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GEOMETRIC INVARIANTS OF RANDOM LINK

In this article we construct the random link with two components as the image of two non-intersecting circles in plane under smooth Gaussian random field. We study the properties of distributions of linking number and average crossing number of obtained link. We propose the results about existence of moments of linking number, average crossing number, Mobius energy of a link.

1. Introduction

Knots and links naturally arise in studying of fluid mechanics and vortex dynamics. The reader can find a detailed survey in [3]. Under the action of small perturbations the image of a link can be viewed as a random link. In this article we propose such model of random link and study it‘s properties.

In the work [1] of A.A. Dorogovtstev the construction of a stationary random knot was proposed. The main advantage of a proposed construction is that the obtained random knot is smooth. This gives an opportunity to study geometrical properties of obtained knot using analysis tools.

In this article we consider the random link consisting of two components and study the existence of moments of linking number, average crossing number and Mobius energy of obtained link.

Let $\vec{\xi} = (\xi_1, \xi_2, \xi_3) : \mathbb{R}^2 \to \mathbb{R}^3$ be a centered Gaussian random field with independent identically distributed coordinates with the covariance

$$\mathbb{E}\xi_1(\vec{v})\xi_1(\vec{u}) = e^{-\frac{1}{2}||\vec{u}-\vec{v}||^2} = G(\vec{u}-\vec{v}). \tag{1.1}$$

Since the covariance is infinitely differentiable, due to Gaussianity, $\vec{\xi}$ has an infinitely differentiable on $\mathbb{R}^2$ modification.

Suppose that $\theta_1, \theta_2 = (x_i + \cos t, y_i + \sin t) : [0, 2\pi] \to \mathbb{R}^2,\ i = 1, 2$ are two non intersecting unit circles on the plane. Then consider $\gamma_i = \vec{\xi}(\theta_i),\ i = 1, 2$. We will prove that $\gamma = \gamma_1 \cup \gamma_2$ is a smooth random link. We will start with the following lemma.

Define on $\mathbb{R}^3$ functions approximating δ-function at $\vec{0}$.

$$h_\varepsilon(\vec{x}) = \frac{1}{\frac{4}{3}\pi\varepsilon^3}\mathbb{1}_{B(\vec{0},\varepsilon)}(\vec{x}),\ \varepsilon > 0$$

Lemma 1.1.

$$\lim_{\varepsilon\to 0+}\int_0^{2\pi}\int_0^{2\pi}\mathbb{E}h_\varepsilon\big(\gamma_1(t_1) - \gamma_2(t_2)\big)dt_1dt_2$$

is finite and equals to

$$\int_0^{2\pi}\int_0^{2\pi}\frac{dt_1dt_2}{(2\pi(2 - 2G(\vec{\theta_1}(t_1) - \vec{\theta_2}(t_2))))^{\frac{3}{2}}}.$$

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Proof. $\forall t_1, t_2 \in [0, 2\pi] : \gamma_1(t_1) - \gamma_2(t_2)$ has density p_{t_1,t_2}

$$p_{t_1,t_2}(\vec{u}) = \frac{1}{(\sqrt{2\pi}\sigma(t_1,t_2))^3} e^{-\frac{1}{2}\sigma(t_1,t_2)^2||\vec{u}||^2},$$

where

$$\sigma(t_1,t_2)^2 = 2 - 2G\big(\vec{\theta_1}(t_1) - \vec{\theta_2}(t_2)\big).$$

Then

$$\mathbb{E}h_\varepsilon\big(\gamma_1(t_1) - \gamma_2(t_2)\big) = \int_{\mathbb{R}^3} h_\varepsilon(\vec{u}) p_{t_1,t_2}(\vec{u}) d\vec{u}.$$

