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WEAKLY PERIODIC GIBBS MEASURES FOR THE POTTS-SOS MODEL ON A CAYLEY TREE OF ORDER TWO

In this paper we consider Potts-SOS model, with spin values 0, 1, 2, on the Cayley tree of order two. We study the weakly periodic Gibbs measures for this model, with respect to normal subgroup of two index of the group representation of a Cayley tree.

1. Introduction

One of the central problems in the theory of Gibbs measures (GMs) is to describe infinite-volume (or limiting) GMs corresponding to a given Hamiltonian. However, a complete analysis of the set of limiting GMs for a specific Hamiltonian is often a difficult problem (see [3, 14, 19, 20]).

In this paper we consider model with nearest neighbour interactions on a Cayley tree (CT). Models on a CT were discussed in Refs. [1, 2, 3, 4, 6, 7, 8, 9, 15, 16, 17, 18, 19].

In [2] all translation-invariant GMs for the Potts model on the CT are described. In [17] periodic GMs, in [7, 8] weakly periodic GMs for the Potts model are studied. In [1] translation-invariant GMs and in [19] periodic GMs for the SOS model on the CT are studied. Weakly periodic GMs for the Ising model in [6, 9, 18], for the Potts model on the CT in [7, 8] are studied.

The model considered in this paper (Potts-SOS model) is a generalization of the Potts and SOS (solid-on-solid) models. In [5] some translation-invariant GMs for the Potts-SOS model on the CT of order $k \geq 2$ are studied, in [10] all translation-invariant GMs for the Potts-SOS model on the CT of order $k = 2$ and in [11, 12] periodic GMs are studied, but weakly periodic GMs for this model on the CT were not studied yet. In this paper we study weakly periodic GMs for this model on the CT of order $k = 2$. We prove that under the considered condition, all weakly periodic Gibbs measures with respect to all two-index normal subgroup are translation-invariant.

2. Notations

2.1. **Cayley tree.** The Cayley tree Γ^k (see, e.g., [13], [15]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y\rangle$.

The distance $d(x,y), x, y \in V$ on the Cayley tree is defined by the formula

$$d(x,y) = \min\{d \mid \exists x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \text{such that} \\ \langle x_0, x_1\rangle, ..., \langle x_{d-1}, x_d\rangle\}.$$

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For the fixed $x^0 \in V$ we set

$$W_n = \{x \in V \mid \ d(x, x^0) = n\}, V_n = \{x \in V \mid \ d(x, x^0) \leq n\},$$

$$L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

It is known (see [13]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e, a_i\}$, $i = 1, ..., k+1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, ..., a_{k+1}$.

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x and $S(x)$ denote the set of all direct successors of x, i.e.,

$$S_1(x) = \{y \in G_k : \langle x, y\rangle\}$$

and

$$S(x) = \{y \in W_{n+1} : \langle x, y\rangle\}, x \in W_n,$$

where $\langle x, y\rangle$ means that the vertex x and y are nearest neighbors.

Let $\{x_\downarrow\} = S_1(x) \setminus S(x)$. Note that $\mid S(x^0) \mid= k+1$ and for all $x \neq x^0$ $\mid S(x) \mid= k$, $x \in V$.

2.2. **The model and a system of functional equations.** Here we shall give main definitions and facts about the model. Consider the model, where the spin takes values in the set $\Phi = \{0, 1, 2, ..., m\}$, $m \geq 1$. For $A \subseteq V$ a spin *configuration* σ_A on A is defined as a function $x \in A \mapsto \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega_A = \Phi^A$. Denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

The Hamiltonian of the Potts-SOS model with nearest-neighbor interactions has the form

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \mid \sigma(x) - \sigma(y) \mid - J_p \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{1}$$

where $J, J_p \in R$ are nonzero coupling constants.

Let $h : x \mapsto h_x = (h_{0,x}, h_{1,x}, ..., h_{m,x}) \in R^{m+1}$ be a real vector-valued function of $x \in V$. We take into consideration the probability distributions $\mu^{(n)}$ on Φ^{V_n} for given $n = 1, 2, ...$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\big(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\big), \tag{2}$$

where $\sigma_n \in \Phi^{V_n}$ and the partition function Z_n can be expressed as follows:

$$Z_n(\widetilde{\sigma}_n) = \sum_{\widetilde{\sigma}_n \in \Phi^{V_n}} \exp\big(-\beta H(\widetilde{\sigma}_n) + \sum_{x \in W_n} h_{\widetilde{\sigma}(x),x}\big).$$

We say that the probability distributions $\mu^{(n)}$ are compatible if $\forall n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n \in \Phi^{W_n}} \mu^{(n)}\big(\sigma_{n-1} \vee \omega_n\big) = \mu^{(n-1)}\big(\sigma_{n-1}\big), \tag{3}$$

here $\big(\sigma_{n-1} \vee \omega_n\big) \in \Phi^{V_n}$ is the concatenation of σ_{n-1} and ω_n.

