) holds. For $s \in \{1, \ldots, n\}$ denote

$$I_s = \{k \in \{1, \ldots, N\} : \ i_k = s\}$$

and let m_s be the number of elements in I_s. Denote by $S_{N,N}(I_1, \ldots, I_n)$ the set of all permutations $\sigma \in S_{N,N}$ such that $\sigma(I_s) = I_s$ for all $s \in \{1, \ldots, n\}$. Let

$$\tilde{f}(y) = \frac{1}{m_1! \ldots m_n!} \sum_{\sigma \in S_{N,N}(I_1, \ldots, I_n)} f \circ \pi_\sigma(y).$$

We note that

$$\int_{\hat{M}^N} \tilde{f}(y) \left(\mu_{i_1} \otimes \ldots \otimes \mu_{i_N}\right)(dy) = 0.$$

By consistency,

$$\begin{aligned}
&\int_{M^N} \tilde{f}(y) \mathsf{P}_t^{(N)}((x_{i_1}, \ldots, x_{i_N}), dy) \\
&\quad = \frac{1}{m_1! \ldots m_n!} \sum_{\sigma \in S_{N,N}(I_1, \ldots, I_n)} \int_{M^N} f \circ \pi_\sigma(y) \mathsf{P}_t^{(N)}((x_{i_1}, \ldots, x_{i_N}), dy) \\
&\quad = \int_{M^N} f(y) \mathsf{P}_t^{(N)}((x_{i_1}, \ldots, x_{i_N}), dy).
\end{aligned}$$

So, we may assume that $f \circ \pi_\sigma = f$ for all $\sigma \in S_{N,N}(I_1, \ldots, I_n)$. We will show that the equality (11) implies $f(z) = 0$ for all $z \in \hat{M}^N$. By Fubini's theorem it is enough to consider the case $n = 1$. In this case $f \in C(\hat{M}^N)$ is symmetric and

$$\int_{\hat{M}^N} f(y) \mu^{\otimes N}(dy) = 0$$

for all finite measures μ on $\hat{M}$. Let $z \in \hat{M}^N$. Then

$$\int_{\hat{M}^N} f(y) \left(p_1 \delta_{z_1} + \ldots + p_N \delta_{z_N}\right)^{\otimes N}(dy) = 0$$

for all $p_1, \ldots, p_N > 0$. Expanding and using symmetry of f, we get

$$\sum_{k_1 + \ldots + k_N = N} \frac{N!}{k_1! \ldots k_N!} p_1^{k_1} \ldots p_N^{k_N} f(\underbrace{z_1, \ldots, z_1}_{k_1}, \ldots, \underbrace{z_N, \ldots, z_N}_{k_N}) = 0.$$

Differentiating in $p_1,\dots,p_N$ at $p_1=\dots=p_N=0$ we find that $f(z)=0$. Correctness of the definition of $\Pi_t^{(n)}(x)$ is verified. Independence of $\Pi_t^{(n)}(x)g$ from the representation of g in the form (8) implies linearity of $\Pi_t^{(n)}(x)$.

It remains to verify that the linear functional $\Pi_t^{(n)}(x):\mathbb{A}_n(\hat{M})\to\mathbb{R}$ is non-negative. Assume that for all $(\mu_1,\dots,\mu_n)\in\mathcal{P}(\hat{M}^n)$

$$g(\mu_1,\dots,\mu_n)=\int_{\hat{M}^N} f(y)\,(\mu_{i_1}\otimes\dots\otimes\mu_{i_N})\,(dy)\ge 0.$$

As before, denote $I_s=\{k\in\{1,\dots,N\}:\ i_k=s\}$, $s\in\{1,\dots,n\}$, and let m_s be the number of elements in I_s. For an integer L denote

$$x^{(L)}=(\underbrace{x_1,\dots,x_1}_{L},\underbrace{x_2,\dots,x_2}_{L},\dots,\underbrace{x_n,\dots,x_n}_{L}).$$

We have

$$\int_{M^{Ln}} g\left(\frac{1}{L}\sum_{j=1}^{L}\delta_{y_j},\frac{1}{L}\sum_{j=1}^{L}\delta_{y_{L+j}},\dots\frac{1}{L}\sum_{j=1}^{L}\delta_{y_{(n-1)L+j}}\right)\mathsf{P}_t^{(Ln)}(x^{(L)},dy)\ge 0.$$

Hence,

$$\frac{1}{L^N}\sum_{j_1,\dots,j_N=1}^{L}\int_{M^{Ln}} f\Big(y_{(i_1-1)L+j_1},\dots,y_{(i_N-1)L+j_N}\Big)\mathsf{P}_t^{(Ln)}(x^{(L)},dy)\ge 0. \tag{12}$$

Assume that for every $s\in\{1,\dots,n\}$ all j_k with $k\in I_s$ are distinct. By consistency,

$$\int_{M^{Ln}} f\Big(y_{(i_1-1)L+j_1},\dots,y_{(i_N-1)L+j_N}\Big)\mathsf{P}_t^{(Ln)}(x^{(L)},dy)$$
$$=\int_{M^N} f(y)\mathsf{P}_t^{(N)}((x_{i_1},\dots,x_{i_N}),dy)=\Pi_t^{(n)}(x)g.$$

So, (12) implies

$$\frac{\prod_{i=1}^n L(L-1)\dots(L-m_i+1)}{L^N}\Pi_t^{(n)}(x)g+R_L\ge 0,$$

where

$$|R_L|\le\left(1-\frac{\prod_{i=1}^n L(L-1)\dots(L-m_i+1)}{L^N}\right)\|f\|.$$

Taking the limit $L\to\infty$, we obtain $\Pi_t^{(n)}(x)g\ge 0$.

□

Lemma 3.1 implies that for every $n\ge 1$, $x\in M^n$ and $t\ge 0$ the linear functional $\Pi_t^{(n)}(x)$ is represented by a probability measure on $\mathcal{P}(\hat{M})^n$. This measure will be also denoted by $\Pi_t^{(n)}(x)$. In particular, the equality

$$\int_{\mathcal{P}(\hat{M})^n}\left(\int_{\hat{M}^N} f(y)\,(\mu_{i_1}\otimes\dots\otimes\mu_{i_N})\,(dy)\right)\Pi_t^{(n)}(x,d\mu)$$
$$=\int_{M^N} f(y)\mathsf{P}_t^{(N)}((x_{i_1},\dots,x_{i_N}),dy)$$

holds for all $f\in C(\hat{M}^N)$, $(i_1,\dots,i_N)\in\{1,\dots,n\}^N$. Next lemmata contain some useful properties of measures $\Pi_t^{(n)}(x)$.

Lemma 3.2. (1) *For all $\sigma\in S_{k,n}$, $1\le k\le n$, and all $x\in M^n$, $t\ge 0$,*

$$\Pi_t^{(n)}(x)\circ\pi_\sigma^{-1}=\Pi_t^{(k)}(\pi_\sigma x).$$

(2) *For all* $x \in M$, $t \geq 0$,
$$\Pi_t^{(2)}((x,x),\Delta) = 1,$$
where $\Delta = \{(\mu,\mu):\ \mu \in \mathcal{P}(\hat{M})\}$.

(3) *For all* $n \geq 1$, $x \in M^n$, $t \geq 0$,
$$\Pi_t^{(n)}(x, \mathcal{P}(M)^n) = 1.$$

(4) *For any* $g \in C(\mathcal{P}(\hat{M})^n)$ *the mapping* $(t,x) \mapsto \Pi_t^{(n)}(x)g$ *is continuous.*

Proof. (1) Let $\sigma \in S_{k,n}$. Consider $g \in \mathbb{A}_k(\hat{M})$ of the form
$$g(\mu_1,\ldots,\mu_k) = \int_{\hat{M}^N} f(y)\,(\mu_{i_1}\otimes\ldots\otimes\mu_{i_N})\,(dy),$$
where $f \in C(\hat{M}^N)$, $(i_1,\ldots,i_N) \in \{1,\ldots,k\}^N$. Then
$$g\circ\pi_\sigma(\mu_1,\ldots,\mu_n) = g(\mu_{\sigma(1)},\ldots,\mu_{\sigma(k)}) = \int_{\hat{M}^N} f(y)\left(\mu_{\sigma(i_1)}\otimes\ldots\otimes\mu_{\sigma(i_N)}\right)(dy).$$
So,
$$\int_{\mathcal{P}(\hat{M})^n} g\circ\pi_\sigma(\mu_1,\ldots,\mu_n)\Pi_t^{(n)}(x,d\mu) = \int_{M^N} f(y)\mathsf{P}_t^{(N)}((x_{\sigma(i_1)},\ldots,x_{\sigma(i_N)}),dy)$$
$$= \int_{\mathcal{P}(\hat{M})^k} g(\mu_1,\ldots,\mu_k)\Pi_t^{(k)}(\pi_\sigma x, d\mu).$$
Equality $\Pi_t^{(n)}(x)\circ\pi_\sigma^{-1} = \Pi_t^{(k)}(\pi_\sigma x)$ is verified.

(2) Let $g \in \mathbb{A}_1(\hat{M})$, $g(\mu) = \int_{\hat{M}^N} f(y)\mu^{\otimes N}(dy)$. Then
$$\Pi_t^{(2)}g^{\otimes 2}((x,x)) = \int_{M^{2N}} f^{\otimes 2}(y)\mathsf{P}_t^{(2N)}(\underbrace{(x,\ldots,x)}_{2N},dy) = \Pi_t^{(1)}g^2(x).$$
By continuity, for all $g_1,g_2 \in C(\mathcal{P}(\hat{M}))$ we have
$$\int_{\mathcal{P}(\hat{M})^2} g_1(\mu_1)g_2(\mu_2)\Pi_t^{(2)}((x,x),d\mu) = \int_{\mathcal{P}(\hat{M})} g_1(\mu)g_2(\mu)\Pi_t^{(1)}(x,d\mu).$$
Hence, for any closed sets $F_1,F_2 \subset \mathcal{P}(\hat{M})$,
$$\Pi_t^{(2)}((x,x),F_1\times F_2) = \Pi_t^{(1)}(x,F_1\cap F_2).$$
It follows that $\Pi_t^{(2)}((x,x),\Delta) = 1$.

(3) Let $x \in M^n$ and $g_k(\mu_1,\ldots,\mu_n) = \prod_{i=1}^n \int_{\hat{M}} \zeta_k(y)\mu_i(dy)$. Then
$$1 \geq \int_{\mathcal{P}(\hat{M})^n} \prod_{i=1}^n \mu_i(M)\Pi_t^{(n)}(x,d\mu) \geq \int_{\mathcal{P}(\hat{M})^n} g_k(\mu_1,\ldots,\mu_n)\Pi_t^{(n)}(x,d\mu)$$
$$= \int_{M^n}\prod_{i=1}^n \zeta_k(y_i)\mathsf{P}_t^{(n)}(x,dy) \geq \mathsf{P}_t^{(n)}(x,L_k^n).$$
Taking the limit $k\to\infty$ we deduce that $\int_{\mathcal{P}(\hat{M})^n}\prod_{i=1}^n \mu_i(M)\Pi_t^{(n)}(x,d\mu) = 1$ and $\mu_1(M) = \ldots = \mu_n(M) = 1$ for $\Pi_t^{(n)}(x)$-a.a. $(\mu_1,\ldots,\mu_n) \in \mathcal{P}(\hat{M})^n$.