Note that, $\forall t_1, t_2 \in [0, 2\pi]$

$$\sup_{\vec{u}\in\mathbb{R}^3} p_{t_1,t_2}(\vec{u}) = p_{t_1,t_2}(\vec{0}),$$

thus $\forall t_1, t_2 \in [0, 2\pi]$

$$\int_{\mathbb{R}^3} h_\varepsilon(\vec{u}) p_{t_1,t_2}(\vec{u}) d\vec{u} \le p_{t_1,t_2}(\vec{0}).$$

Due to construction of h_ε

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}^3} h_\varepsilon(\vec{u}) p_{t_1,t_2}(\vec{u}) d\vec{u} = p_{t_1,t_2}(\vec{0}).$$

Then by Lebesgue dominated convergence theorem

$$\lim_{\varepsilon\to 0+} \int_0^{2\pi}\int_0^{2\pi} \mathbb{E}h_\varepsilon\big(\gamma_1(t_1) - \gamma_2(t_2)\big) dt_1 dt_2 = \int_0^{2\pi}\int_0^{2\pi} p_{t_1,t_2}(\vec{0}) dt_1 dt_2,$$

which is the integral from the statement of the lemma. This integral converge because θ_1 and θ_2 are non intersecting. □

Theorem 1.1. *$\gamma = \gamma_1 \cup \gamma_2$ is a smooth random link with two components.*

Proof. Due to the Theorem 1.1 from [1] γ_1 and γ_2 are random knots. Then it is enough to prove that they are non intersecting with probability 1.
Consider

$$A = \{\omega \mid \exists\, t_1, t_2 \in [0, 2\pi] : \gamma_1(t_1) = \gamma_2(t_2)\}. \tag{1.2}$$

$\vec{\xi}$ has an infinitely differentiable on $\mathbb{R}^2$ modification. Then with probability 1 γ_1 and γ_2 are smooth random curves. Hence

$$\Omega \setminus A = \{\omega \mid \exists\, n \ge 1 : \ \forall\, r_1, r_2 \in \mathbb{Q} \cap [0, 2\pi] : ||\gamma_1(r_1) - \gamma_2(r_2)|| > \frac{1}{n}\}.$$

Consequently, A is a random event. Suppose that $\mathrm{P}(A) > 0$. Take $\omega \in A$. Then $\exists\, t_1^0, t_2^0 \in [0, 2\pi]$:

$$\gamma_1(t_1^0)(\omega) = \gamma_2(t_2^0)(\omega).$$

Consider integral

$$\int_0^{2\pi}\int_0^{2\pi} h_\varepsilon\big(\gamma_1(t_1)(\omega) - \gamma_2(t_2)(\omega)\big) dt_1 dt_2.$$

Due to differentiability of γ, as $t_1 \to t_1^0, t_2 \to t_2^0$

$$\gamma_1(t_1) - \gamma_2(t_2) = \gamma_1'(t_1^0)(t_1 - t_1^0) + \gamma_2'(t_2^0)(t_2 - t_2^0) + o(t_1 - t_1^0) + o(t_2 - t_2^0).$$

Take $C_0(\omega) = \max\{|\gamma_1'(t_1^0)|, |\gamma_2'(t_2^0)|\}$. With probability 1: $C_0(\omega) > 0$. Then there exists $\varepsilon_0 > 0, \forall \varepsilon < \varepsilon_0$:

$$\forall t_i \in [0, 2\pi], |t_i - t_i^0| < \frac{\varepsilon}{3C_0(\omega)}, i = 1, 2 : ||\gamma_1(t_1) - \gamma_2(t_2)|| < \varepsilon.$$

Then

$$\int_0^{2\pi}\int_0^{2\pi} h_\varepsilon\big(\gamma_1(t_1)(\omega) - \gamma_2(t_2)(\omega)\big) dt_1 dt_2 \ge \frac{1}{\frac{4}{3}\pi\varepsilon^3}\frac{\varepsilon^2}{9C_0(\omega)^2} = \frac{C(\omega)}{\varepsilon}.$$