Definition 2.1. If probability distribution $\mu^{(n)}$ on Φ^{V_n} holds the equality (3), then, by the Kolmogorov extension theorem, there exists a unique probability measure μ on the measurable space $(\Phi^V, \mathcal{F})$, where $\mathcal{F}$ is the sigma-algebra generated by the cylinder sets of Φ^V, such that for all n and $\sigma_n \in \Phi^{V_n}$,

$$\mu\big(\{\sigma \in \Phi^V \mid \sigma \mid_{V_n} = \sigma_n\}\big) = \mu^{(n)}(\sigma_n).$$

This measure μ is called a **splitting Gibbs measure** (SGM) corresponding to the Hamiltonian H and a boundary condition function $h : x \mapsto h_x$ for all $x \neq x^0$.

The next theorem expresses the requirement on function h that the distributions $\mu^{(n)}(\sigma_n)$ hold the compatibility conditions.

Theorem 2.1. [5] *Defined as in (2) the probability distributions* $\mu^{(n)}(\sigma_n), n = 1, 2, ...,$ *satisfy the compatibility condition if and only if*

$$h^*_x = \sum_{y\in S(x)} F\big(h^*_y, m, \theta, r\big) \tag{4}$$

is satisfied for any $x \in V \setminus \{x^0\}$, *where*

$$\theta = \exp(J\beta), \quad r = \exp(J_p\beta) \tag{5}$$

and also $\beta = 1/T$ *is the inverse temperature. Here* h^*_x *represents the vector* $(h_{0,x} - h_{m,x}, h_{1,x} - h_{m,x}, ..., h_{m-1,x} - h_{m,x})$ *and the vector function* $F(., m, \theta, r) : R^m \to R^m$ *is defined as follows*

$$F(h, m, \theta, r) = \big(F_0(h, m, \theta, r), F_1(h, m, \theta, r), ..., F_{m-1}(h, m, \theta, r)\big)$$

where

$$F_i(h, m, \theta, r) = \ln \frac{\sum_{j=0}^{m-1} \theta^{|i-j|} r^{\delta_{ij}} e^{h_j} + \theta^{m-i} r^{\delta_{mi}}}{\sum_{j=0}^{m-1} \theta^{m-j} r^{\delta_{mj}} e^{h_j} + r}, \tag{6}$$

$h = (h_0, h_1, ..., h_{m-1}), i = 0, 1, 2, ..., m - 1.$

Namely, for any collection of functions satisfying the functional equation (4) there exists a unique splitting Gibbs measure, the correspondence being one-to-one.

Let $G_k/K = \{K_0, K_1..., K_{r-1}\}$ factor group, where K is a normal subgroup of index $r \geq 1$.

Definition 2.2. A collection of vectors $h = \{h_x \in R^m : x \in G_k\}$ is said to be K-*periodic*, if $h_x = h_i$ for $x \in K_i$ for all $x \in G_k$. A G_k-periodic collections is said to be *translation-invariant*.

Definition 2.3. A collection of vectors $h = \{h_x \in R^m : x \in G_k\}$ is said to be K-*weakly periodic*, if $h_x = h_{ij}$ for $x_\downarrow \in K_i$ and $x \in K_j$ for any $x \in G_k$.

Definition 2.4. A Gibbs measure μ is called K-*(weakly) periodic*, if it corresponds to the K-*(weakly) periodic* collection of h. A G_k-periodic measure is said to be *translation-invariant*.

Note that in [10, 5] translation-invariant splitting Gibbs measures, in [11, 12] periodic Gibbs measures for the Potts-SOS model are studied. Our aim is to describe a set of weakly periodic Gibbs measures for the Potts-SOS model on the Cayley tree of order two.