(4) Let $g \in \mathbb{A}_n(\hat{M})$ be of the form
$$g(\mu_1,\ldots,\mu_n) = \int_{\hat{M}^N} f(y)\,(\mu_{i_1}\otimes\ldots\otimes\mu_{i_N})\,(dy),$$
where $f \in C(\hat{M}^N)$, $(i_1,\ldots,i_N) \in \{1,\ldots,n\}^N$. Then
$$\Pi_t^{(n)}(x)g = \int_{M^N} f(y)\mathsf{P}_t^{(N)}((x_{i_1},\ldots,x_{i_N}),dy).$$

Fix $l \geq 1$ and $T \geq 0$. By Feller property of $\mathsf{P}^{(N)}$ (Lemma 2.1) for each $\varepsilon > 0$ there exists $j \geq 1$ such that

$$\inf_{t\in[0,T],z\in L_l^N} \mathsf{P}_t^{(N)}(z, L_j^N) \geq 1-\varepsilon.$$

For the function

$$g_j(\mu_1,\ldots,\mu_n) = \int_{\hat{M}^N} f(y)\zeta_j^{\otimes N}(y)\,(\mu_{i_1}\otimes\ldots\otimes\mu_{i_N})\,(dy)$$

we have an estimate

$$\sup_{t\in[0,T],x\in L_l^n}\left|\Pi_t^{(n)}(x)g - \Pi_t^{(n)}(x)g_j\right|$$

$$= \sup_{t\in[0,T]x\in L_l^n}\left|\int_{M^N\setminus L_j^N} f(y)(1-\zeta_j^{\otimes N}(y))\mathsf{P}_t^{(N)}((x_{i_1},\ldots,x_{i_N}),dy)\right| \leq \|f\|\varepsilon.$$

On the other hand, the equality

$$\Pi_t^{(n)}(x)g_j = \mathsf{P}_t^{(N)}\left(f\zeta_j^{\otimes N}\right)(x_{i_1},\ldots,x_{i_N})$$

implies that the function $(t,x)\mapsto\Pi_t^{(n)}(x)g_j$ is continuous. Since $\varepsilon > 0$ is arbitrary, we deduce that the function $(t,x)\mapsto\Pi_t^{(n)}(x)g$ is continuous on $[0,T]\times L_l^n$ and thus on $[0,\infty)\times M^n$.

□

Denote $\Delta_\varepsilon^c = \{(\mu_1,\mu_2)\in\mathcal{P}(M)^2 : d(\mu_1,\mu_2)\geq\varepsilon\}$.

Lemma 3.3. *For any compact $C\subset M$, $T\geq 0$ and $\varepsilon>0$*

$$\lim_{r\to 0+}\sup_{\substack{t\in[0,T],(x,y)\in C^2\\ \rho(x,y)\leq r}}\Pi_t^{(2)}((x,y),\Delta_\varepsilon^c) = 0.$$

Proof. Assume the result does not hold. Then there is $\alpha>0$ and a sequence $(x_k,y_k,t_k)\in C^2\times[0,T]$, such that $\lim_{k\to\infty}\rho(x_k,y_k)=0$ and

$$\Pi_{t_k}^{(2)}((x_k,y_k),\Delta_\varepsilon^c)\geq\alpha.$$

We may and do assume that $\lim_{k\to\infty}x_k = \lim_{k\to\infty}y_k = x\in C$, and $\lim_{k\to\infty}t_k = t\in[0,T]$. Property (4) of Lemma 3.2 implies that $\Pi_{t_k}^{(2)}((x_k,y_k))\to\Pi_t^{(2)}((x,x))$ weakly as probability measures on $\mathcal{P}(\hat{M})^2$, and as probability measures on $\mathcal{P}(M)^2$. The Portmanteau theorem implies

$$\alpha\leq\limsup_{k\to\infty}\Pi_{t_k}^{(2)}((x_k,y_k),\Delta_\varepsilon^c)\leq\Pi_t^{(2)}((x,x),\Delta_\varepsilon^c)=0,$$

since $\Pi_t^{(2)}((x,x))$ is concentrated on Δ (property (2) of Lemma 3.2). Obtained contradiction proves the result.

□

3.2. **Approximating procedure.** The measure ν_t can be viewed as the distribution of a measure-valued process $(K_{0,t}(x) : x\in M)$. Let Z be an at most countable dense set in M. The idea of the construction is to define properly the joint distribution of $(K_{0,t}(z) : z\in Z)$ and to recover the measure ν_t by certain limit procedure. We note that for any Polish space $\mathcal{X}$ there exists a measurable mapping $\ell : \mathcal{X}^{\mathbb{N}}\to\mathcal{X}$ with the following property: for any relatively compact sequence $x=(x_n : n\in\mathbb{N})$, $\ell(x)$ is a limit point of x. This is known for compact spaces $\mathcal{X}$ from [8, L 1.1]. The generalization to Polish spaces was proved in [14, L. 7.1]. The proof is based on the fact that any Polish space is homeomorphic to a Borel subset of some compact metric space [16, Rem. 2.2.8]. In what follows ℓ denotes such a mapping in the case $\mathcal{X}=\mathcal{P}(M)$.

Recall the exhaustive sequence $(L_j : j \geq 1)$ defined in the beginning of Section 2. Lemma 3.3 implies that there exists a sequence of positive numbers $(\varepsilon_j : j \geq 1)$ that is strictly decreasing to zero and is such that

$$(t, x, y) \in [0, j] \times L_j^2,\ \rho(x, y) \leq \varepsilon_j \Rightarrow \Pi_t^{(2)}((x, y), \Delta^c_{2^{-j}}) \leq 2^{-j}.$$

Let $m \mapsto z_m$ be a bijection between a subset I of $\mathbb{N}$ and the set Z. For any $x \in M$ and any $j \geq 1$ we define

$$m_j^x = \inf\{m \in I : \rho(x, z_m) < \varepsilon_j/2\}. \tag{13}$$

Note that $(m_j^x : j \geq 1)$ is a sequence in I, and each mapping $x \mapsto m_j^x$ is measurable.

Define mappings $i : \mathcal{P}(M)^I \to E$, $e : E \to \mathcal{P}(M)^I$, $\mathfrak{p} : E \to E$ as follows:

$$i(\mu)(x) = \ell\left(\left(\mu_{m_j^x} : j \geq 1\right)\right),\ e(K) = (K(z_m) : m \in I),\ \mathfrak{p} = i \circ e.$$

Lemma 3.4. *Mappings $(x, \mu) \mapsto i(\mu)(x)$, $K \mapsto e(K)$, $(K, x) \mapsto \mathfrak{p}(K)(x)$ are measurable. Composition $e \circ i$ is the identity mapping on $\mathcal{P}(M)^I$. Mapping $\mathfrak{p}$ satisfies the property $\mathfrak{p} \circ \mathfrak{p} = \mathfrak{p}$.*

Proof. By definition, $i(\mu)(x) = \ell\left(\left(\mu_{m_j^x} : j \geq 1\right)\right)$. To prove the measurability of $(x, \mu) \mapsto i(\mu)(x)$, it is enough to prove that mappings $(x, \mu) \mapsto \mu_{m_j^x} \in \mathcal{P}(M)$ are measurable. This follows from the measurability of $x \mapsto m_j^x$ and the equality

$$\{(x, \mu) : \mu_{m_j^x} \in B\} = \bigcup_{r \in I} \{(x, \mu) : m_j^x = r,\ \mu_r \in B\},\ B \in \mathcal{B}(\mathcal{P}(M)).$$

The measurability of e is obvious. Further, if $x = z_m$, then $m_j^x = m$ as soon as $\varepsilon_j/2 \leq \min_{n \in I, n < m} \rho(z_m, z_n)$. So, $i(\mu)(z_m) = \mu_m$ and $e(i(\mu))_m = i(\mu)(z_m) = \mu_m$. In particular, $\mathfrak{p} \circ \mathfrak{p} = \mathfrak{p}$. The equality $\mathfrak{p}(K)(x) = i(e(K))(x)$ proves the measurability of the mapping $(K, x) \mapsto \mathfrak{p}(K)(x)$.

□

For $n \geq 1$ define mappings $\Phi_n : (\mathcal{P}(M)^I)^n \to E$, $\Psi_n : M \times (\mathcal{P}(M)^I)^n \to \mathcal{P}(M)$ by formulas

$$\Phi_n(\mu^1, \ldots, \mu^n)(x) = i(\mu^1) \ldots i(\mu^n)(x) = \Psi_n(x, \mu^1, \ldots, \mu^n).$$

Lemma 3.5. *For all $n \geq 1$ mappings Φ_n and Ψ_n are well-defined and measurable.*

Proof. We note that the mapping $(\mu, K) \mapsto \mu\mathfrak{p}(K)$ is measurable. By induction, it follows that

$$\Psi_n(x, \mu^1, \ldots, \mu^n) = \Psi_{n-1}(x, \mu^2, \ldots, \mu^n) i(\mu^n) = \Psi_{n-1}(x, \mu^2, \ldots, \mu^n)\mathfrak{p}(i(\mu^n))$$

is measurable.

□

3.3. Probability measures Π_t.

By Kolmogorov's theorem, for every $t \geq 0$ there exists a unique probability measure Π_t on $\mathcal{P}(M)^I$, such that for any finite set $J \subset I$ and $B \in \mathcal{B}(\mathcal{P}(M)^{|J|})$

$$\Pi_t\{\mu : \mu|_J \in B\} = \Pi_t^{(|J|)}((z_m)_{m \in J}, B).$$

Proposition 3.1. *For any $(i_1, \ldots, i_n) \in I^n$ and $B \in \mathcal{B}(\mathcal{P}(M)^n)$*

$$\Pi_t\{\mu : (\mu_{i_1}, \ldots, \mu_{i_n}) \in B\} = \Pi_t^{(n)}((z_{i_1}, \ldots, z_{i_n}), B).$$

Remark 3.1. Note that some indices among $i_1, \ldots, i_n$ may coincide.

Proof. The proof follows from statements (1) and (2) of Lemma 3.2.

Let $\bigcup_{j=1}^n \{i_j\} = \{k_1, \ldots, k_p\} \subset I$ with $k_1 < \ldots < k_p$. Denote $J_l = \{j \in \{1, \ldots, n\} : i_j = k_l\}$, $1 \le l \le p$. Then $J_1, \ldots, J_p$ is a partition of $\{1, \ldots, n\}$ into non-empty subsets. Let $\sigma \in S_{p,k_p}$ be the injection $\sigma(l) = k_l$, $1 \le l \le p$.

Consider the mapping $h : \mathcal{P}(M)^p \to \mathcal{P}(M)^n$ given by

$$h(\mu)_j = \mu_l, \ j \in J_l, 1 \le l \le p.$$

Take $B = \prod_{j=1}^n B_j$, where $B_j \in \mathcal{B}(\mathcal{P}(M))$, $1 \le j \le n$. The equality $h(\mu_{k_1}, \ldots, \mu_{k_p}) = (\mu_{i_1}, \ldots, \mu_{i_n})$ implies

$$\begin{aligned}\Pi_t\{\mu : (\mu_{i_1}, \ldots, \mu_{i_n}) \in B\} &= \Pi_t\{\mu : (\mu_{k_1}, \ldots, \mu_{k_p}) \in h^{-1}(B)\} \\ &= \Pi_t^{(p)}\left((z_{k_1}, \ldots, z_{k_p}), h^{-1}(B)\right).\end{aligned}$$

For every $l \in \{1, \ldots, p\}$ we choose $j(l) \in J_l$, and set $C_l = \cap_{j \in J_l} B_j$. Consider injections $\alpha \in S_{p,n}$, $\alpha(l) = j(l)$, $1 \le l \le p$, and $\beta_{j_1,j_2;l} \in S_{2,n}$, $\beta_{j_1,j_2;l}(i) = j_i$, $i = 1, 2$. Here $j_1, j_2 \in J_l$, $j_1 \ne j_2$. We note that

$$\begin{aligned}\Pi_t^{(n)}((z_{i_1}, \ldots, z_{i_n}), \beta^{-1}_{j_1,j_2;l}(\mathcal{P}(M)^2 \setminus \Delta)) &= \Pi^{(2)}((z_{i_{j_1}}, z_{i_{j_2}}), \mathcal{P}(M)^2 \setminus \Delta) \\ &= \Pi^{(2)}((z_{k_l}, z_{k_l}), \mathcal{P}(M)^2 \setminus \Delta) = 0.\end{aligned}$$