With Fatou lemma this gives us

$$\liminf_{\varepsilon\to 0+}\int_0^{2\pi}\int_0^{2\pi}\mathbb{E}h_\varepsilon\big(\gamma_1(t_1)-\gamma_2(t_2)\big)dt_1dt_2 \geq \lim_{\varepsilon\to 0+}\frac{\mathbb{E}C(\omega)}{\varepsilon}.$$

If $\mathrm{P}(A) > 0$, then $\mathbb{E}C(\omega) > 0$(or equal to $+\infty$) and the following limit doesn't exist, what contradicts Lemma 1.1. Hence $P(A) = 0$. □

2. Geometric invariants of a deterministic link

For a link γ of class C^1 consisting of two components γ_1, γ_2 linking number $link(\gamma_1, \gamma_2)$ is defined by Gauss linking integral ([2])

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\det(\dot{\gamma_1}(t),\dot{\gamma_2}(s),\gamma_1(t)-\gamma_2(s))}{||\gamma_1(t)-\gamma_2(s)||^3}\,dt\,ds. \tag{2.1}$$

Another representation of a linking number is the following. Suppose that location and orientation of a link $\gamma = \gamma_1 \cup \gamma_2$ in space are given by parametric equations $\gamma_1(t), \gamma_2(s)$. Consider the projection of a link on a plane with at most finitely many transversal double points such that orientation of a projection is agreed with orientation of a link. For each transversal double point consisting of two local arcs, one can give heights between these two arcs with respect to the projecting vector. We call the higher arc *overcrossing* and the lower arc *undercrossing*. The projection of a link with over/under information is called a diagram. Each over-crossing or under-crossing arises either from γ_1 alone, from γ_2 alone or from an intersection involving both γ_1 and γ_2. Depending on the orientation of the over-crossing and under-crossing of a crossing c, it can be of two types: *positive and negative crossings with signs +1 and -1, respectively.* Then the half of the sum of signs of crossings consisting of over and under-crossings from different components is exactly the linking number. The proof of equivalence of these two definitions and other definitions of linking number can be found in [2].

In the work [5] of Freedman, Michael H. and Zheng-Xu He average crossing number of a link was proposed. Suppose that link γ consists of two components γ_1, γ_2, and $\mathbb{S}^2$ is a unit sphere in $\mathbb{R}^3$. For $\vec{p} \in \mathbb{S}^2$ let $n(\gamma, \vec{p})$ be the number of crossings between γ_1 and γ_2 in a diagram of a link, when link γ is orthogonally projected on some plane, which is orthogonal to $\vec{p}$. Then average crossing number $c(\gamma_1, \gamma_2)$ is equal to

$$\frac{1}{4\pi}\iint\limits_{\vec{p}\in\mathbb{S}^2} n(\gamma,\vec{p})\,dS.$$

This integral is well defined due to the fact that all directions, for which the number of crossing in a diagram is infinite, are exactly the critical values of Gauss map([2]) and by Sard's theorem([7]) the set of critical values have zero measure.

In [5] it was proved that average crossing number of a smooth link has representation similar to Gauss linking integral.

Proposition 2.1. *([5])*

$$c(\gamma_1,\gamma_2) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{|\det(\dot{\gamma_1}(t),\dot{\gamma_2}(s),\gamma_1(t)-\gamma_2(s))|}{||\gamma_1(t)-\gamma_2(s)||^3}\,dt\,ds \tag{2.2}$$

Using this formula we can get useful property of average crossing number.