2.3. **Weakly periodic SGMs.** Let

$$H_A = \{x \in G_k : \sum_{i\in A} \omega_x(a_i) \ \text{ is an even number}\},$$

where $\emptyset \neq A \subseteq N_k = \{1, 2, 3, ..., k + 1\}$, and $\omega_x(a_i)$ is the number of letters a_i in a word $x \in G_k$. Note that H_A is a normal subgroup of the G_k (see [13]). We consider a quotient group $G_k/H_A = \{H_A, G_k \setminus H_A\}$, where

$$G_k \setminus H_A = \{x \in G_k : \sum_{i\in A} \omega_x(a_i) \ \text{ is an odd number}\}.$$

By (4), the H_A-weakly periodic collection h_x has the following form

$$h_x = \begin{cases} h_{00}, & \text{if } x_{\downarrow} \in H_A, \quad x \in H_A, \\ h_{01}, & \text{if } x_{\downarrow} \in H_A, \quad x \in G_k \setminus H_A, \\ h_{10}, & \text{if } x_{\downarrow} \in G_k \setminus H_A, \quad x \in H_A, \\ h_{11}, & \text{if } x_{\downarrow} \in G_k \setminus H_A, \quad x \in G_k \setminus H_A. \end{cases} \tag{7}$$

Clarify that a weakly periodic Gibbs measure depends to a normal subgroup, i.e. relying on the selection of the normal subgroup, different weakly periodic Gibbs measures are found. The set of weakly periodic Gibbs measures also includes the set of periodic (in particular translation-invariant) Gibbs measures.

Let $G_k^{(2)}$ be the subgroup in G_k consisting of all words of even length, i.e. $G_k^{(2)} = \{x \in G_k :\mid x \mid \text{ is even}\}$. Clearly, $G_k^{(2)}$ is a normal subgroup of index 2 (see [13]). If $\mid A \mid= k+1$, where $\mid A \mid$ is the number of elements of the set A, i.e. $A = N_k$, then the concept of weakly periodicity coincides with ordinary periodicity. Therefore, we consider $A \subset N_k$ such that $A \neq N_k$, i.e. $H_A \neq G_k^{(2)}$.

3. Main results

Theorem 3.1. *Let $k = 2, \mid A \mid= 1, r = \theta^2$. Then all H_A-weakly periodic Gibbs measures for the Potts-SOS model are translation-invariant.*

Theorem 3.2. *Let $k = 2, \mid A \mid= 2, r = \theta^2$. Then all H_A-weakly periodic Gibbs measures for the Potts-SOS model are translation-invariant.*

4. Proofs

Proof. **(Proof of Theorem 3.1.)** We consider the case $\mid A \mid= 1$ and a quotient group $G_k/H_A = \{H_A, G_k \setminus H_A\}$, where

$$H_A = \{x \in G_k : w_x(a_i) \text{ is an even number}\},$$

$$G_k \setminus H_A = \{x \in G_k : w_x(a_i) \text{ is an odd number}\}.$$

Let $k = 2, m = 2$. Then for H_A-weakly periodic collection (7) from (4) we have the following system of equations:

$$\begin{cases} h_{00} = F(h_{00}, \theta, r) + F(h_{01}, \theta, r), \\ h_{01} = 2F(h_{11}, \theta, r), \\ h_{10} = 2F(h_{00}, \theta, r), \\ h_{11} = F(h_{10}, \theta, r) + F(h_{11}, \theta, r), \end{cases} \tag{8}$$

where $h_{ij} = (h_{ij}^{(1)}, h_{ij}^{(2)})$ and

$$F(h_{ij}, \theta, r) = \left(\ln \frac{r \exp h_{ij}^{(1)} + \theta \exp h_{ij}^{(2)} + \theta^2}{\theta^2 \exp h_{ij}^{(1)} + \theta \exp h_{ij}^{(2)} + r}, \ln \frac{\theta \exp h_{ij}^{(1)} + r \exp h_{ij}^{(2)} + \theta}{\theta^2 \exp h_{ij}^{(1)} + \theta \exp h_{ij}^{(2)} + r} \right).$$