So,

$$\begin{aligned}\Pi_t^{(n)}((z_{i_1}, \ldots, z_{i_n}), B) &= \Pi_t^{(n)}\left((z_{i_1}, \ldots, z_{i_n}), B \cap \left(\bigcap_{l=1}^p \bigcap_{\substack{(j_1,j_2) \in J_l^2 \\ j_1 \ne j_2}} \beta^{-1}_{j_1,j_2;l}(\Delta)\right)\right) \\ &= \Pi_t^{(n)}\left((z_{i_1}, \ldots, z_{i_n}), \alpha^{-1}\left(\prod_{l=1}^p C_l\right) \cap \left(\bigcap_{l=1}^p \bigcap_{\substack{(j_1,j_2) \in J_l^2 \\ j_1 \ne j_2}} \beta^{-1}_{j_1,j_2;l}(\Delta)\right)\right) \\ &= \Pi_t^{(n)}\left((z_{i_1}, \ldots, z_{i_n}), \alpha^{-1}\left(\prod_{l=1}^p C_l\right)\right) = \Pi_t^{(p)}\left((z_{k_1}, \ldots, z_{k_p}), \prod_{l=1}^p C_l\right) \\ &= \Pi_t^{(p)}\left((z_{k_1}, \ldots, z_{k_p}), h^{-1}(B)\right) = \Pi_t\{\mu : (\mu_{i_1}, \ldots, \mu_{i_n}) \in B\}.\end{aligned}$$

□

The measure Π_t must be understood as the distribution of $(K_{0,t}(z) : z \in Z)$. We will recover the distribution ν_t by approximating the distribution of $K_{0,t}(x)$ with the distributions of $(K_{0,t}(z_{m_j^x}) : j \ge 1)$, where m_j^x was defined in (13). To do this we need several estimates on the speed of approximation.

Lemma 3.6. *Let $C \subset M$ be compact and $t \ge 0$. There exists $j_0 \ge 1$ such that for all $j \ge j_0$ and all $x \in C$*

$$\Pi_t\{\mu : d(\mu_{m_j^x}, \mu_{m_{j+1}^x}) \ge 2^{-j}\} \le 2^{-j}.$$

Proof. There is $l \ge 1$ such that $\{u \in M : \rho(u, C) \le 1\} \subset L_l$. Take $j_0 \ge t \vee l$ such that $\varepsilon_{j_0} < 1$. If $x \in C$ and $j \ge j_0$, then

$$\rho(z_{m_j^x}, x) < \frac{\varepsilon_j}{2} < 1, \ \rho(z_{m_{j+1}^x}, x) < \frac{\varepsilon_{j+1}}{2} < 1.$$

So, $(t, z_{m_j^x}, z_{m_{j+1}^x}) \in [0, j] \times L_j^2$. Since $\rho(z_{m_j^x}, z_{m_{j+1}^x}) < \varepsilon_j$, we deduce that

$$\Pi_t\{\mu : d(\mu_{m_j^x}, \mu_{m_{j+1}^x}) \ge 2^{-j}\} = \Pi_t^{(2)}((z_{m_j^x}, z_{m_{j+1}^x}), \Delta^c_{2^{-j}}) \le 2^{-j}.$$

□

Lemma 3.7. *For all $x \in M$ and Π_t-a.a $\mu \in \mathcal{P}(M)^I$,*

$$\lim_{j\to\infty} \mu_{m_j^x} = i(\mu)(x).$$

Proof. By the Lemma 3.6, for all $j \geq j_0$

$$\Pi_t\{\mu : d(\mu_{m_j^x}, \mu_{m_{j+1}^x}) \geq 2^{-j}\} \leq 2^{-j}.$$

By the Borel-Cantelli lemma, for Π_t-a.a $\mu \in \mathcal{P}(M)^I$, $\sum_{j=1}^{\infty} d(\mu_{m_j^x}, \mu_{m_{j+1}^x}) < \infty$. So, for Π_t-a.a. $\mu \in \mathcal{P}(M)^I$ the limit $\lim_{j\to\infty} \mu_{m_j^x}$ exists and necessarily coincides with $i(\mu)(x)$. □

3.4. **Feller convolution semigroup** $(\nu_t : t \geq 0)$**.** Define $\nu_t = \Pi_t \circ i^{-1}$. ν_t is a regular probability measure on $(E, \mathcal{E})$ with the measurable presentation $\mathfrak{p}$. Indeed, the mapping

$$\mathfrak{p}\,(K)\,(x) = i(e(K))(x)$$

is $\mathcal{E} \otimes \mathcal{B}(M)/\mathcal{B}(M)$-measurable (Lemma 3.4). Further, for every $x \in M$

$$\begin{aligned}\nu_t\{K : \mathfrak{p}(K)(x) = K(x)\} &= \nu_t\{K : i(e(K))(x) = K(x)\}\\ &= \Pi_t\{\mu : i(e(i(\mu))(x) = i(\mu)(x)\}\\ &= \Pi_t\{\mu : i(\mu)(x) = i(\mu)(x)\} = 1,\end{aligned}$$

since $e \circ i$ is the identity mapping on $\mathcal{P}(M)^I$ (Lemma 3.4).

Consider $x \in M^N$, $t \geq 0$, and $f \in C_0(M^N)$. Using Proposition 3.1, Lemma 3.7, dominated convergence theorem and the Feller property of $(\mathsf{P}^{(n)} : n \geq 1)$, we obtain

$$\begin{aligned}&\int_E \left(\int_{M^N} f(y)\left(\otimes_{r=1}^N K(x_r)\right)(dy)\right)\nu_t(dK)\\ &\quad= \int_{\mathcal{P}(M)^I}\left(\int_{M^N} f(y)\left(\otimes_{r=1}^N i(\mu)(x_r)\right)(dy)\right)\Pi_t(d\mu)\\ &\quad= \lim_{j\to\infty}\int_{\mathcal{P}(M)^I}\left(\int_{M^N} f(y)\left(\otimes_{r=1}^N \mu_{m_j^{x_r}}\right)(dy)\right)\Pi_t(d\mu)\\ &\quad= \lim_{j\to\infty}\int_{\mathcal{P}(M)^N}\left(\int_{M^N} f(y)\left(\otimes_{r=1}^N \mu_r\right)(dy)\right)\Pi_t^{(N)}((z_{m_j^{x_1}}, \ldots, z_{m_j^{x_N}}), d\mu)\\ &\quad= \lim_{j\to\infty} \mathsf{P}_t^{(N)} f(z_{m_j^{x_1}}, \ldots, z_{m_j^{x_N}}) = \mathsf{P}_t^{(N)} f(x_1, \ldots, x_N).\end{aligned} \tag{14}$$

Now we can verify that $(\nu_t : t \geq 0)$ is the needed Feller convolution semigroup in the space of kernels on M. Let $t, s \geq 0$. From the Lemma 2.3 it is enough to verify that integrals of functions

$$K \mapsto \int_{M^N} f(y)\,(K(x_{i_1}) \otimes \ldots \otimes K(x_{i_N}))\,(dy),$$

where $x \in M^n$, $f \in C_0(M^N)$, $(i_1, \ldots, i_N) \in \{1, \ldots, n\}^N$, coincide for distributions $\nu_t * \nu_s$ and ν_{t+s}. Using Fubini's theorem, we have

$$\int_E \left(\int_{M^N} f(y) \left(\otimes_{r=1}^N K(x_{i_r}) \right) (dy) \right) (\nu_t * \nu_s)(dK)$$
$$= \int_E \int_E \left(\int_{M^N} f(y) \left(\otimes_{r=1}^N K_1 \mathfrak{p}(K_2)(x_{i_r}) \right) (dy) \right) \nu_t(dK_1) \nu_s(dK_2)$$
$$= \int_E \int_E \left(\int_{M^N} \int_{M^N} f(z) \left(\otimes_{r=1}^N \mathfrak{p}(K_2)(y_r) \right) (dz) \left(\otimes_{r=1}^N K_1(x_{i_r}) \right) (dy) \right) \nu_t(dK_1) \nu_s(dK_2)$$
$$= \int_E \int_{M^N} P_s^{(N)} f(y) \left(\otimes_{r=1}^N K_1(x_{i_r}) \right) (dy) \nu_t(dK_1) = P_t^{(N)} P_s^{(N)} f(x_{i_1}, \ldots, x_{i_N})$$
$$= P_{t+s}^{(N)} f(x_{i_1}, \ldots, x_{i_N}) = \int_E \left(\int_{M^N} f(y) \left(\otimes_{r=1}^N K(x_{i_r}) \right) (dy) \right) \nu_{t+s}(dK).$$

The equality $\nu_t * \nu_s = \nu_{t+s}$ is proved.

We verify conditions (2) and (3) of the Definition 2.3. Let $f \in C_0(M)$ and $\varepsilon > 0$. Then

$$\sup_{x \in M} \nu_t \{K : |Kf(x) - f(x)| \geq \varepsilon\} \leq \varepsilon^{-2} \sup_{x \in M} \int_E (Kf(x) - f(x))^2 \nu_t(dK)$$
$$= \varepsilon^{-2} \sup_{x \in M} \left(\mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) - 2f(x) \mathsf{P}^{(1)} f(x) + f^2(x) \right) \to 0,$$

as $t \to 0+$. Further,

$$\nu_t \{K : |Kf(y) - Kf(x)| \geq \varepsilon\} \leq \varepsilon^{-2} \int_E (Kf(y) - Kf(x))^2 \nu_t(dK)$$
$$= \varepsilon^{-2} \left(\mathsf{P}_t^{(2)} f^{\otimes 2}(y, y) - 2\mathsf{P}_t^{(2)} f^{\otimes 2}(x, y) + \mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) \right) \to 0, y \to x.$$

Finally,

$$\nu_t \{K : |Kf(y)| \geq \varepsilon\} \leq \varepsilon^{-2} \int_E (Kf(y))^2 \nu_t(dK) = \varepsilon^{-2} \mathsf{P}_t^{(2)} f^{\otimes 2}(y, y) \to 0, y \to \infty.$$

Equation (14) implies that the consistent sequence of Feller transition functions that corresponds to $(\nu_t : t \geq 0)$ is exactly $(\mathsf{P}^{(n)} : n \geq 1)$. This finishes the proof of Theorem 2.1.

In the next section we will need the following result.

Lemma 3.8. *For all* $t_1, \ldots, t_n \geq 0$

$$(\Pi_{t_1} \otimes \ldots \otimes \Pi_{t_n}) \circ \Phi_n^{-1} = \nu_{t_1 + \ldots + t_n}.$$

Proof. The proof is by induction on $n \geq 1$. For $n = 1$ the statement reduces to the definition of ν_t. Assume the result is proved for $n-1$ and let $A \in \mathcal{E}$. We note that the map $(K, \mu) \mapsto Ki(\mu) = K\mathfrak{p}(i(\mu))$ is $\mathcal{E} \times \mathcal{B}(\mathcal{P}(M)^I)/\mathcal{E}$-measurable. Using Fubini's theorem, we get

$$(\Pi_{t_1} \otimes \ldots \otimes \Pi_{t_n}) \left(\Phi_n^{-1}(A) \right)$$
$$= (\Pi_{t_1} \otimes \ldots \otimes \Pi_{t_n}) \left\{ (\mu^1, \ldots, \mu^n) \in (\mathcal{P}(M)^I)^n : i(\mu^1) \ldots i(\mu^{n-1}) \mathfrak{p}(i(\mu^n)) \in A \right\}$$
$$= \int_E \nu_{t_1 + \ldots + t_{n-1}} \{K_1 : K_1 \mathfrak{p}(K_2) \in A\} \nu_{t_n}(dK_2) = (\nu_{t_1 + \ldots + t_{n-1}} * \nu_{t_n})(A)$$
$$= \nu_{t_1 + \ldots + t_n}(A).$$

□

4. Proof of Theorem 2.2

4.1. **Probability space** $(\Omega, \mathcal{A}, \mathsf{P})$**.** As before, Z is an at most countable dense set in M and $m \mapsto z_m$ is a bijection between a subset $I \subset \mathbb{N}$ and the set Z. Recall the probability measure Π_t on $(\mathcal{P}(M)^I, \mathcal{B}(\mathcal{P}(M))^{\otimes I})$ constructed in Section 3.3. We will use the mappings $i : \mathcal{P}(M)^I \to E$, $e : E \to \mathcal{P}(M)^I$, $\Phi_n : (\mathcal{P}(M)^I)^n \to E$, $\Psi_n : M \times (\mathcal{P}(M)^I)^n \to \mathcal{P}(M)$, $\mathfrak{p} : E \to E$, defined in Section 3.2. We recall that $\mathfrak{p} = i \circ e$ is a measurable presentation of every measure ν_t (Section 3.4) and that $\mathfrak{p} \circ \mathfrak{p} = \mathfrak{p}$ (Lemma 3.4).