Proposition 2.2. *Suppose that link γ is of class C^2 and consists of two components γ_1, γ_2, and $c(\gamma_1, \gamma_2) = 0$. Then γ_1 and γ_2 are non-intersecting trivial knots on some plane in $\mathbb{R}^3$.*

Proof. By Proposition 2.1,

$$\forall\, t,s \in [0,2\pi]:\ \det(\dot{\gamma_1}(t), \dot{\gamma_2}(s), \gamma_1(t)-\gamma_2(s)) = 0. \tag{2.3}$$

Differentiating this equality by parameter t we get

$$\forall\, t,s \in [0,2\pi]:\ \det(\ddot{\gamma_1}(t), \dot{\gamma_2}(s), \gamma_1(t)-\gamma_2(s)) = 0. \tag{2.4}$$

Similarly

$$\forall\, t,s \in [0,2\pi]:\ \det(\dot{\gamma_1}(t), \ddot{\gamma_2}(s), \gamma_1(t)-\gamma_2(s)) = 0. \tag{2.5}$$

Suppose that for any fixed t, s $P_{t,s}$ is a linear subspace generated by vectors $\dot{\gamma_2}(s), \gamma_1(t)-\gamma_2(s)$. From equations (2.3) - (2.5) follows that vectors $\ddot{\gamma_1}(t), \ddot{\gamma_2}(s), \dot{\gamma_1}(t), \dot{\gamma_2}(s), \gamma_1(t)-\gamma_2(s)$ lies in $P_{t,s}$. Then the determinant of any 3 vectors from these 5 equals to 0. We will use it later.

Consider parametric surface $\mathbb{X} = \{\vec{f}(t,s)|\ t,s \in [0,2\pi]\} = \{\gamma_1(t)-\gamma_2(s)|\ t,s \in [0,2\pi]\}$. At point $\vec{f}(t,s)$ we can define a normal vector $\vec{n}$ to a surface as

$$\vec{n}(t,s) = \frac{d\vec{f}}{dt} \times \frac{d\vec{f}}{ds} = \dot{\gamma_1}(t) \times -\dot{\gamma_2}(s).$$

Suppose that the second fundamental form of surface $\mathbb{X}$ at point $\vec{f}(t,s)$ equals to

$$L\, dt^2 + 2M\, dt\, ds + N\, ds^2.$$

We can find coefficients L and N using the next equalities.

$$L(t,s) = \frac{d^2\vec{f}}{dt^2} \cdot \vec{n} = \ddot{\gamma_1}(t) \cdot (\dot{\gamma_1}(t) \times -\dot{\gamma_2}(s)) = -\det(\ddot{\gamma_1}(t), \dot{\gamma_1}(t), \dot{\gamma_2}(s))$$
$$N(t,s) = \frac{d^2\vec{f}}{ds^2} \cdot \vec{n} = -\ddot{\gamma_2}(s) \cdot (\dot{\gamma_1}(t) \times -\dot{\gamma_2}(s)) = \det(\ddot{\gamma_2}(s), \dot{\gamma_1}(t), \dot{\gamma_2}(s))$$

From equations (2.3), (2.4), (2.5) follows that these two determinants are 0. Finally

$$M(t,s) = \frac{d^2\vec{f}}{dt\, ds} \cdot \vec{n} = \vec{0} \cdot \vec{n} = 0.$$

Since the second fundamental form of $\mathbb{X}$ is zero at any point $\mathbb{X}$ is a plane. Then γ lies in a plane.
γ_1 and γ_2 are plane knots and by Schoenflies theorem([8]) γ_1 and γ_2 are isotopic to a circle, so γ_1 and γ_2 are trivial knots. Hence γ is a trivial link that lies in a plane. Proof is finished. □

Remark 2.1. The same idea works in a case when $\gamma_1 = \gamma_2$. Hence, if $c(\gamma_1, \gamma_1) = 0$ and γ_1 is of class C^2, then γ_1 is trivial knot in some plane.

