Theory of Stochastic Processes
Vol. 29 (45), no. 1, 2025, pp. 74–79

H. V. NAVROTSKYI

PROPERTIES OF GROWING CROSS SECTIONS OF ISOTROPIC GAUSSIAN FIELD

The article is devoted to the asymptotic properties of Gaussian random field on the plane. We find the conditions for convergence of the number of upcrossings in the weak sense.

Our interest is in limiting behavior of topological characteristics of smooth Gaussian random field. In this paper we discuss the geometric characteristics of smooth periodic Gaussian processes which arise as a restriction of smooth Gaussian random field on $\mathbb{R}^2$ on circles of radius R with a center in origin. The interest to such objects is due to the paper [4] where the following model of the random knot was proposed. Let $\xi : \mathbb{R}^2 \to \mathbb{R}^3$ be a centered Gaussian random field with independent coordinates which have covariance

$$E\xi_i(u)\xi_i(v) = e^{-|u-v|^2}$$

Let γ be a smooth closed curve in $\mathbb{R}^2$ without self-intersections. Then the random curve

$$\Gamma = \xi(\gamma)$$

with probability one has no self-intersections , i.e. is a random knot. It was proved in [4], that Γ can have arbitrary topological type with positive probability. The investigation of topology of Γ naturally leads to the investigation of the number of bridges over the certain plane. Consequently it is reduced to the behaviour of the number of upcrossings of a level by the fixed coordinate of ξ. The goal is to find the asymptotic growth of the complexity of the random knot.

Definition 1. Define $h : [0, 1] \to \mathbb{R}, h \in C^1([0, 1]), c \in \mathbb{R}$. Then $t \in [0, 1]$ is called a *point of crossing* of the level c by function h, if

$$h(t) = c,\ h'(t) \neq 0.$$

Definition 2. $t \in [0, 1]$ is called an *upcrossing point* of the level c by function h, if

$$h(t) = c,\ h'(t) > 0.$$

For each $R > 0$ denote as $\mathcal{C}_R$ the circle with center at 0 and radius R in the parameter plane. For $\mathcal{C}_R$, we use the parametrization $f(t) = (R\cos t, R\sin t)$. For a fixed $R > 0$, consider the restriction of ξ onto $\mathcal{C}_R$. From here on ξ is one dimensional. This restriction can be described as a random process

$$\eta_R(t) = \xi(R\cos t, R\sin t)$$

It is natural to presume that with the growth of radius R, the number of upcrossings of fixed level c by η_R will increase. The covariance of process η_R is:

$$E\eta_R(t)\eta_R(s) = e^{-R^2(2-2\cos(t-s))}$$

If $t \neq s,\ |t - s| \neq 2\pi k, k \in \mathbb{Z}_+$, then when $R \to \infty$

$$E\eta_R(t)\eta_R(s) = e^{-R^2(2-2\cos(t-s))} \to 0.$$

2020 *Mathematics Subject Classification.* 60B05 .
Key words and phrases. Gaussian random field,number of upcrossings, Rice's formula.
This work was supported by a grant from the Simons Foundation (SFI-PD-Ukraine-00014586, N.H.).

Thus, for large values of R, the values of the process η_R with different t,s become almost independent. Let us modify the process to obtain a smooth covariance in the limit when $R \to \infty$:

$$\zeta_R(t) = \eta_R(\frac{t}{R}),\ t \geq 0$$

Then

$$E\zeta_R(t)\zeta_R(s) = e^{-2R^2(1-\cos\frac{t-s}{R})} \to e^{-(t-s)^2},\ \text{as } R \to \infty$$

Let us denote as ζ_∞ a centered Gaussian random process with covariance function

$$Cov(\zeta_\infty(t), \zeta_\infty(s)) = e^{-(t-s)^2}$$

Note that for any $R > 0$ the processes $\zeta_R(t), \zeta'_R(t)$ have continuous modifications (the corresponding sufficient condition can be found in paragraph **5** of chapter **III** in [6]) .

Further we will consider the continuous modifications of ζ_R, ζ'_R.