For each $n \geq 0$ consider the probability space

$$(S_n, \mathcal{S}_n, \mathsf{P}_n) = (\mathcal{P}(M)^I, \mathcal{B}(\mathcal{P}(M))^{\otimes I}, \Pi_{2^{-n}})^{\otimes \mathbb{Z}}.$$

Note that $\mathcal{S}_n$ is the Borel σ-field of the complete separable metric space S_n. Denote $D_n = 2^{-n}\mathbb{Z}$, $D = \bigcup_{n=0}^{\infty} D_n$.

Remark 4.1. If $\omega^n \in S_n$, then we intuitively understand $i(\omega^n_l)$ as the random kernel $K_{l2^{-n},(l+1)2^{-n}}$ from the future flow.

Consider the mappings

$$\pi_{n-1,n} : S_n \to S_{n-1},\ \pi_{n-1,n}(\omega^n) = \left(e\left(i(\omega^n_{2l}) i(\omega^n_{2l+1})\right)\right)_{l \in \mathbb{Z}}.$$

Mappings $\pi_{n-1,n}$ are measurable and surjective. To show measurability we note that the $l-$th component of $\pi_{n-1,n}$ equals $e\left(i(\omega^n_{2l}) i(\omega^n_{2l+1})\right) \in \mathcal{P}(M)^I$. Its element that corresponds to $m \in I$ is

$$i(\omega^n_{2l}) i(\omega^n_{2l+1})(z_m) = \Psi_2\left(z_m, \omega^n_{2l}, \omega^n_{2l+1}\right) \in \mathcal{P}(M),$$

and the mapping Ψ_2 is measurable (Lemma 3.5). Surjectivity of $\pi_{n-1,n}$ follows from the following Lemma.

Lemma 4.1. *Consider $\mu_0 = (\delta_{z_m})_{m \in I}$. Then for each $x \in M$, $i(\mu_0)(x) = \delta_x$. In particular, the kernel $x \mapsto \delta_x$ is invariant under $\mathfrak{p}$.*

Proof. For each $x \in M$ we have $\lim_{j \to \infty} z_{m^x_j} = x$. Hence,

$$i(\mu_0)(x) = \ell\left(\left(\delta_{z_{m^x_j}} : j \geq 1\right)\right) = \delta_x.$$

Denote $K_0(x) = \delta_x$. Then

$$\mathfrak{p}(K_0)(x) = i(\mu_0)(x) = \delta_x = K_0(x).$$

□

From Lemma 4.1 we deduce that $i(\mu_0)K = K$ for each kernel $K \in E$. For given $\omega^{n-1} \in S_{n-1}$ define $\omega^n_{2l} = \mu_0$, $\omega^n_{2l+1} = \omega^{n-1}_l$. Then

$$(\pi_{n-1,n}(\omega^n))_l = e\left(i(\mu_0) i(\omega^{n-1}_l)\right) = e \circ i(\omega^{n-1}_l) = \omega^{n-1}_l.$$

This proves surjectivity of $\pi_{n-1,n}$.

We note that $\mathsf{P}_n \circ \pi^{-1}_{n-1,n} = \mathsf{P}_{n-1}$. Indeed, under the measure $\mathsf{P}_n \circ \pi^{-1}_{n-1,n}$ components of ω^{n-1} are independent, and the law of ω^{n-1}_l equals (Lemma 3.8)

$$(\Pi_{2^{-n}} \otimes \Pi_{2^{-n}}) \circ (e \circ \Phi_2)^{-1} = \nu_{2^{-(n-1)}} \circ e^{-1} = \Pi_{2^{-(n-1)}} \circ i^{-1} \circ e^{-1} = \Pi_{2^{-(n-1)}}.$$

Let the set Ω be the inverse limit

$$\Omega = \left\{\omega = (\omega^n)_{n \geq 0} \in \prod_{n=0}^{\infty} S_n :\ \forall n \geq 1\ \pi_{n-1,n}(\omega^n) = \omega^{n-1}\right\}$$

(in the terminology of K.R. Parthasarathy [13, Sec. 2, Ch. V]). Let the mapping $\pi_n : \Omega \to S_n$ be the projection, $\pi_n(\omega) = \omega^n$, and the σ-field $\mathcal{A}$ on Ω be the smallest σ-field under which all projections π_n, $n \geq 0$, are measurable. There exists a unique probability measure P on $(\Omega, \mathcal{A})$, such that for all $n \geq 0$ and $C \in \mathcal{S}_n$,

$$\mathsf{P}(\pi_n^{-1}(C)) = \mathsf{P}_n(C)$$

[13, Th. 3.2, Ch. V].

For $(s,t) \in D^2$, $s \leq t$, let $\mathcal{A}_{s,t}$ be the σ-field generated by mappings $\omega \mapsto \omega_u^n$, where $n \geq 0$ and $u \in \mathbb{Z}$ are such that $(s,t) \in D_n^2$ and $u2^{-n} \in [s,t)$. We note that $\mathcal{A}_{s,s}$ is the trivial σ-field $\{\varnothing, \Omega\}$.

Lemma 4.2. *For all $0 \leq n \leq k$ and any $u \in \mathbb{Z}$, ω_u^n is a measurable function of $\omega^k_{2^{k-n}u}, \ldots, \omega^k_{2^{k-n}u+2^{k-n}-1}$.*

Proof. The proof is by induction on $k - n \geq 0$. If $k = n$, then the statement is obvious. Assume that $k < n$ and that the statement is proved for $k-1-n$. Let $u \in \mathbb{Z}$. By the inductive hypothesis, there exists a measurable function $F : (\mathcal{P}(M)^I)^{2^{k-1-n}} \to \mathcal{P}(M)^I$, such that

$$\omega_u^n = F\left(\omega^{k-1}_{2^{k-1-n}u}, \ldots, \omega^{k-1}_{2^{k-1-n}u+2^{k-1-n}-1}\right).$$

Then

$$\omega_u^n = F\left(e\left(i\left(\omega^k_{2^{k-n}u}\right) i\left(\omega^k_{2^{k-n}u+1}\right)\right), \ldots, e\left(i\left(\omega^k_{2^{k-n}u+2^{k-n}-2}\right) i\left(\omega^k_{2^{k-n}u+2^{k-n}-1}\right)\right)\right)$$

is a measurable function of $\omega^k_{2^{k-n}u}, \ldots, \omega^k_{2^{k-n}u+2^{k-n}-1}$.

□

Lemma 4.3. *If $(r,s,t) \in D^3$, $r \leq s \leq t$, then $\sigma-$fields $\mathcal{A}_{r,s}$ and $\mathcal{A}_{s,t}$ are independent. If $(r_1,r_2,r_3,r_4) \in D^4$, $r_1 \leq r_2 \leq r_3 \leq r_4$, then $\mathcal{A}_{r_2,r_3} \subset \mathcal{A}_{r_1,r_4}$.*

Proof. Let $(r,s,t) \in D^3$, $r < s < t$. Consider $n_1, \ldots, n_k \geq 0$, $u_1, \ldots, u_k \in \mathbb{Z}$, $m_1, \ldots, m_l \geq 0$, $v_1, \ldots, v_l \in \mathbb{Z}$, such that $(r,s) \in D^2_{n_i}$, $u_1 2^{-n_1}, \ldots, u_k 2^{-n_k} \in [r,s)$, $(s,t) \in D^2_{m_j}$, $v_1 2^{-m_1}, \ldots, v_l 2^{-m_l} \in [s,t)$. Denote $N = \max(n_1, \ldots, n_k, m_1, \ldots, m_l)$. By the Lemma 4.2, each $\omega^{n_i}_{u_i}$ is a measurable function of $\omega^N_{2^{N-n_i}u_i}, \ldots, \omega^N_{2^{N-n_i}u_i+2^{N-n_i}-1}$, and each $\omega^{m_j}_{v_j}$ is a measurable function of $\omega^N_{2^{N-m_j}v_j}, \ldots, \omega^N_{2^{N-m_j}v_j+2^{N-m_j}-1}$. Since $s \in D_{n_i}$, we write $s = 2^{-n_i}a$, $a \in \mathbb{Z}$. Inequality $u_i 2^{-n_i} < s$ implies $u_i + 1 \leq a$. Hence,

$$2^{N-n_i}u_i + 2^{N-n_i} - 1 \leq 2^{N-n_i}a - 1 = 2^N s - 1.$$

Since $s \in D_{m_j}$, we write $s = 2^{-m_j}b$, $b \in \mathbb{Z}$. Inequality $v_j 2^{-m_j} \geq s$ implies $v_j \geq b$. Hence,

$$2^{N-m_j}v_j \geq 2^{N-m_j}b = 2^N s.$$

Independence of $\mathcal{A}_{r,s}$ and $\mathcal{A}_{s,t}$ now follows from independence of $(\omega^N_k)_{k\in\mathbb{Z}}$.

Let $(r_1,r_2,r_3,r_4) \in D^4$, $r_1 \leq r_2 < r_3 \leq r_4$. Let $(r_2,r_3) \in D^2_n$, $u2^{-n} \in [r_2,r_3)$. Take $N \geq n$, such that $(r_1,r_2,r_3,r_4) \in D^4_N$. ω^n_u is a measurable function of $\omega^N_{2^{N-n}u}$, …, $\omega^N_{2^{N-n}u+2^{N-n}-1}$. We note that

$$2^{N-n}u2^{-N} = 2^{-n}u \geq r_2 \geq r_1,$$

and, since $u + 1 \leq 2^n r_3$,

$$\left(2^{N-n}u + 2^{N-n} - 1\right)2^{-N} = 2^{-n}(u+1) - 2^{-N} \leq r_3 - 2^{-N} < r_3 \leq r_4.$$

It follows that ω^n_u is $\mathcal{A}_{r_1,r_4}$-measurable.

□

4.2. **Random kernels $K^D_{s,t}$ for $(s,t)\in D^2$, $s\le t$.**

Definition 4.1. (1) For $(s,t)\in D^2_n$, $s<t$, and $\omega\in\Omega$, we define

$$K^{D,n}_{s,t}(\omega)=\Phi_{(t-s)2^n}\left(\omega^n_{s2^n},\ldots,\omega^n_{t2^n-1}\right).$$

(2) For $(s,t)\in D^2$, $s<t$, we define

$$K^D_{s,t}=\mathfrak{p}\left(K^{D,n}_{s,t}\right),$$

where $n\ge 0$ is the minimal non-negative integer, such that $(s,t)\in D^2_n$.