In the work [4] the Mobius energy of the smooth knot was defined. For two fixed points $\gamma_1(t), \gamma_1(s)$ we will denote by $D(\gamma_1(t), \gamma_1(s))$ the distance between them on the curve; i.e., the minimum of the lengths of subarcs of γ_1 with one endpoint at $\gamma_1(t)$ and the other at $\gamma_1(s)$. Then Mobius energy of a knot γ_1 equals to

$$\mathcal{E}(\gamma_1,\gamma_1) = \int_0^{2\pi}\int_0^{2\pi}\Big(\frac{1}{||\gamma_1(t)-\gamma_1(s)||^2} - \frac{1}{D(\gamma_1(t),\gamma_1(s))}\Big)||\dot{\gamma_1}(t)|| \cdot ||\dot{\gamma_1}(s)||\, dt\, ds. \tag{2.6}$$

Then, the total energy of a link $\gamma = \gamma_1 \cup \gamma_2$ equals to

$$\mathcal{E}(\gamma) = \mathcal{E}(\gamma_1,\gamma_1) + \mathcal{E}(\gamma_2,\gamma_2) + \frac{1}{2}\mathcal{E}(\gamma_1,\gamma_2), \tag{2.7}$$

where $\mathcal{E}(\gamma_1,\gamma_2)$ is defined as

$$\mathcal{E}(\gamma_1,\gamma_2) = \int_0^{2\pi}\int_0^{2\pi} \frac{||\dot{\gamma_1}(t)|| \cdot ||\dot{\gamma_2}(s)||}{||\gamma_1(t)-\gamma_2(s)||^2}\, dt\, ds.$$

In [4] it was proved that $\mathcal{E}(\gamma)$ is invariant under affine transformations and under Mobius transformations.

3. Main results

In this section we will consider random link γ consisting of two components γ_1 and γ_2 that was defined in section 1. Due to existence of infinitely differentiable modification of $\vec{\xi}$ all proposed invariants are well defined.

We will start with properties of distributions of $link(\gamma_1, \gamma_2)$ and $c(\gamma_1, \gamma_2)$.

Theorem 3.1. *$link(\gamma_1, \gamma_2)$ has symmetric distribution. In other words*

$$\forall\, k \in \mathbb{Z} : \mathrm{P}\{link(\gamma_1, \gamma_2) = k\} = \mathrm{P}\{link(\gamma_1, \gamma_2) = -k\}.$$

Proof. Let $-\gamma_1$ and $-\gamma_2$ be the results of reflection of γ_1 and γ_2 with respect to $\vec{0}$, respectively. $\vec{\xi}$ is a rotational invariant and isotropic random field. Then pairs (γ_1, γ_2) and $(-\gamma_1, -\gamma_2)$ have the same distribution. At the same time by formula (2.2)

$$link(\gamma_1, \gamma_2) = -link(-\gamma_1, -\gamma_2).$$

Then $link(\gamma_1, \gamma_2) \overset{d}{=} -link(\gamma_1, \gamma_2)$ and conclusion follows. □

Theorem 3.2. $\mathrm{P}\{c(\gamma_1, \gamma_2) = 0\} = 0$.

Proof. Following the arguments in the proof of Proposition 2.2, we obtain

$$\mathrm{P}\{c(\gamma_1, \gamma_2) = 0\} = \mathrm{P}\{\ \exists \text{ plane } L : \gamma_1 \text{ and } \gamma_2 \subset L\} \leq$$
$$\leq \mathrm{P}\{\ \exists \text{ plane } L : \gamma_1 \subset L\} + \mathrm{P}\{\ \exists \text{ plane } L : \gamma_2 \subset L\}.$$

Let‘s prove that each of those probabilities is 0.

$$\mathrm{P}\{\ \exists \text{ plane } L : \gamma_1 \subset L\} \leq \mathrm{P}\{\det\big(\vec{\xi}(\vec{v}_1) - \vec{\xi}(\vec{v}_4), \vec{\xi}(\vec{v}_2) - \vec{\xi}(\vec{v}_4), \vec{\xi}(\vec{v}_3) - \vec{\xi}(\vec{v}_4)\big) = 0\},$$

where $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \theta_1$ are fixed and distinct.

Covariation function G from (1.1) is positive definite. Then random vector $\big(\vec{\xi}(\vec{v}_1) - \vec{\xi}(\vec{v}_4), \vec{\xi}(\vec{v}_2) - \vec{\xi}(\vec{v}_4), \vec{\xi}(\vec{v}_3) - \vec{\xi}(\vec{v}_4)\big)$ has density $p(x_{11}, x_{12}, x_{13}, ..., x_{33})$.