Denote by $N_{\zeta_R}([0,T],c)$ the number of upcrossings of level c before time T by the process $\zeta_R(t)$.To calculate the expectation of $N_{\zeta_R}([0,T],c)$ Rice's formula [2] will be used:

$$\mathbb{E}N_{\zeta_R}([0,T],c) = \int_0^T \int_0^\infty y q_t(c,y)\mathrm{d}y\mathrm{d}t$$

where $q_t(c,y)$ is a joint density of $\zeta_R(t), \zeta'_R(t)$. Then, due to the process $\zeta_R(t)$ being stationary and Gaussian, $\zeta_R(t)$ and $\zeta'_R(t)$ are independent. So, the joint density of the random values $\zeta_R(t)$ and $\zeta'_R(t)$ can be calculated as a product of their respective densities. Since the process ζ_R is stationary, the joint density $q_t(c,y)$ is not dependent on time:

$$q_t(c,y) = \frac{1}{\sqrt{2}2\pi}e^{-y^2/4-c^2/2}$$

Thus

$$\mathbb{E}N_{\zeta_R}([0,T],c) = T\int_0^\infty yq(c,y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}}T \cdot e^{-c^2/2}$$

For ζ_∞ the corresponding probability density of the pair $\zeta_\infty(t), \zeta'_\infty(t)$ is:

$$p_t(c,y) = \frac{1}{\sqrt{2}2\pi}e^{-y^2/4-c^2/2}$$

Therefore we conclude that

$$\mathbb{E}N_{\zeta_\infty}([0,T],c) = \frac{1}{\sqrt{2\pi}}T \cdot e^{-c^2/2} \tag{a}$$

Let us find the asymptotics of growth of $N_{\zeta_R}([0,T],c)$ with respect to R, T.

Lemma 1. *Let $c > 0$. Then*

$$\frac{1}{T}N_{\zeta_R}([0,T],c) \to \frac{1}{2\pi R}N_{\zeta_R}([0,2\pi R],c), \quad \text{as } T \to \infty$$

almost surely.

Proof. Without loss of generality, we can assume:

$$\zeta_R(2k\pi R) \neq c,\ k \in \mathbb{N}$$

since

$$P(\{\exists k \in \mathbb{N} : \zeta_R(a_k) = c\}) = 0$$

for any sequence $\{a_k,\ k \in \mathbb{N}\}$ of points in $\mathbb{R}$.

Due to the periodicity

$$N_{\zeta_R}([0,2\pi R],c) = N_{\zeta_R}([2\pi R, 4\pi R],c) = ... = N_{\zeta_R}([2n\pi R, 2(n+1)\pi R],c)$$

Therefore,

$$\frac{1}{2\pi nR}N_{\zeta_R}([0,[2\pi nR],c) = \frac{1}{2\pi R}N_{\zeta_R}([0,2\pi R],c)$$

Moreover,

$$N_{\zeta_R}([0,2\pi R],c)-1 \leq \sup_{0<t<2\pi R} N_{\zeta_R}([0,t],c) = N_{\zeta_R}([0,2\pi R],c)$$

Consequently,

$$\frac{1}{T}N_{\zeta_R}([0,T],c) = \frac{1}{T}N_{\zeta_R}([0,\lfloor\frac{T}{2\pi R}\rfloor 2\pi R],c) + \frac{1}{T}N_{\zeta_R}([0,T-\lfloor\frac{T}{2\pi R}\rfloor 2\pi R],c) \to$$
$$\to \frac{1}{2\pi R}N_{\zeta_R}([0,2\pi R],c) \text{ as } T\to\infty.$$

□

Since we are investigating the convergence of the number of crossings $N_{\zeta_R}([0,T],c)$, we first need to determine the conditions for the convergence of the number of level crossings for a deterministic function.

Lemma 2. *Consider a sequence of continuously differentiable functions f_k such that $f_k \to f, f'_k \to f', k\to\infty$ uniformly on the interval $[0,T]$. For any $t\in[0,T]$, if $f(t)=c$, then $f'(t)\neq 0$. If the number of upcrossings of level c by the function f, which we will denote as $U([0,T])$, is finite, then the number of upcrossings of level $c>0$ on the interval $[0,T]$ by the function f_k (denoted as $U_k([0,T])$) has a limit as $k\to\infty$, equal to $U([0,T])$.*

$$\lim_{k\to\infty} U_k([0,T]) = U([0,T])$$

Proof. Denote as $(t_1,\dots t_n)$ all points of crossings of level c by function f. For any point of crossing t_k of level c by the function f, there exists a number $\delta_k>0$ such that $\exists N\ \forall s>N\ \forall t\in B(t_k,\delta_k\ ) : |f_s^{'}(t)|>0$. Hence, the function f_s is strictly increasing or decreasing and in each such neighborhood there can be only one point of upcrossing of the level c.