From (8), we have the following system of equations

$$
(9)\qquad \begin{cases}
h_{00}^{(1)} = \ln \dfrac{r\exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + \theta^2}{\theta^2 \exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + r} + \ln \dfrac{r\exp h_{01}^{(1)} + \theta \exp h_{01}^{(2)} + \theta^2}{\theta^2 \exp h_{01}^{(1)} + \theta \exp h_{01}^{(2)} + r},\\
h_{00}^{(2)} = \ln \dfrac{\theta\exp h_{00}^{(1)} + r \exp h_{00}^{(2)} + \theta}{\theta^2 \exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + r} + \ln \dfrac{\theta\exp h_{01}^{(1)} + r \exp h_{01}^{(2)} + \theta}{\theta^2 \exp h_{01}^{(1)} + \theta \exp h_{01}^{(2)} + r},\\
h_{01}^{(1)} = 2\ln \dfrac{r\exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + \theta^2}{\theta^2 \exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + r},\\
h_{01}^{(2)} = 2\ln \dfrac{\theta\exp h_{11}^{(1)} + r \exp h_{11}^{(2)} + \theta}{\theta^2 \exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + r},\\
h_{10}^{(1)} = 2\ln \dfrac{r\exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + \theta^2}{\theta^2 \exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + r},\\
h_{10}^{(2)} = 2\ln \dfrac{\theta\exp h_{00}^{(1)} + r \exp h_{00}^{(2)} + \theta}{\theta^2 \exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + r},\\
h_{11}^{(1)} = \ln \dfrac{r\exp h_{10}^{(1)} + \theta \exp h_{10}^{(2)} + \theta^2}{\theta^2 \exp h_{10}^{(1)} + \theta \exp h_{10}^{(2)} + r} + \ln \dfrac{r\exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + \theta^2}{\theta^2 \exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + r},\\
h_{11}^{(2)} = \ln \dfrac{\theta\exp h_{10}^{(1)} + r \exp h_{10}^{(2)} + \theta}{\theta^2 \exp h_{10}^{(1)} + \theta \exp h_{10}^{(2)} + r} + \ln \dfrac{\theta\exp h_{11}^{(1)} + r \exp h_{11}^{(2)} + \theta}{\theta^2 \exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + r}.
\end{cases}
$$

Let $r = \theta^2$. Since the general analysis of the system of equations (9) is very complicated, we solve it under the condition $r = \theta^2$. Then the system of equations (9) have the following form:

$$
(10)\qquad \begin{cases}
h_{00}^{(1)} = \ln \dfrac{\theta^2\exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + \theta^2}{\theta^2 \exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + \theta^2} + \ln \dfrac{\theta^2\exp h_{01}^{(1)} + \theta \exp h_{01}^{(2)} + \theta^2}{\theta^2 \exp h_{01}^{(1)} + \theta \exp h_{01}^{(2)} + \theta^2},\\
h_{00}^{(2)} = \ln \dfrac{\theta\exp h_{00}^{(1)} + \theta^2 \exp h_{00}^{(2)} + \theta}{\theta^2 \exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + \theta^2} + \ln \dfrac{\theta\exp h_{01}^{(1)} + \theta^2 \exp h_{01}^{(2)} + \theta}{\theta^2 \exp h_{01}^{(1)} + \theta \exp h_{01}^{(2)} + \theta^2},\\
h_{01}^{(1)} = 2\ln \dfrac{\theta^2\exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + \theta^2}{\theta^2 \exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + \theta^2},\\
h_{01}^{(2)} = 2\ln \dfrac{\theta\exp h_{11}^{(1)} + \theta^2 \exp h_{11}^{(2)} + \theta}{\theta^2 \exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + \theta^2},\\
h_{10}^{(1)} = 2\ln \dfrac{\theta^2\exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + \theta^2}{\theta^2 \exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + \theta^2},\\
h_{10}^{(2)} = 2\ln \dfrac{\theta\exp h_{00}^{(1)} + \theta^2 \exp h_{00}^{(2)} + \theta}{\theta^2 \exp h_{00}^{(1)} + \theta \exp h_{00}^{(2)} + \theta^2},\\
h_{11}^{(1)} = \ln \dfrac{\theta^2\exp h_{10}^{(1)} + \theta \exp h_{10}^{(2)} + \theta^2}{\theta^2 \exp h_{10}^{(1)} + \theta \exp h_{10}^{(2)} + \theta^2} + \ln \dfrac{\theta^2\exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + \theta^2}{\theta^2 \exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + \theta^2},\\
h_{11}^{(2)} = \ln \dfrac{\theta\exp h_{10}^{(1)} + \theta^2 \exp h_{10}^{(2)} + \theta}{\theta^2 \exp h_{10}^{(1)} + \theta \exp h_{10}^{(2)} + \theta^2} + \ln \dfrac{\theta\exp h_{11}^{(1)} + \theta^2 \exp h_{11}^{(2)} + \theta}{\theta^2 \exp h_{11}^{(1)} + \theta \exp h_{11}^{(2)} + \theta^2}.
\end{cases}
$$