(3) For all $t\in D$ and $x\in M$, we define

$$K^D_{t,t}(x)=\delta_x.$$

Proposition 4.1. (1) *For all $(s,t)\in D^2$, $s\le t$, we have $\mathfrak{p}(K^D_{s,t})=K^D_{s,t}$.*

(2) *Let $s\in D_n$. Then $K^D_{s,s+2^{-n}}(\omega)=K^{D,n}_{s,s+2^{-n}}(\omega)=\mathfrak{p}\left(K^{D,n}_{s,s+2^{-n}}(\omega)\right)=i\left(\omega^n_{s2^n}\right)$, $\omega\in\Omega$.*

(3) *If $(s,t)\in D^2_n$, $s<t$, then $K^{D,n}_{s,t}$ is a measurable function of $K^{D,n}_{s,s+2^{-n}},\ldots,K^{D,n}_{t-2^{-n}t}$, and is $\mathcal{A}_{s,t}/\mathcal{E}$-measurable. If $(s,t)\in D^2$, $s\le t$, then $K^D_{s,t}$ is a measurable function of*

$$\left\{K^{D,n}_{r,r+2^{-n}}:n\ge 0,r\in D_n,[r,r+2^{-n})\subset[s,t)\right\}$$

and is $\mathcal{A}_{s,t}/\mathcal{E}$-measurable.

(4) *If $(s,t)\in D^2_n$, $s\le t$, then the distribution of $K^{D,n}_{s,t}$ in the space of kernels $(E,\mathcal{E})$ coincides with ν_{t-s}. If $(s,t)\in D^2$, $s\le t$, then the distribution of $K^D_{s,t}$ in the space of kernels $(E,\mathcal{E})$ coincides with ν_{t-s}.*

(5) *If $(r,s,t)\in D^3_n$, $r<s<t$, then $K^{D,n}_{r,t}=K^{D,n}_{r,s}K^{D,n}_{s,t}$.*

Proof. (1) If $s<t$, then the result follows from the fact that $\mathfrak{p}\circ\mathfrak{p}=\mathfrak{p}$. Let $s=t$. Let $\mu_0:=e(K^D_{t,t})=(\delta_{z_m})_{m\in I}$. Then $i(\mu_0)=K^D_{t,t}$ (Lemma 4.1) and, since $e\circ i$ is the identity mapping on $\mathcal{P}(M)^I$,

$$\mathfrak{p}(K^D_{t,t})=i\circ e\circ i(\mu_0)=i(\mu_0)=K^D_{t,t}.$$

(2) Let $m=\inf\{k\ge 0:(s,s+2^{-n})\in D^2_k\}\le n$. Write $s=j2^{-n}$ with $j\in\mathbb{Z}$, $s+2^{-n}=(j+1)2^{-n}$. Assume that $m<n$. Then $j2^{-n}=k2^{-m}$, $(j+1)2^{-n}=l2^{-m}$, $(l-k)2^{-m}=2^{-n}$. It follows that $l-k=2^{m-n}\in(0,1)$, which is impossible. So, $m=n$. Further,

$$K^D_{s,s+2^{-n}}(\omega)=\mathfrak{p}\left(K^{D,n}_{s,s+2^{-n}}(\omega)\right)=\mathfrak{p}\circ i(\omega^n_{s2^n})=i(\omega^n_{s2^n}).$$

Here we again use the property $\mathfrak{p}\circ i=i$.

(3) The random mapping $K^{D,n}_{s,s+2^{-n}}(\omega)=i(\omega^n_{s2^n})$ is $\mathcal{A}_{s,s+2^{-n}}/\mathcal{E}$-measurable. The needed result follows from the equality

$$K^{D,n}_{s,t}=\Phi_{(t-s)2^n}\left(e\left(K^{D,n}_{s,s+2^{-n}}\right),\ldots,e\left(K^{D,n}_{t-2^{-n},t}\right)\right).$$

(4) Follows from Lemma 3.8, definitions of $K^{D,n}_{s,t}$ and $K^D_{s,t}$, and the fact that distributions of K and $\mathfrak{p}(K)$ with respect to any measure ν_t coincide.

(5) Follows from the definition of $K^{D,n}$.

□

Lemma 4.4. *Let $(s,t)\in D^2_n$, $s<t$. For any $\mathcal{P}(M)$-valued random element $\mathcal{M}$, $\mathcal{M}K^{D,n}_{s,t}$ and $\mathcal{M}K^D_{s,t}$ are random elements in $\mathcal{P}(M)$.*

Proof. We note that the mapping $(\mu, K) \mapsto \mu\mathfrak{p}(K)$ is measurable. This implies the measurability of $\mathcal{M}K^D_{s,t}$, since $\mathfrak{p}(K^D_{s,t}) = K^D_{s,t}$. If $t = s + 2^{-n}$, we have $\mathfrak{p}(K^{D,n}_{s,s+2^{-n}}) = K^{D,n}_{s,s+2^{-n}}$. So, $\mathcal{M}K^{D,n}_{s,s+2^{-n}}$ is a random element in $\mathcal{P}(M)$. By induction,

$$\mathcal{M}K^{D,n}_{s,t} = \mathcal{M}K^{D,n}_{s,t-2^{-n}}K^{D,n}_{t-2^{-n},t}$$

is a random element in $\mathcal{P}(M)$.

□

Proposition 4.2. *Let $(s,t) \in D^2_n$, $s < t$. For any $\mathcal{P}(M)$-valued random element $\mathcal{M}$ independent from $\mathcal{A}_{s,t}$,*

$$\mathcal{M}K^{D,n}_{s,t} = \mathcal{M}K^D_{s,t} \quad a.s.$$

Proof. Denote by Π the distribution of $\mathcal{M}$ in $\mathcal{P}(M)$. We show that $\mathcal{M}K^{D,n}_{s,t} = \mathcal{M}K^{D,n+1}_{s,t}$ a.s. For every $x \in M$ statements (1) and (2) of Proposition 4.1 imply

$$\begin{aligned} K^{D,n}_{s,s+2^{-n}}(\omega)(x) &= i(\omega^n_{s2^n})(x) = i \circ e\left(i(\omega^{n+1}_{s2^{n+1}})i(\omega^{n+1}_{s2^{n+1}+1})\right)(x) \\ &= \mathfrak{p}\left(\Phi_2\left(\omega^{n+1}_{s2^{n+1}}, \omega^{n+1}_{s2^{n+1}+1}\right)\right)(x) = \mathfrak{p}\left(K^{D,n+1}_{s,s+2^{-n}}(\omega)\right)(x) \\ &= K^{D,n+1}_{s,s+2^{-n}}(\omega)(x) \quad \text{a.s.} \end{aligned}$$

We note that

$$\begin{aligned} K^{D,n+1}_{s,s+2^{-n}}(\omega)(x) &= K^{D,n+1}_{s,s+2^{-n-1}}(\omega)K^{D,n+1}_{s+2^{-n-1},s+2^{-n}}(\omega)(x) \\ &= \mathfrak{p}\left(K^{D,n+1}_{s,s+2^{-n-1}}(\omega)\right)\mathfrak{p}\left(K^{D,n+1}_{s+2^{-n-1},s+2^{-n}}(\omega)\right)(x) \\ &= \mathfrak{p}\left(K^{D,n+1}_{s,s+2^{-n-1}}(\omega)\right)\mathfrak{p}\circ\mathfrak{p}\left(K^{D,n+1}_{s+2^{-n-1},s+2^{-n}}(\omega)\right)(x). \end{aligned}$$

Mappings

$$(x, K_1, K_2, K_3) \mapsto 1_{\mathfrak{p}(K_1)(x)=\mathfrak{p}(K_2)\mathfrak{p}\circ\mathfrak{p}(K_3)(x)}$$

and

$$(\mu, K_1, K_2, K_3) \mapsto \mu\{x : \mathfrak{p}(K_1)(x) = \mathfrak{p}(K_2)\mathfrak{p}\circ\mathfrak{p}(K_3)(x)\}$$

are measurable. Fubini's theorem implies

$$\begin{aligned} &\mathsf{E}\mathcal{M}\{x : K^{D,n}_{s,s+2^{-n}}(x) = K^{D,n+1}_{s,s+2^{-n}}(x)\} \\ &= \int_{\mathcal{P}(M)} \mathsf{E}\mu\{x : K^{D,n}_{s,s+2^{-n}}(x) = K^{D,n+1}_{s,s+2^{-n}}(x)\}\Pi(d\mu) \\ &= \int_{\mathcal{P}(M)}\int_M \mathsf{P}\left(K^{D,n}_{s,s+2^{-n}}(x) = K^{D,n+1}_{s,s+2^{-n}}(x)\right)\mu(dx)\Pi(d\mu) = 1. \end{aligned}$$

It follows that a.s. for $\mathcal{M}$-a.a. $x \in M$,

$$K^{D,n}_{s,s+2^{-n}}(x) = K^{D,n+1}_{s,s+2^{-n}}(x),$$

and a.s.

$$\mathcal{M}K^{D,n}_{s,s+2^{-n}} = \mathcal{M}K^{D,n+1}_{s,s+2^{-n}}.$$

Assume the result is proved for $s+(k-1)2^{-n} = t-2^{-n}$. Statement (5) of Proposition 4.1 implies that

$$K^{D,n}_{s,t} = K^{D,n}_{s,t-2^{-n}}K^{D,n}_{t-2^{-n},t}, \quad K^{D,n+1}_{s,t} = K^{D,n+1}_{s,t-2^{-n}}K^{D,n+1}_{t-2^{-n},t}.$$

By inductive hypothesis, a.s.

$$\begin{aligned} \mathcal{M}K^{D,n}_{s,t} &= \left(\mathcal{M}K^{D,n}_{s,t-2^{-n}}\right)K^{D,n}_{t-2^{-n},t} = \left(\mathcal{M}K^{D,n+1}_{s,t-2^{-n}}\right)K^{D,n}_{t-2^{-n},t} \\ &= \left(\mathcal{M}K^{D,n+1}_{s,t-2^{-n}}\right)K^{D,n+1}_{t-2^{-n},t} = \mathcal{M}K^{D,n+1}_{s,t}. \end{aligned}$$

Here we used independency of $\mathcal{M}K^{D,n+1}_{s,t-2^{-n}}$ from $\mathcal{A}_{t-2^{-n},t}$, which follows from the representation

$$\mathcal{M}K^{D,n+1}_{s,t-2^{-n}} = \mathcal{M}\mathfrak{p}\left(K^{D,n+1}_{s,s+2^{-n-1}}\right)\ldots\mathfrak{p}\left(K^{D,n+1}_{t-3\times 2^{-n-1},t-2^{-n}}\right).$$

Mappings

$$(x, K_1, \ldots, K_{(t-s)2^n}) \mapsto 1_{\mathfrak{p}(K_1)\ldots\mathfrak{p}(K_{(t-s)2^n})(x)=\mathfrak{p}\left(\mathfrak{p}(K_1)\ldots\mathfrak{p}(K_{(t-s)2^n})\right)(x)}$$

and

$$(\mu, K_1, \ldots, K_{(t-s)2^n}) \mapsto \mu\{x : \mathfrak{p}(K_1)\ldots\mathfrak{p}(K_{(t-s)2^n})(x) = \mathfrak{p}\left(\mathfrak{p}(K_1)\ldots\mathfrak{p}(K_{(t-s)2^n})\right)(x)\}$$

are measurable.