If it is increasing, in each neighborhood $B(t_k,\delta_k\ )$ there exists local infimum $a_k<c$ and supremum $b_k>c$ for function f. By choosing M such that $\forall m>M\ \forall t\in[0,T]$

$$|f_m(t)-f(t)| < \min_k |a_k-c|/2, |b_k-c|/2$$

we can conclude that there exists a point of upcrossing in $B(t_k,\delta_k\ )$ by functions f_m.

On the other hand, consider the compact set

$$J=[0,T]\backslash\bigcup_k B(t_k,\delta_k).$$

Due to the continuity of the function f, there exists $\min_J |f(t)-c|>0$ on this compact set. Then starting from some $n>0$, $\min_J |f_n(t)-c|>0$. □

Due to the fact that the processes $\zeta_R(t)$ are Gaussian, from the pointwise convergence of covariance functions it follows that the finite-dimensional distributions of the process converge to the distribution of the process ζ_∞, as $R\to\infty$. Yet it is not sufficient for the convergence in distribution. The sufficient conditions are (paragraph **2.4** in [1]):

(1) $$\sup_{R\geq 1}\mathbb{E}(\zeta_R(0)^2) < +\infty$$

(2) $$\exists\alpha,\beta,C>0 : \mathbb{E}|\zeta_R(t)-\zeta_R(s)|^\alpha \leq C_T|t-s|^{1+\beta}$$

for $\alpha,\beta>0$,$\forall\, T,\forall\, 0<t,s<T; R>1$ and some C_T depending on T.

Lemma 3. *For every fixed $T>0$ the following convergence in distribution holds: $\zeta_R\to\zeta_\infty$ and $\zeta'_R\to\zeta'_\infty$ in $C([0,T])$ as $R\to\infty$.*

Proof. Let us check the condition (1):

$$\sup_{R\geq 1} \mathbb{E}(\zeta_R(0)^2) \leq 1$$

and

$$\sup_{R\geq 1} \mathbb{E}(\zeta_R'(0)^2) \leq 2.$$

Next, the following inequality for $\xi_R(t)$ holds uniformly w.r.t. $R > 0$

$$\mathbb{E}|\zeta_R(t) - \zeta_R(s)|^2 \leq 2|t-s|^2$$

To prove this we use the inequalities

$$\cos(\frac{t-s}{R}) \geq 1 - 2(\frac{t-s}{2R})^2 \tag{1}$$

and

$$\exp(-2R^2(1-\cos\frac{t-s}{R})) \geq \exp(-(t-s)^2) \geq 1-(t-s)^2 \tag{2}$$

Hence

$$\begin{aligned}\mathbb{E}|\zeta_R(t) - \zeta_R(s)|^2 &= 2 - 2\exp(-2R^2(1-\cos\frac{t-s}{R})) \leq \\ &\leq 2 - 2\exp(-|t-s|^2) \\ &\leq 2|t-s|^2.\end{aligned}$$

We find the covariance function of the process ζ_R'

$$\mathbb{E}\zeta_R'(t)\zeta_R'(s) = 2e^{-2R^2(1-\cos\frac{t-s}{R})}\,(\cos\frac{t-s}{R} - 2R^2(\sin\frac{t-s}{R})^2)$$

Then we use the formulas $(1), (2)$ to obtain

$$\begin{aligned}\mathbb{E}|\zeta_R'(t) - \zeta_R'(s)|^2 &\leq 4(1-(1-(t-s)^2)(1-\frac{(t-s)^2}{2R^2} - 2(t-s)^2)) \\ &= 4(\frac{(t-s)^2}{2R^2} - \frac{(t-s)^4}{2R^2} - 2(t-s)^4 + 3(t-s)^2) \\ &\leq 4(4(t-s)^2 - 2(t-s)^4)\end{aligned}$$

for small enough $|t-s|$ when $R > 1$. □

If there is a convergence in distribution of processes ζ_R to a process ζ_∞, it is natural to presume that there is a convergence in distribution of the number of upcrossings $N_{\zeta_R}([0,T], c)$. To prove that, we need to check the conditions of lemma **2** for $\zeta_\infty(t)$.

The a.s. continuity of trajectories of the processes $\zeta_\infty(t), \zeta_\infty'(t)$ follows from lemma **3**.