From (10) we obtain:

$$
(11) \qquad \begin{cases}
h_{00}^{(1)} = \ln 1 + \ln 1 = 0, \\
h_{00}^{(2)} = \ln \dfrac{\theta \exp h_{00}^{(2)} + 2}{\exp h_{00}^{(2)} + 2\theta} + \ln \dfrac{\theta \exp h_{01}^{(2)} + 2}{\exp h_{01}^{(2)} + 2\theta}, \\
h_{01}^{(1)} = 2\ln 1 = 0, \\
h_{01}^{(2)} = 2\ln \dfrac{\theta \exp h_{11}^{(2)} + 2}{\exp h_{11}^{(2)} + 2\theta}, \\
h_{10}^{(1)} = 2\ln 1 = 0, \\
h_{10}^{(2)} = 2\ln \dfrac{\theta \exp h_{00}^{(2)} + 2}{\exp h_{00}^{(2)} + 2\theta}, \\
h_{11}^{(1)} = \ln 1 + \ln 1 = 0, \\
h_{11}^{(2)} = \ln \dfrac{\theta \exp h_{10}^{(2)} + 2}{\exp h_{10}^{(2)} + 2\theta} + \ln \dfrac{\theta \exp h_{11}^{(2)} + 2}{\exp h_{11}^{(2)} + 2\theta}.
\end{cases}
$$

Denoting $t_1 = \exp h_{00}^{(2)}, t_2 = \exp h_{01}^{(2)}, t_3 = \exp h_{10}^{(2)}, t_4 = \exp h_{11}^{(2)}$, where $t_i > 0, i = \overline{1,4}$, we get:

$$
(12) \qquad \begin{cases}
t_1 = \frac{\theta t_1 + 2}{t_1 + 2\theta} \cdot \frac{\theta t_2 + 2}{t_2 + 2\theta}, \\
t_2 = \left(\frac{\theta t_4 + 2}{t_4 + 2\theta}\right)^2, \\
t_3 = \left(\frac{\theta t_1 + 2}{t_1 + 2\theta}\right)^2, \\
t_4 = \frac{\theta t_3 + 2}{t_3 + 2\theta} \cdot \frac{\theta t_4 + 2}{t_4 + 2\theta}.
\end{cases}
$$

In the system of equations (12), we put the expression of t_2 in the second equation into the first equation, the expression of t_3 in the third equation into the fourth equation, and we have the following system of equations with respect to the variables t_1 and t_4:

$$
(13) \qquad \begin{cases}
t_1 = \frac{\theta t_1 + 2}{t_1 + 2\theta} \cdot \frac{\theta\left(\frac{\theta t_4 + 2}{t_4 + 2\theta}\right)^2 + 2}{\left(\frac{\theta t_4 + 2}{t_4 + 2\theta}\right)^2 + 2\theta}, \\
t_4 = \frac{\theta\left(\frac{\theta t_1 + 2}{t_1 + 2\theta}\right)^2 + 2}{\left(\frac{\theta t_1 + 2}{t_1 + 2\theta}\right)^2 + 2\theta} \cdot \frac{\theta t_4 + 2}{t_4 + 2\theta}.
\end{cases}
$$

We simplify each equation of the system of equations (13):

$$
(14) \qquad \begin{cases}
t_1(t_1 + 2\theta)\big((\theta t_4 + 2)^2 + 2\theta(t_4 + 2\theta)^2\big) - \\
-(\theta t_1 + 2)(\theta(\theta t_4 + 2)^2 + 2(t_4 + 2\theta)^2) = 0, \\
t_4(t_4 + 2\theta)\big((\theta t_1 + 2)^2 + 2\theta(t_1 + 2\theta)^2\big) - \\
-(\theta t_4 + 2)(\theta(\theta t_1 + 2)^2 + 2(t_1 + 2\theta)^2) = 0.
\end{cases}
$$

In the system of equations (14), we subtract the second equation from the first equation and divide the expression on the left side of the resulting equation into multipliers:

$$
(15) \qquad (t_1 - t_4) \cdot \big((\theta^4 t_1 - 2\theta^3 t_1 + 4\theta^2 t_1 + 6\theta t_1 + 10\theta^3 + 8)t_4 +
$$

$$+2(5\theta^3 t_1 + 4t_1 + 8\theta^4 - 4\theta^3 + 2\theta^2 + 12\theta)\big) = 0.$$

From (15) we have the following cases:

$$t_4 = t_1 \tag{16}$$

or

$$t_4 = \frac{-2(5\theta^3 t_1 + 4t_1 + 8\theta^4 - 4\theta^3 + 2\theta^2 + 12\theta)}{\theta^4 t_1 - 2\theta^3 t_1 + 4\theta^2 t_1 + 6\theta t_1 + 10\theta^3 + 8}. \tag{17}$$