Consider

$$f(x_{11}, ..., x_{33}) = \sum_{\sigma} (-1)^{|\sigma|} x_{1\sigma(1)} x_{2\sigma(2)} x_{3\sigma(3)},$$

where σ runs through all permutations of $\{1, 2, 3\}$. f is infinitely differentiable so it‘s locally Lipshitz. By cofactor formula

$$f(x_{11}, ..., x_{33}) = \sum_{i=1}^{3} (-1)^{i+j} x_{ij} M_{ij}, j = 1, 2, 3,$$

where M_{ij} is ij minor of matrix (x_{ij}). Then

$$||\nabla f(\vec{x})|| = \sqrt{\sum_{i,j=1}^{3} M_{ij}^2(\vec{x})}.$$

Hence $||\nabla f(\vec{x})|| > 0$ almost surely.

Denote $D = \det(\vec{\xi}(\vec{v_1}), \vec{\xi}(\vec{v_2}), \vec{\xi}(\vec{v_3})) = f\big(\vec{\xi}(\vec{v_1}), \vec{\xi}(\vec{v_2}), \vec{\xi}(\vec{v_3})\big)$. Suppose that $a \in \mathbb{R}$, then for all $s \geq a$

$$\mathrm{P}\{D \in (a, s)\} = \int_{A_s} p(\vec{x}) d\vec{x},$$

where $A_s = f^{-1}((a, s))$. Then for all $s \geq a$ by coarea formula [6]

$$\mathrm{P}\{D \in (a, s)\} = \int_a^s \bigg(\int_{f^{-1}(t)} \frac{p(\vec{x})}{||\nabla f(\vec{x})||} dH_8(\vec{x}) \bigg) dt,$$

where H_8 is 8 dimensional Hausdorff measure. Differentiating by s we get that D has density $\tilde{p}$ and

$$\tilde{p}(s) = \int_{f^{-1}(s)} \frac{p(\vec{x})}{||\nabla f(\vec{x})||} dH_8(\vec{x}).$$

Then required probability is 0 and proof is finished. □

For a two component link in $\mathbb{R}^3$, it must have even numbers of crossings between different components. Hence the following corollary is true.

Corollary 3.1. *With probability 1 there exists plane L such that orthogonal projection of $\gamma = \gamma_1 \cup \gamma_2$ on it has at least 2 crossings between γ_1 and γ_2.*

Using Remark 2.1 and the same idea as in proof of Theorem 3.2 we can get the following corollary.

Corollary 3.2. *With probability 1 there exists plane L such that orthogonal projection of γ_1 on it has at least 1 self-crossing.*

Next results are devoted to the question of the existence of moments of defined invariants.

Theorem 3.3. *For any $p \in [0, \frac{3}{2})$*

$$\mathbb{E}(c(\gamma_1, \gamma_2))^p \text{ is finite}$$

Proof. By Proposition 2.1

$$c(\gamma_1, \gamma_2)^p = \frac{1}{(4\pi)^p} \Big(\int_0^{2\pi} \int_0^{2\pi} \frac{|(\dot{\gamma_1}(t), \dot{\gamma_2}(s), \gamma_1(t) - \gamma_2(s))|}{||\gamma_1(t) - \gamma_2(s)||^3} \, dt \, ds \Big)^p.$$

By Holder inequality

$$c(\gamma_1, \gamma_2)^p \le C_1 \int_0^{2\pi} \int_0^{2\pi} \frac{|(\dot{\gamma_1}(t), \dot{\gamma_2}(s), \gamma_1(t) - \gamma_2(s))|^p}{||\gamma_1(t) - \gamma_2(s)||^{3p}} \, dt \, ds.$$