Let us calculate the covariance

$$\begin{aligned}&cov(\zeta_\infty(t), \zeta_\infty(s)) = e^{-(t-s)^2} \\ &cov(\zeta_\infty(t), \zeta_\infty'(s)) = 2(t-s)e^{-(t-s)^2} \\ &cov(\zeta_\infty'(t), \zeta_\infty'(s)) = 2e^{-(t-s)^2} - 4(t-s)^2e^{-(t-s)^2}\end{aligned}$$

Lemma 4.

$$P\{\exists\, t > 0\ : \zeta_\infty(t) = c,\ \zeta_\infty'(t) = 0\} = 0$$

Proof. Since the probability density of $\zeta_\infty(t)$ is bounded and the derivative $\zeta_\infty'(t)$ is a.s. continuous, then $P\{\exists\, t \in [0,1]\ : \zeta_\infty(t) = c,\ \zeta_\infty'(t) = 0\} = 0$ (**4.5** in [3]).

Since the process $\xi(t)$ is stationary and

$$[0,\infty) = \bigcup_{n=0}^{\infty} [n, n+1]$$

we have the statement of the lemma.

□

Theorem 1. *There exists the limit in distribution*

$$\lim_{R\to\infty} N_{\zeta_R}([0,T],c) = N_{\zeta_\infty}([0,T],c).$$

Proof. Due to lemma **3** and the Skorokhod's theorem [5], there exists a family of random values κ_R such that

$$\kappa_R = \zeta_R$$

in distribution and there exists a limit with probability 1

$$\lim_{R\to\infty} \kappa_R = \kappa_\infty.$$

Then, by lemma **2** there exists the a.s. limit of the number of crossings of level c $N_{\kappa_R}([0,T],c)$ and since

$$N_{\kappa_R}([0,T],c) = N_{\zeta_R}([0,T],c)$$

in distribution, there exists a limit in distribution of $N_{\zeta_R}([0,T],c)$ as $R\to\infty$, which is equal to $N_{\zeta_\infty}([0,T],c)$.

□

Further, we study the asymptotic properties of $N_{\zeta_\infty}([0,T),c)$ as $T\to\infty$..

Lemma 5. *The process $\zeta_\infty(t)$ is ergodic.*

Proof. Using results of chapter **7.11** in [3] we verify the conditions for ergodicity of the process ζ_∞:

(1)The trajectories of the process ζ_∞ are a.s. continuous.

(2) The spectral function of the process $\zeta_\infty(t)$ is equal to

$$G(\lambda) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\lambda} e^{-s^2/4}ds$$

and is continuous with respect to λ.

□

Then the following statement follows from lemma **5** (the proof can be found in chapter **11.5** of [3]) and the formula (a) :

Proposition 1. *There exists the limit with probability* 1

$$\lim_{T\to\infty} \frac{N_{\zeta_\infty}([0,T),c)}{T} = \frac{e^{-c^2/2}}{\sqrt{2\pi}}$$

Therefore, we conclude that for the process ζ_∞ and a fixed level $c>0$ the growth of the number of upcrossings $N_{\zeta_\infty}([0,T),c)$ as $T\to\infty$ is asymptotically linear with probability 1.

References

1. Ioannis Karatzas and Steven Shreve, *Brownian motion and stochastic calculus; 2nd ed.*, Springer-Verlag New York, Inc. 1998. https://doi.org/10.1007/978-1-4612-0949-2
2. Robert J. Adler and Jonathan E. Taylor, *Random Fields and Geometry*, Springer New York, NY 2007. https://doi.org/10.1007/978-0-387-48116-6
3. H. Cramer and M. Leadbetter, *Stationary and Related Stochastic Processes*, Izdatelstvo 'MIR', Moskva 1969.
4. A.A. Dorogovtsev, *On a Stationary Random Knot*, Journal of Stochastic Analysis **4** (2023), no. 3, Article 4. https://doi.org/10.31390/josa.4.3.04
5. P. Billingsley, *Convergence of probability measures; 2nd ed.*, Wiley, Hoboken, NJ, 1999. https://doi.org/10.1002/9780470316962

6. I. I. Gihman and A.V. Skorokhod, *The theory of stochastic processes*, Vol. 1, Springer-Verlag, 2004. https://doi.org/10.1007/978-3-642-61943-4

Institute of Mathematics of NAS of Ukraine, Tereschenkivska st. 3, 01024 Kyiv, Ukraine
E-mail address: navrotskyi1992@gmail.com
E-mail address: navrotskyih@imath.kiev.ua