First, we consider case (16). Substituting (16) into the first equation of (14), simplifying, and then factoring, we get:

$$\theta\Big((\theta+2)t_1 + 4\theta + 2\Big)\Big(t_1^3 + (-\theta^2 + 4\theta)t_1^2 + (4\theta^2 - 4\theta)t_1 - 4\Big) = 0. \tag{18}$$

Considering $\theta > 0$ and $t_1 > 0$, from the equation (18) we get the following equation:

$$t_1^3 + (-\theta^2 + 4\theta)t_1^2 + (4\theta^2 - 4\theta)t_1 - 4 = 0. \tag{19}$$

The solutions of equation (19) characterize the TISGMs for the Potts-SOS model on a second-order Cayley tree. This equation was studied in Lemma 3 of [10].

Now we consider case (17). Substituting (17) into the second equation of (14) and simplifying, we have the following equation with respect to the variable t_1:

$$\begin{aligned}
&(\theta^6 - 4\theta^5 + 4\theta^4 + 16\theta^3 - 8\theta^2 + 2\theta + 16)t_1^4 + (2\theta^7 - 9\theta^6 + 8\theta^5 + 52\theta^4 - 42\theta^3 + 28\theta^2 + 96\theta)t_1^3 \\
&+(-24\theta^6 + 90\theta^5 - 60\theta^4 + 156\theta^2)t_1^2 + (-32\theta^7 + 96\theta^6 - 128\theta^5 - 28\theta^4 + 192\theta^3 - 112\theta^2 \\
&-96\theta)t_1 - 64\theta^6 + 64\theta^5 - 16\theta^4 - 136\theta^3 + 32\theta^2 - 32\theta - 64 = 0.
\end{aligned} \tag{20}$$

We determine the signs of the coefficients of the polynomial in equation (20):

N	Coeff.	Sign of coeff.
1	$\theta^6 - 4\theta^5 + 4\theta^4 + 16\theta^3 - 8\theta^2 + 2\theta + 16$	$+$
2	$2\theta^7 - 9\theta^6 + 8\theta^5 + 52\theta^4 - 42\theta^3 + 28\theta^2 + 96\theta$	$+$
3	$-24\theta^6 + 90\theta^5 - 60\theta^4 + 156\theta^2$	$+$, if $\theta < \theta_c$, $-$, if $\theta > \theta_c$
4	$-32\theta^7 + 96\theta^6 - 128\theta^5 - 28\theta^4 + 192\theta^3 - 112\theta^2 - 96\theta$	$-$
5	$-64\theta^6 + 64\theta^5 - 16\theta^4 - 136\theta^3 + 32\theta^2 - 32\theta - 64$	$-$

where $\theta_c \approx 3.165136214$.

The number of sign exchanges of the coefficients of that polynomial is equal to one in arbitrary theta. Therefore, according to Descartes' theorem, equation (20) has at most one positive solution. Suppose equation (20) has one positive solution. Let us denote this positive solution of the equation (20) by A_1. Let the corresponding solutions be as follows: $t_2 = B_1, t_3 = C_1, t_4 = D_1$.

We assume that $D_1 > 0$. Note that in the system of equations (12), variables t_1 and t_4, t_2 and t_3 are mutually symmetric. Therefore, since (A_1, B_1, C_1, D_1) is a solution of the system of equations (12), it follows that (D_1, C_1, B_1, A_1) is also a solution. But according to the Descartes' theorem $D_1 < 0$. It contradicts our assumption.

Therefore, the system of equations (12) does not have mutually unequal positive solutions. From this we get the proof of Theorem 3.1. □

Proof. **(Proof of Theorem 3.2.)** We consider the case $k=2, m=2, \mid A \mid = 2$. For H_A-weakly periodic collection (7) from (4) we have the following system of equations:

$$
\begin{cases}
h_{00}=2F(h_{01},\theta,r),\\
h_{01}=F(h_{10},\theta,r)+F(h_{11},\theta,r),\\
h_{10}=F(h_{00},\theta,r)+F(h_{01},\theta,r),\\
h_{11}=2F(h_{10},\theta,r),
\end{cases}
\tag{21}
$$

where $h_{ij}=(h_{ij}^{(1)},h_{ij}^{(2)})$ and

$$
F(h_{ij},\theta,r)=\left(\ln\frac{r\exp h_{ij}^{(1)}+\theta\exp h_{ij}^{(2)}+\theta^2}{\theta^2\exp h_{ij}^{(1)}+\theta\exp h_{ij}^{(2)}+r},\ln\frac{\theta\exp h_{ij}^{(1)}+r\exp h_{ij}^{(2)}+\theta}{\theta^2\exp h_{ij}^{(1)}+\theta\exp h_{ij}^{(2)}+r}\right).
$$