Substituting $K_j = K^{D,n}_{s+(j-1)2^{-n},s+j2^{-n}}$, $1 \le j \le (t-s)2^n$, and using Fubini's theorem, we get

$$\begin{aligned}
&\mathsf{E}\mathcal{M}\{x : K^{D,n}_{s,t}(x) = \mathfrak{p}\left(K^{D,n}_{s,t}\right)(x)\}\\
&= \int_{\mathcal{P}(M)} \mathsf{E}\mu\left\{x : K^{D,n}_{s,t}(x) = \mathfrak{p}\left(K^{D,n}_{s,t}\right)(x)\right\}\Pi(d\mu)\\
&= \int_{\mathcal{P}(M)}\int_M \mathsf{P}\left(K^{D,n}_{s,t}(x) = \mathfrak{p}\left(K^{D,n}_{s,t}\right)(x)\right)\mu(dx)\Pi(d\mu) = 1.
\end{aligned}$$

It follows that a.s. for $\mathcal{M}$-a.a. $x \in M$,

$$K^{D,n}_{s,t}(x) = \mathfrak{p}\left(K^{D,n}_{s,t}\right)(x),$$

and a.s.

$$\mathcal{M}K^{D,n}_{s,t} = \mathcal{M}\mathfrak{p}\left(K^{D,n}_{s,t}\right).$$

This finishes the proof. □

Remark 4.2. Let $(r, s, t) \in D^3$, $r \le s \le t$. For all $\mathcal{P}(M)$-valued random elements $\mathcal{M}$ independent from $\mathcal{A}_{r,t}$,

$$\mathcal{M}K^D_{r,t} = \mathcal{M}K^D_{r,s}K^D_{s,t} \text{ a.s.}$$

Lemma 4.5. *Let $f \in C(\mathcal{P}(\hat{M})^2)$. For any compact $\mathcal{C} \subset \mathcal{P}(M)$ the function*

$$(s, t, u, v, \mu, \nu) \mapsto \mathsf{E}f\left(\mu K^D_{s,t}, \nu K^D_{u,v}\right)$$

can be continuously extended to $\{(s, t) \in \mathbb{R}^2 : s \le t\}^2 \times \mathcal{C}^2$.

Proof. By the Stone-Weierstrass theorem it is enough to consider functions $f \in C(\mathcal{P}(\hat{M})^2)$ of the form $f(\mu, \nu) = g(\mu)h(\nu)$, where $g, h \in \mathbb{A}_1(\hat{M})$,

$$g(\varkappa) = \int_{\hat{M}^N} a(z)\varkappa^{\otimes N}(dz),\ h(\varkappa) = \int_{\hat{M}^N} b(z)\varkappa^{\otimes N}(dz),$$

and $a, b \in C(\hat{M}^N)$.

At first we consider the case when a and b have compact supports in M^N, in particular $a, b \in C_0(M^N)$. We will prove that there exists a continuous function $F : \{(s, t) \in \mathbb{R}^2 : s \le t\}^2 \times \mathcal{P}(M)^2 \to \mathbb{R}$, such that for all $(s, t, u, v, \mu, \nu) \in D^4 \times \mathcal{P}(M)^2$, $s \le t$, $u \le v$,

$$F(s, t, u, v, \mu, \nu) = \mathsf{E}f\left(\mu K^D_{s,t}, \nu K^D_{u,v}\right) = \mathsf{E}g\left(\mu K^D_{s,t}\right)h\left(\nu K^D_{u,v}\right).$$

Denote

$$\begin{aligned}
H(s, t, u, v, \mu, \nu) &= \mathsf{E}g\left(\mu K^D_{s,t}\right)h\left(\nu K^D_{u,v}\right)\\
&= \mathsf{E}\int_{M^N}\left(K^D_{s,t}\right)^{\otimes N} a(x)\mu^{\otimes N}(dx)\int_{M^N}\left(K^D_{u,v}\right)^{\otimes N} b(y)\nu^{\otimes N}(dy)\\
&= \int_{M^N}\int_{M^N}\left[\mathsf{E}\left(K^D_{s,t}\right)^{\otimes N}_{s,t} a(x)\left(K^D_{u,v}\right)^{\otimes N} b(y)\right]\mu^{\otimes N}(dx)\nu^{\otimes N}(dy).
\end{aligned}$$

We evaluate the function H for different displacements of (s,t,u,v).

Case 1: $s \le t \le u \le v$.

$$\begin{aligned} H(s,t,u,v,\mu,\nu) &= \int_{M^N}\int_{M^N}\left[\mathsf{E}\left(K^D_{s,t}\right)^{\otimes N} a(x)\left(K^D_{u,v}\right)^{\otimes N} b(y)\right]\mu^{\otimes N}(dx)\nu^{\otimes N}(dy) \\ &= \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{t-s}a(x)\mathsf{P}^{(N)}_{v-u}b(y)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy). \end{aligned}$$

The right-hand side is continuous on the set $\{(s,t,u,v,\mu,\nu)\in\mathbb{R}^4\times\mathcal{P}(M)^2 : s\le t\le u\le v\}$.

Case 2: $s \le u \le t \le v$.

$$\begin{aligned} &H(s,t,u,v,\mu,\nu) \\ &= \int_{M^N}\int_{M^N}\left[\mathsf{E}\left(K^D_{s,u}\right)^{\otimes N}\left(K^D_{u,t}\right)^{\otimes N} a(x)\left(K^D_{u,t}\right)^{\otimes N}\left(K^D_{t,v}\right)^{\otimes N} b(y)\right]\mu^{\otimes N}(dx)\nu^{\otimes N}(dy) \\ &= \int_{M^N}\int_{M^N}\mathsf{P}^{(N)}_{u-s}\left[\mathsf{P}^{(2N)}_{t-u}\left(a\otimes \mathsf{P}^{(N)}_{v-t}b\right)(\cdot,y)\right](x)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy). \end{aligned}$$

We check that the right-hand side is continuous on the set $\{(s,t,u,v,\mu,\nu)\in\mathbb{R}^4\times\mathcal{P}(M)^2 : s\le u\le t\le v\}$. Let $(s_n,t_n,u_n,v_n,\mu_n,\nu_n)\to(s,t,u,v,\mu,\nu)$, $s_n\le u_n\le t_n\le v_n$. Denote $G_{t,u,v}(x,y) = \mathsf{P}^{(2N)}_{t-u}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)(x,y)\in C_0(M^{2N})$, where $x,y\in M^N$. We have uniform estimates

$$\begin{aligned} &\sup_{(x,y)\in M^{2N}}|G_{t_n,u_n,v_n}(x,y)-G_{t,u,v}(x,y)| \\ &\le \sup_{(x,y)\in M^{2N}}\left|\mathsf{P}^{(2N)}_{t_n-u_n}\left(a\otimes\mathsf{P}^{(N)}_{v_n-t_n}b\right)(x,y)-\mathsf{P}^{(2N)}_{t_n-u_n}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)(x,y)\right| \\ &\quad+\sup_{(x,y)\in M^{2N}}\left|\mathsf{P}^{(2N)}_{t_n-u_n}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)(x,y)-\mathsf{P}^{(2N)}_{t-u}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)(x,y)\right| \\ &\le \|a\|\|\mathsf{P}^{(N)}_{v_n-t_n}b-\mathsf{P}^{(N)}_{v-t}b\| + \|\mathsf{P}^{2N}_{t_n-u_n}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)-\mathsf{P}^{2N}_{t-u}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)\|; \end{aligned}$$

$$\begin{aligned} &\sup_{(x,y)\in M^{2N}}\left|\mathsf{P}^{(N)}_{u_n-s_n}\left[G_{t_n,u_n,v_n}(\cdot,y)\right](x)-\mathsf{P}^{(N)}_{u-s}\left[G_{t,u,v}(\cdot,y)\right](x)\right| \\ &\le \|a\|\|\mathsf{P}^{(N)}_{v_n-t_n}b-\mathsf{P}^{(N)}_{v-t}b\| + \|\mathsf{P}^{2N}_{t_n-u_n}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)-\mathsf{P}^{2N}_{t-u}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)\| \\ &\quad+\|(\mathsf{P}^{(N)}_{u_n-s_n}\otimes I)G_{t,u,v}-(\mathsf{P}^{(N)}_{u-s}\otimes I)G_{t,u,v}\|, \end{aligned}$$

where I is the identity operator on $C_0(M^N)$. Finally,

$$\begin{aligned} &|H(s_n,t_n,u_n,v_n,\mu_n,\nu_n)-H(s,t,u,v,\mu,\nu)| \\ &= \left|\int_{M^N}\int_{M^N}\mathsf{P}^{(N)}_{u_n-s_n}\left[G_{t_n,u_n,v_n}(\cdot,y)\right](x)\mu_n^{\otimes N}(dx)\nu_n^{\otimes N}(dy)\right. \\ &\quad\left.-\int_{M^N}\int_{M^N}\mathsf{P}^{(N)}_{u-s}\left[G_{t,u,v}(\cdot,y)\right](x)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy)\right| \\ &\le \|a\|\|\mathsf{P}^{(N)}_{v_n-t_n}b-\mathsf{P}^{(N)}_{v-t}b\| + \|\mathsf{P}^{2N}_{t_n-u_n}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)-\mathsf{P}^{2N}_{t-u}\left(a\otimes\mathsf{P}^{(N)}_{v-t}b\right)\| \\ &\quad+\|(\mathsf{P}^{(N)}_{u_n-s_n}\otimes I)G_{t,u,v}-(\mathsf{P}^{(N)}_{u-s}\otimes I)G_{t,u,v}\| \\ &\quad+\left|\int_{M^N}\int_{M^N}\mathsf{P}^{(N)}_{u-s}\left[G_{t,u,v}(\cdot,y)\right](x)\mu_n^{\otimes N}(dx)\nu_n^{\otimes N}(dy)\right. \\ &\quad\left.-\int_{M^N}\int_{M^N}\mathsf{P}^{(N)}_{u-s}\left[G_{t,u,v}(\cdot,y)\right](x)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy)\right| \to 0,\ n\to\infty. \end{aligned}$$

Case 3: $s \le u \le v \le t$.

$$H(s,t,u,v,\mu,\nu)$$
$$= \int_{M^N}\int_{M^N} \left[\mathsf{E}\left(K^D_{s,u}\right)^{\otimes N}\left(K^D_{u,v}\right)^{\otimes N}\left(K^D_{v,t}\right)^{\otimes N} a(x)\left(K^D_{u,v}\right)^{\otimes N} b(y)\right]\mu^{\otimes N}(dx)\nu^{\otimes N}(dy)$$
$$= \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{u-s}\left[\mathsf{P}^{(2N)}_{v-u}\left(\mathsf{P}^{(N)}_{t-v}a\otimes b\right)(\cdot,y)\right](x)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy).$$

Similarly to the Case 2 we get continuity of the right-hand side on the set

$$\{(s,t,u,v,\mu,\nu)\in\mathbb{R}^4\times\mathcal{P}(M)^2 : s\le u\le v\le t\}.$$

Case 4: $u \le v \le s \le t$ is identical to the Case 1.

$$H(s,t,u,v,\mu,\nu) = \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{t-s}a(x)\mathsf{P}^{(N)}_{v-u}b(y)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy).$$

Case 5: $u \le s \le v \le t$ is identical to the Case 2.

$$H(s,t,u,v,\mu,\nu) = \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{s-u}\left[\mathsf{P}^{(2N)}_{v-s}\left(\mathsf{P}^{(N)}_{t-v}a\otimes b\right)(x,\cdot)\right](y)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy).$$

Case 6: $u \le s \le t \le v$ is identical to the Case 3.

$$H(s,t,u,v,\mu,\nu) = \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{s-u}\left[\mathsf{P}^{(2N)}_{t-s}\left(a\otimes \mathsf{P}^{(N)}_{v-t}b\right)(x,\cdot)\right](y)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy).$$

We note that the function $F(s,t,u,v,\mu,\nu) =$

$$= \begin{cases} \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{t-s}a(x)\mathsf{P}^{(N)}_{v-u}b(y)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy), \ s\le t\le u\le v, \\ \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{u-s}\left[\mathsf{P}^{(2N)}_{t-u}\left(a\otimes \mathsf{P}^{(N)}_{v-t}b\right)(\cdot,y)\right](x)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy), \ s\le u\le t\le v, \\ \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{u-s}\left[\mathsf{P}^{(2N)}_{v-u}\left(\mathsf{P}^{(N)}_{t-v}a\otimes b\right)(\cdot,y)\right](x)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy), \ s\le u\le v\le t, \\ \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{t-s}a(x)\mathsf{P}^{(N)}_{v-u}b(y)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy), \ u\le v\le s\le t, \\ \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{s-u}\left[\mathsf{P}^{(2N)}_{v-s}\left(\mathsf{P}^{(N)}_{t-v}a\otimes b\right)(x,\cdot)\right](y)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy), \ u\le s\le v\le t, \\ \int_{M^N}\int_{M^N} \mathsf{P}^{(N)}_{s-u}\left[\mathsf{P}^{(2N)}_{t-s}\left(a\otimes \mathsf{P}^{(N)}_{v-t}b\right)(x,\cdot)\right](y)\mu^{\otimes N}(dx)\nu^{\otimes N}(dy), \ u\le s\le t\le v \end{cases}$$

is well-defined. Hence, F is continuous on its domain and gives the needed continuous extension of H.