From (21), we have the following system of equations:

$$
\begin{cases}
h_{00}^{(1)}=2\ln\dfrac{r\exp h_{01}^{(1)}+\theta\exp h_{01}^{(2)}+\theta^2}{\theta^2\exp h_{01}^{(1)}+\theta\exp h_{01}^{(2)}+r},\\
h_{00}^{(2)}=2\ln\dfrac{\theta\exp h_{01}^{(1)}+r\exp h_{01}^{(2)}+\theta}{\theta^2\exp h_{01}^{(1)}+\theta\exp h_{01}^{(2)}+r},\\
h_{01}^{(1)}=\ln\dfrac{r\exp h_{10}^{(1)}+\theta\exp h_{10}^{(2)}+\theta^2}{\theta^2\exp h_{10}^{(1)}+\theta\exp h_{10}^{(2)}+r}+\ln\dfrac{r\exp h_{11}^{(1)}+\theta\exp h_{11}^{(2)}+\theta^2}{\theta^2\exp h_{11}^{(1)}+\theta\exp h_{11}^{(2)}+r},\\
h_{01}^{(2)}=\ln\dfrac{\theta\exp h_{10}^{(1)}+r\exp h_{10}^{(2)}+\theta}{\theta^2\exp h_{10}^{(1)}+\theta\exp h_{10}^{(2)}+r}+\ln\dfrac{\theta\exp h_{11}^{(1)}+r\exp h_{11}^{(2)}+\theta}{\theta^2\exp h_{11}^{(1)}+\theta\exp h_{11}^{(2)}+r},\\
h_{10}^{(1)}=\ln\dfrac{r\exp h_{00}^{(1)}+\theta\exp h_{00}^{(2)}+\theta^2}{\theta^2\exp h_{00}^{(1)}+\theta\exp h_{00}^{(2)}+r}+\ln\dfrac{r\exp h_{01}^{(1)}+\theta\exp h_{01}^{(2)}+\theta^2}{\theta^2\exp h_{01}^{(1)}+\theta\exp h_{01}^{(2)}+r},\\
h_{10}^{(2)}=\ln\dfrac{\theta\exp h_{00}^{(1)}+r\exp h_{00}^{(2)}+\theta}{\theta^2\exp h_{00}^{(1)}+\theta\exp h_{00}^{(2)}+r}+\ln\dfrac{\theta\exp h_{01}^{(1)}+r\exp h_{01}^{(2)}+\theta}{\theta^2\exp h_{01}^{(1)}+\theta\exp h_{01}^{(2)}+r},\\
h_{11}^{(1)}=2\ln\dfrac{r\exp h_{10}^{(1)}+\theta\exp h_{10}^{(2)}+\theta^2}{\theta^2\exp h_{10}^{(1)}+\theta\exp h_{10}^{(2)}+r},\\
h_{11}^{(2)}=2\ln\dfrac{\theta\exp h_{10}^{(1)}+r\exp h_{10}^{(2)}+\theta}{\theta^2\exp h_{10}^{(1)}+\theta\exp h_{10}^{(2)}+r}.
\end{cases}
\tag{22}
$$

At arbitrary values of θ and r, $h_{00}^{(1)}=h_{01}^{(1)}=h_{10}^{(1)}=h_{11}^{(1)}=0$ satisfies the system of equations (22). Therefore, we have the following system of equations:

$$
\begin{cases}
h_{00}^{(2)}=2\ln\dfrac{r\exp h_{01}^{(2)}+2\theta}{\theta^2+\theta\exp h_{01}^{(2)}+r},\\
h_{01}^{(2)}=\ln\dfrac{r\exp h_{10}^{(2)}+2\theta}{\theta^2+\theta\exp h_{10}^{(2)}+r}+\ln\dfrac{r\exp h_{11}^{(2)}+2\theta}{\theta^2+\theta\exp h_{11}^{(2)}+r},\\
h_{10}^{(2)}=\ln\dfrac{r\exp h_{00}^{(2)}+2\theta}{\theta^2+\theta\exp h_{00}^{(2)}+r}+\ln\dfrac{r\exp h_{01}^{(2)}+2\theta}{\theta^2+\theta\exp h_{01}^{(2)}+r},\\
h_{11}^{(2)}=2\ln\dfrac{r\exp h_{10}^{(2)}+2\theta}{\theta^2+\theta\exp h_{10}^{(2)}+r}.
\end{cases}
\tag{23}
$$

In the equation (23), we introduce the following notations:

$$
t_1=\exp h_{00}^{(2)}, t_2=\exp h_{01}^{(2)}, t_3=\exp h_{10}^{(2)}, t_4=\exp h_{11}^{(2)}.
$$

Then from the system of equations (23) we get the following system of equations:

$$\left\{\begin{array}{l} t_1=\left(\frac{rt_2+2\theta}{\theta^2+\theta t_2+r}\right)^2, \\ t_2=\frac{rt_3+2\theta}{\theta^2+\theta t_3+r}\cdot\frac{rt_4+2\theta}{\theta^2+\theta t_4+r}, \\ t_3=\frac{rt_1+2\theta}{\theta^2+\theta t_1+r}\cdot\frac{rt_2+2\theta}{\theta^2+\theta t_2+r}, \\ t_4=\left(\frac{rt_3+2\theta}{\theta^2+\theta t_3+r}\right)^2. \end{array}\right. \tag{24}$$

Let $r=\theta^2$. After several substitutions and simplifications in the (24), we obtain the following equations with respect to the variable t_3:

$$\left(t_3+\frac{2(2\theta+1)}{\theta+2}\right)\left(t_3^3+\theta(4-\theta)t_3^2+4\theta(\theta-1)t_3-4\right)=0 \tag{25}$$

or

$$\begin{gathered}(\theta^{12}+2\theta^{11}+12\theta^{10}+34\theta^9+80\theta^8+128\theta^7+144\theta^6+136\theta^5+112\theta^4+64\theta^3+16\theta^2)t_3^6\\ +(36\theta^{11}+112\theta^{10}+568\theta^9+1216\theta^8+1784\theta^7+1952\theta^6+1480\theta^5+928\theta^4+480\theta^3\\ +160\theta^2+32\theta)t_3^5+(16\theta^{12}+48\theta^{11}+1196\theta^{10}+3240\theta^9+8240\theta^8+10872\theta^7\\ +7504\theta^6+5920\theta^5+4368\theta^4+1504\theta^3+576\theta^2+256\theta)t_3^4+(960\theta^{11}+3072\theta^{10}\\ +17760\theta^9+25472\theta^8+18880\theta^7+22272\theta^6+16320\theta^5+5120\theta^4+4480\theta^3+2176\theta^2\\ +128)t_3^3+(256\theta^{12}+11520\theta^{11}+210560\theta^{10}+268800\theta^9+226400\theta^8+415680\theta^7+288640\theta^6\\ +107200\theta^5+142080\theta^4+60160\theta^3+5760\theta^2+10240\theta)t_3^2+(138240\theta^{11}+133120\theta^{10}+133120\theta^9\\ +348160\theta^8+210560\theta^7+120320\theta^6+198400\theta^5+71680\theta^4+19200\theta^3+25600\theta^2+1280\theta)t_3\\ +40960\theta^{12}+20480\theta^{11}+30720\theta^{10}+117760\theta^9+51200\theta^8+43520\theta^7+90240\theta^6+29440\theta^5\\ +17920\theta^4+21760\theta^3+2560\theta^2=0.\end{gathered} \tag{26}$$

Since $\theta>0$ and $t_3>0$, from equation (25) we get the following equation:

$$t_3^3+\theta(4-\theta)t_3^2+4\theta(\theta-1)t_3-4=0. \tag{27}$$

The solutions of equation (27) characterize the TISGMs for the Potts-SOS model on a second-order Cayley tree. This equation was studied in Lemma 3 of [10].

Now we are interested in the solutions of equation (26). It is easy to see that all the coefficients of the polynomial in (26) are positive. The number of sign exchanges of the coefficients of the polynomial in (26) is zero. Therefore, according to the Descartes' theorem, this expression does not have a positive solution.

Combining above results, we have the proof of Theorem 3.2. $\square$

Remark 4.1. Let $k=2, \mid A\mid=3$. Then all H_A-weakly periodic Gibbs measures for the Potts-SOS model are H_A-periodic, i.e. $G_2^{(2)}$-periodic. These Gibbs measures are studied in [11].

In summary, the all weakly periodic Gibbs measures with respect to the normal subgroup of two index under the condition $r=\theta^2$ on the Cayley tree of order two are studied completely.

Remark 4.2. From Theorems 3.1 and 3.2, we have seen that under the condition $r=\theta^2$, all weakly periodic Gibbs measures with respect to the normal subgroup of two index coincide with translation-invariant Gibbs measures. The issue of extremality or non-extremality of these measures was studied in [10].

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