Consider the general case $a,b\in C(\hat{M}^N)$. Recall the exhaustive sequences $(L_j : j\ge 1)$, $(\zeta_j : j\ge 1)$ introduced in the beginning of Section 2. Let $a_j = a\times\zeta_j^{\otimes N}$, $b_j = b\times\zeta_j^{\otimes N}$,

$$g_j(\varkappa) = \int_{M^N} a_j(z)\varkappa^{\otimes N}(dz), \ h_j(\varkappa) = \int_{M^N} b_j(z)\varkappa^{\otimes N}(dz).$$

There exists a continuous function $F_j : \{(s,t)\in\mathbb{R}^2 : \ s\le t\}^2\times\mathcal{P}(M)^2\to\mathbb{R}$, such that

$$F_j(s,t,u,v,\mu,\nu) = \mathsf{E}g_j(\mu K^D_{s,t})h_j(\nu K^D_{u,v})$$

for $(s,t,u,v,\mu,\nu)\in D^4\times\mathcal{P}(M)^2$, $s\le t$, $u\le v$.

Fix $\varepsilon>0$, $T>0$ and a compact set $\mathcal{C}\subset\mathcal{P}(M)$. There exists a compact $C\subset M$, such that

$$\inf_{\varkappa\in\mathcal{C}}\varkappa(C)\ge 1-\varepsilon.$$

Continuity of the mapping $(t,z)\mapsto\Pi^{(1)}_t(z)\in\mathcal{P}(\mathcal{P}(M))$ implies that there exists a compact $L\subset M$, such that [13, Ch. II, Th. 6.7]

$$\inf_{t\in[0,2T],z\in C}\Pi^{(1)}_t(z,\{\varkappa : \ \varkappa(L)\ge 1-\varepsilon\})\ge 1-\varepsilon.$$

If $L \subset L_j$, we estimate for all $(s,t,u,v,\mu,\nu) \in (D\cap[-T,T])^4 \times \mathcal{C}^2$, $s \le t$, $u \le v$:

$$\begin{aligned}
&\left|\mathsf{E}g(\mu K^D_{s,t})h(\nu K^D_{u,v}) - F_j(s,t,u,v,\mu,\nu)\right| \\
&\le \|a\|\mathsf{E}\int_{M^N} \left(K^D_{u,v}\right)^{\otimes N}\left[|b|(1-\zeta_j^{\otimes N})\right](y)\nu^{\otimes N}(dy) \\
&\quad + \|b\|\mathsf{E}\int_{M^N} \left(K^D_{s,t}\right)^{\otimes N}\left[|a|(1-\zeta_j^{\otimes N})\right](x)\mu^{\otimes N}(dx) \\
&\le \|a\|\|b\|\left(2 - \mathsf{E}\left(\int_M K^D_{s,t}\zeta_j(x)\mu(dx)\right)^N - \mathsf{E}\left(\int_M K^D_{u,v}\zeta_j(y)\nu(dy)\right)^N\right) \\
&\le \|a\|\|b\|\left(2 - \left(\mathsf{E}\int_M K^D_{s,t}\zeta_j(x)\mu(dx)\right)^N - \left(\mathsf{E}\int_M K^D_{u,v}\zeta_j(y)\nu(dy)\right)^N\right) \\
&\le 2\|a\|\|b\|\left(1-(1-\varepsilon)^{3N}\right),
\end{aligned}$$

where the last inequality follows from relations

$$\begin{aligned}
\mathsf{E}\int_M K^D_{s,t}\zeta_j(x)\mu(dx) &\ge \int_C \mathsf{E}K^D_{s,t}\zeta_j(x)\mu(dx) \ge \int_C \mathsf{E}K^D_{s,t}(x,L)\mu(dx) \\
&= \int_C\int_{\mathcal{P}(M)} \varkappa(L)\Pi^{(1)}_{t-s}(x,d\varkappa)\mu(dx) \ge (1-\varepsilon)^3.
\end{aligned}$$

It follows that the function F is uniformly continuous on $\{(s,t)\in(D\cap[-T,T])^2 :\ s \le t\}^2 \times \mathcal{C}^2$ and can be continuously extended to $\{(s,t)\in[-T,T]^2 :\ s\le t\}^2\times\mathcal{C}^2$.

□

Proposition 4.3. *For any $T>0$, compact $\mathcal{C}\subset\mathcal{P}(M)$ and $\varepsilon>0$ there exists $\delta>0$ such that for all $(\mu,\nu)\in\mathcal{C}^2$, $((s,t),(u,v))\in(D\cap[-T,T])^4$ with $s\le t$, $u\le v$, $|s-u|\le\delta$, $|t-v|\le\delta$, $d(\mu,\nu)\le\delta$,*

$$\mathsf{P}\{\hat d(\mu K^D_{s,t},\nu K^D_{u,v})\ge\varepsilon\}\le\varepsilon.$$

Proof. Assume this is not true. Then for some $\varepsilon>0$ there exist sequences $-T\le s_n\le t_n\le T$, $-T\le u_n\le v_n\le T$, $(\mu_n,\nu_n)\in\mathcal{C}^2$, such that $|s_n-u_n|\to 0$, $|t_n-v_n|\to 0$, $d(\mu_n,\nu_n)\to 0$, but

$$\mathsf{P}\{\hat d(\mu_n K^D_{s_n,t_n},\nu_n K^D_{u_n,v_n})\ge\varepsilon\}\ge\varepsilon.$$

We may and do assume that $(s_n,t_n,u_n,v_n,\mu_n,\nu_n)\to(s,t,s,t,\mu,\mu)\in[-T,T]^4\times\mathcal{C}^2$.

Consider closed set

$$\Delta^c_\varepsilon = \{(\varkappa_1,\varkappa_2)\in\mathcal{P}(\hat M)^2 :\ \hat d(\varkappa_1,\varkappa_2)\ge\varepsilon\}$$

and a function

$$f(\varkappa_1,\varkappa_2) = (1 - R\hat d_2((\varkappa_1,\varkappa_2),\Delta^c_\varepsilon))_+,$$

where $\hat d_2((\varkappa_1,\varkappa_2),(\nu_1,\nu_2)) = \hat d(\varkappa_1,\nu_1)+\hat d(\varkappa_2,\nu_2)$ and $R>\frac{1}{\varepsilon}$. Then $f\in C(\mathcal{P}(\hat M)^2)$. Denote $F(s,t,u,v,\varkappa_1,\varkappa_2)=\mathsf{E}f(\varkappa_1K^D_{s,t},\varkappa_2K^D_{u,v})$. By the Lemma 4.5 the function F has a continuous extension on $\{(s,t)\in\mathbb{R}^2 : s\le t\}^2\times\mathcal{C}^2$. We have

$$\begin{aligned}
\varepsilon &\le \mathsf{P}\{(\mu_nK^D_{s_n,t_n},\nu_nK^D_{u_n,v_n})\in\Delta^c_\varepsilon\}\le\mathsf{E}f(\mu_nK^D_{s_n,t_n},\nu_nK^D_{u_n,v_n}) \\
&= F(s_n,t_n,u_n,v_n,\mu_n,\nu_n)\to F(s,t,s,t,\mu,\mu),\ n\to\infty.
\end{aligned}$$

However,

$$F(s,t,s,t,\mu,\mu) = \lim_{n\to\infty}F(s_n,t_n,s_n,t_n,\mu,\mu) = 0,$$

since

$$\hat d_2((\mu K^D_{s_n,t_n},\mu K^D_{s_n,t_n}),\Delta^c_\varepsilon)\ge\inf_{(\varkappa_1,\varkappa_2)\in\Delta^c_\varepsilon}\hat d(\varkappa_1,\varkappa_2)\ge\varepsilon>\frac{1}{R}$$

and $f(\mu K^D_{s_n,t_n},\mu K^D_{s_n,t_n})=0$. Obtained contradiction proves the Proposition.

□

4.3. **Stochastic flow of kernels** $(K_{s,t} : -\infty < s \le t < \infty)$**.** Proposition 4.3 implies that there exists a strictly increasing sequence of positive integers $(n_j : j \ge 1)$, such that for each $j \ge 1$ and all $(s,t,u,v,\mu,\nu) \in (D \cap [-j,j])^4 \times \mathcal{P}(L_j)^2$ with $s \le t$, $u \le v$, $|s-u| \le 2^{-n_j}$, $|t-v| \le 2^{-n_j}$, $d(\mu,\nu) \le 2^{-n_j}$,

$$\mathsf{P}\{\hat{d}(\mu K^D_{s,t}, \nu K^D_{u,v}) \ge 2^{-j}\} \le 2^{-j}.$$

Given $t \in \mathbb{R}$ define $t_j = \max\{s \in D_{n_j} : s \le t\}$. We note that $0 \le t - t_j < 2^{-n_j}$, and $s \le t \Rightarrow s_j \le t_j$.

Fix a measurable mapping $\hat{\ell} : \mathcal{P}(\hat{M})^{\mathbb{N}} \to \mathcal{P}(\hat{M})$ with the following property: for any sequence $\mu = (\mu_n : n \in \mathbb{N})$ in $\mathcal{P}(\hat{M})$, $\hat{\ell}(\mu)$ is a limit point of μ (see [14, L. 7.1] and explanations in the beginning of Section 3.2 for the existence of such mapping). Fix $x_0 \in M$ and consider measurable mappings $\Phi : E^{\mathbb{N}} \to E$, $\Psi : M \times E^{\mathbb{N}} \to \mathcal{P}(M)$,

$$\Phi(K)(x) = \Psi(x,K) = \begin{cases} \hat{\ell}\left((\mathfrak{p}(K_n)(x) : n \ge 1)\right), & \text{if } \hat{\ell}\left((\mathfrak{p}(K_n)(x) : n \ge 1)\right)(M) = 1 \\ \delta_{x_0}, & \text{otherwise} \end{cases}.$$

Now we have everything ready to construct the needed stochastic flow of kernels. We will use random kernels $(K^D_{s,t} : -\infty < s \le t < \infty,\ (s,t) \in D^2)$ constructed in the Section 4.2.

Definition 4.2. For real $s \le t$ we define random kernels

$$\tilde{K}_{s,t} = \Phi((K^D_{s_j,t_j} : j \ge 1)),\ K_{s,t} = \mathfrak{p}(\tilde{K}_{s,t}).$$

In the subsequent sections we verify that the family $(K_{s,t} : -\infty < s \le t < \infty)$ satisfies all conditions stated in the Theorem 2.2.

4.3.1. *Consistency.* We check that Definitions 4.2 and 4.1 are consistent. Let $(s,t) \in D^2$, $s \le t$. Then $s_j = s$, $t_j = t$ for all large enough j. The needed statement follows from equalities

$$\tilde{K}_{s,t}(x) = \hat{\ell}\left(\left(\mathfrak{p}(K^D_{s_j,t_j})(x) : j \ge 1\right)\right) = \mathfrak{p}(K^D_{s,t})(x) = K^D_{s,t}(x).$$

(see statement (1) of Proposition 4.1). In what follows we identify $K^D_{s,t}$ with $K_{s,t}$ for $(s,t) \in D^2$, $s \le t$.

4.3.2. *Case when* $s = t$. If $s = t$, then $s_j = t_j$ for all $j \ge 1$. Since the kernel $x \mapsto \delta_x$ is invariant under $\mathfrak{p}$ (Lemma 4.1) and $K_{s,s}(x) = \delta_x$ (Definition 4.1), we deduce that

$$\tilde{K}_{s,s}(x) = \hat{\ell}\left(\left(\mathfrak{p}(K_{s_j,s_j})(x) : j \ge 1\right)\right) = \delta_x.$$

It follows that $K_{s,s}(x) = \delta_x$ without exceptions.

4.3.3. *Invariance under* $\mathfrak{p}$. Property $\mathfrak{p} \circ \mathfrak{p} = \mathfrak{p}$ and Definition 4.2 immediately imply that $\mathfrak{p}(K_{s,t}) = K_{s,t}$.

4.3.4. *Measurability.* Mappings $(s,t,\omega) \mapsto K_{s_j,t_j}(\omega)$ are measurable. From measurability of Φ we immediately obtain the measurability of $(s,t,\omega) \mapsto K_{s,t}(\omega)$.

4.3.5. *Convergence of approximations and distribution of* $K_{s,t}$.

Proposition 4.4. (1) *For all real* $s \le t$ *and all* $\varkappa \in \mathcal{P}(M)$,

$$\lim_{j\to\infty} \varkappa K_{s_j,t_j} = \varkappa K_{s,t} \quad a.s.$$

(2) *The law of* $K_{s,t}$ *coincides with with* ν_{t-s}.

Proof. We start by showing that for each $x \in M$ a.s. the limit $\lim_{j\to\infty} K_{s_j,t_j}(x)$ exists in $\mathcal{P}(M)$. Fix $T > 0$, such that $-T \le s \le t \le T$. If $[-T-1, T+1] \subset [-j, j]$ and $x \in L_j$, then under conditions $(s,t,u,v) \in (D \cap [-j,j])^4$, $s \le t$, $u \le v$, $|s-u| \le 2^{-n_j}$, $|t-v| \le 2^{-n_j}$, we have

$$\mathsf{P}\{\hat{d}(K_{s,t}(x), K_{u,v}(x)) \ge 2^{-j}\} \le 2^{-j}.$$

We note that for all large enough j, $(s_j, t_j, s_{j+1}, t_{j+1}) \in (D \cap [-j,j])^4$. Further $0 \le s_{j+1} - s_j < 2^{-n_j}$, $0 \le t_{j+1} - t_j < 2^{-n_j}$. Hence,

$$\mathsf{P}\{\hat{d}(K_{s_j,t_j}(x), K_{s_{j+1},t_{j+1}}(x)) \ge 2^{-j}\} \le 2^{-j}.$$

It follows that with probability 1 for all large enough j, $\hat{d}(K_{s_j,t_j}(x), K_{s_{j+1},t_{j+1}}(x)) < 2^{-j}$. In particular, the limit $\lim_{j\to\infty} K_{s_j,t_j}(x)$ a.s. exists in $\mathcal{P}(\hat{M})$. The law of $\lim_{j\to\infty} K_{s_j,t_j}(x)$ is $\Pi^{(1)}_{t-s}(x) \in \mathcal{P}(\mathcal{P}(M))$. So, a.s. $\lim_{j\to\infty} K_{s_j,t_j}(x)$ is concentrated on $\mathcal{P}(M)$.

The proved convergence implies that the distribution of $\tilde{K}_{s,t}$ coincides with ν_{t-s}. Since $\mathfrak{p}$ is a measurable presentation of ν_{t-s}, $\tilde{K}_{s,t}(x) = K_{s,t}(x)$ a.s. and the distribution of $K_{s,t}$ coincides with ν_{t-s}. The first statement of the Proposition follows from Fubini's theorem. □

4.3.6. *Idependent increments.* Let $t^{(1)} \le t^{(2)} \le \ldots \le t^{(m)}$. Then for each $j \ge 1$ random kernels $K_{t_j^{(1)},t_j^{(2)}}, \ldots, K_{t_j^{(m-1)},t_j^{(m)}}$ are independent (Proposition 4.1). Distribution of $(K_{t^{(1)},t^{(2)}}, \ldots, K_{t^{(m-1)},t^{(m)}})$ in $(E^{m-1}, \mathcal{E}^{\otimes(m-1)})$ is completely determined by distributions of $(K_{t^{(k)},t^{(k+1)}}(x_r) : 1 \le k \le m-1, 1 \le r \le l)$, where $x \in M^l$, $l \ge 1$. Proposition 4.4 implies that random kernels $K_{t^{(1)},t^{(2)}}, \ldots, K_{t^{(m-1)},t^{(m)}}$ are independent as well.

4.3.7. *Evolutionary property.*

Proposition 4.5. *For all real* $r \le s \le t$ *and* $x \in M$,

$$K_{r,s}K_{s,t}(x) = K_{r,t}(x) \quad a.s.$$

Proof. Case when $r = s$ or $s = t$ is trivial. Assume that $r < s < t$. Let $\mathcal{M}_j = K_{r_j,s_j}(x)$, $\mathcal{M} = K_{r,s}(x)$. Sequence $(\mathcal{M}_j : j \ge 1)$ is independent from $K_{s,t}$ and converges a.s. to $\mathcal{M}$ (Proposition 4.4).

Choose $T > 0$ such that $[r,t] \subset [-T+1, T]$. Given $\alpha > 0$ find compact $\mathcal{C} \subset \mathcal{P}(M)$, such that $\mathsf{P}\{\mathcal{M} \in \mathcal{C}\} > 1 - \alpha$ and $\mathsf{P}\{\mathcal{M}_j \in \mathcal{C}\} > 1 - \alpha$ for all j. By Proposition 4.3 there is a strictly increasing sequence of positive integers $(j_l : l \ge 1)$, such that for all $((w,z),(u,v)) \in (D \cap [-T,T])^4$ with $w \le z$, $u \le v$, $|w-u| \le 2^{-n_{j_l}}$, $|z-v| \le 2^{-n_{j_l}}$, and all $\varkappa \in \mathcal{C}$,

$$\mathsf{P}\{\hat{d}(\varkappa K_{w,z}, \varkappa K_{u,v}) \ge 2^{-l}\} \le 2^{-l}.$$

It follows that for $\varkappa \in \mathcal{C}$,

$$\mathsf{P}\{\hat{d}(\varkappa K_{s_{j_l},t_{j_l}}, \varkappa K_{s_{j_{l+1}},t_{j_{l+1}}}) \ge 2^{-l}\} \le 2^{-l}.$$

Proposition 4.4 implies

$$\mathsf{P}\{\hat{d}(\varkappa K_{s_{j_l},t_{j_l}},\varkappa K_{s,t})>2^{-l+1}\}\le\liminf_{L\to\infty}\mathsf{P}\{\hat{d}(\varkappa K_{s_{j_l},t_{j_l}},\varkappa K_{s_{j_L},t_{j_L}})\ge 2^{-l+1}\}$$
$$\le\liminf_{L\to\infty}\mathsf{P}\left\{\hat{d}(\varkappa K_{s_{j_l},t_{j_l}},\varkappa K_{s_{j_L},t_{j_L}})\ge\sum_{m=l}^{L-1}2^{-m}\right\}$$
$$\le\liminf_{L\to\infty}\sum_{m=l}^{L-1}\mathsf{P}\left\{\hat{d}(\varkappa K_{s_{j_m},t_{j_m}},\varkappa K_{s_{j_{m+1}},t_{j_{m+1}}})\ge 2^{-m}\right\}\le\lim_{L\to\infty}\sum_{m=l}^{L-1}2^{-m}=2^{-l+1}.$$

When $2^{-l+1}\le\alpha$, we have

$$\begin{aligned}&\mathsf{P}\{\hat{d}(\mathcal{M}_{j_l}K_{s_{j_l},t_{j_l}},\mathcal{M}K_{s,t})>2\alpha\}\\&\le\mathsf{P}\{\hat{d}(\mathcal{M}_{j_l}K_{s_{j_l},t_{j_l}},\mathcal{M}_{j_l}K_{s,t})>\alpha\}+\mathsf{P}\{\hat{d}(\mathcal{M}_{j_l}K_{s,t},\mathcal{M}K_{s,t})>\alpha\}\\&\le\alpha+\sup_{\varkappa\in\mathcal{C}}\mathsf{P}\{\hat{d}(\varkappa K_{s_{j_l},t_{j_l}},\varkappa K_{s,t})>2^{-l+1}\}+\mathsf{P}\{\hat{d}(\mathcal{M}_{j_l}K_{s,t},\mathcal{M}K_{s,t})>\alpha\}\\&\le\alpha+2^{-l+1}+\mathsf{P}\{\hat{d}(\mathcal{M}_{j_l}K_{s,t},\mathcal{M}K_{s,t})>\alpha\}\le2\alpha+\mathsf{P}\{\hat{d}(\mathcal{M}_{j_l}K_{s,t},\mathcal{M}K_{s,t})>\alpha\}.\end{aligned}\tag{15}$$

By Proposition 4.3 there exists $\delta>0$, such that for all $(u,v))\in(D\cap[-T,T])^2$ with $u\le v$, and all $(\varkappa_1,\varkappa_2)\in\mathcal{C}^2$ with $d(\varkappa_1,\varkappa_2)\le\delta$,

$$\mathsf{P}\{\hat{d}(\varkappa_1K_{u,v},\varkappa_2K_{u,v})\ge\alpha\}\le\alpha.$$

From Proposition 4.4 it follows that for all $(\varkappa_1,\varkappa_2)\in\mathcal{C}^2$ with $d(\varkappa_1,\varkappa_2)\le\delta$,

$$\mathsf{P}\{\hat{d}(\varkappa_1K_{s,t},\varkappa_2K_{s,t})>\alpha\}\le\alpha.$$

We estimate

$$\begin{aligned}&\mathsf{P}\{\hat{d}(\mathcal{M}_{j_l}K_{s,t},\mathcal{M}K_{s,t})>\alpha\}\\&\le2\alpha+\mathsf{P}\{d(\mathcal{M}_{j_l},\mathcal{M})>\delta\}+\sup_{\substack{(\varkappa_1,\varkappa_2)\in\mathcal{C}^2\\d(\varkappa_1,\varkappa_2)\le\delta}}\mathsf{P}\{\hat{d}(\varkappa_1K_{s,t},\varkappa_2K_{s,t})>\alpha\}\\&\le3\alpha+\mathsf{P}\{d(\mathcal{M}_{j_l},\mathcal{M})>\delta\}.\end{aligned}\tag{16}$$

Substituting (16) into (15), we get

$$\mathsf{P}\{\hat{d}(\mathcal{M}_{j_l}K_{s_{j_l},t_{j_l}},\mathcal{M}K_{s,t})>2\alpha\}\le5\alpha+\mathsf{P}\{d(\mathcal{M}_{j_l},\mathcal{M})>\delta\},$$

when $2^{-l+1}\le\alpha$. Since $\lim_{l\to\infty}\mathsf{P}\{d(\mathcal{M}_{j_l},\mathcal{M})>\delta\}=0$, we deduce that

$$\mathcal{M}_{j_l}K_{s_{j_l},t_{j_l}}\to\mathcal{M}K_{s,t},\ l\to\infty,$$

in probability. On the other hand, a.s.

$$\mathcal{M}_{j_l}K_{s_{j_l},t_{j_l}}=K_{r_{j_l},t_{j_l}}(x)\to K_{r,t}(x),\ j\to\infty$$

(Remark 4.2). It follows that $K_{r,t}(x)=\mathcal{M}K_{s,t}=K_{r,s}K_{s,t}(x)$ a.s. This finishes the proof of Proposition 4.5 and of Theorem 2.2 as well.

□

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