Note that $\forall \vec{a_1}, \vec{a_2}, \vec{a_3} \in \mathbb{R}^3 : |(\vec{a_1}, \vec{a_2}, \vec{a_3})| \le ||\vec{a_1}|| \cdot ||\vec{a_2}|| \cdot ||\vec{a_3}||$. Then

$$c(\gamma_1, \gamma_2)^p \le C_1 \int_0^{2\pi} \int_0^{2\pi} \frac{||\dot{\gamma_1}(t)||^p \cdot ||\dot{\gamma_2}(s)||^p}{||\gamma_1(t) - \gamma_2(s)||^{2p}} \, dt \, ds.$$

If $p < \frac{3}{2}$ then $\exists \, \varepsilon > 0 : p < \frac{3}{2}(1 - \varepsilon)$. By Holder inequality with exponents $1 - \varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}$

$$\mathbb{E} \frac{||\dot{\gamma_1}(t)||^p \cdot ||\dot{\gamma_2}(s)||^p}{||\gamma_1(t) - \gamma_2(s)||^{2p}} \le E_1 E_2 E_3,$$

where

$$E_i(t) = \big(\mathbb{E} ||\dot{\gamma_i}(t)||^{\frac{2p}{\varepsilon}} \big)^{\frac{\varepsilon}{2}}, i = 1, 2,$$

$$E_3(t, s) = \Big(\mathbb{E} \frac{1}{||\gamma_1(t) - \gamma_2(s)||^{\frac{2p}{1-\varepsilon}}} \Big)^{1-\varepsilon}.$$

Consequently,

$$\mathbb{E}(c(\gamma_1, \gamma_2))^p \le C_1 \int_0^{2\pi} \int_0^{2\pi} E_1(t) E_2(s) E_3(t, s) \, dt \, ds.$$

Firstly, note that $\forall t \in [0, 2\pi], i = 1, 2$

$$\dot{\gamma_i}(t) \sim N(0, I).$$

Indeed

$$\dot{\gamma_i}(t) = l.i.m. \ \frac{\gamma_i(t + h) - \gamma_i(t)}{h}, h \to 0.$$

Then $\dot{\gamma_i}(t)$ is Gaussian random variable with 0 mean and variance

$$\tilde{\sigma}^2 = \lim_{h\to 0} \frac{2 - 2G(\vec{\theta}_i(t+h) - \vec{\theta}_i(h))}{h^2}.$$

As $||\vec{\theta}_i(t+h) - \theta_i\vec{(t)}|| = 2 - 2\cos h$, standard calculations give that $\tilde{\sigma} = 1$. Then $\forall\, t, s \in [0, 2\pi] : E_1(t) = E_2(s) = C_2 < +\infty$. Now consider

$$\vec{\eta}(t,s) = \gamma_1(t) - \gamma_2(s).$$

Standard calculations give that

$$\vec{\eta}(t,s) \sim N(0, \sigma(t,s)^2 I),$$
$$\sigma(t,s)^2 = 2 - 2G(\vec{\theta}_1(t) - \vec{\theta}_2(s)).$$

For the next calculations we will omit t, s. $||\vec{\eta}||$ has density that equals to

$$p_{||\vec{\eta}||}(x) = \int_{S_x} \frac{1}{(\sqrt{2\pi}\sigma)^3} exp\Big(-\frac{||\vec{u}||^2}{2\sigma^2}\Big)\nu(d\vec{u}),$$

where $S_x = \{\vec{u} \in \mathbb{R}^3 : ||\vec{u}|| = x\}$ and ν is surface measure on S_x. This integral equals to

$$p_{||\vec{\eta}||}(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} x^2 e^{-\frac{x^2}{2\sigma^2}}.$$

Then

$$\mathbb{E}||\vec{\eta}||^{\frac{-2p}{1-\varepsilon}} = \int_0^\infty x^{\frac{-2p}{1-\varepsilon}} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} x^2 e^{-\frac{x^2}{2\sigma^2}}\, dx = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \int_0^\infty x^{2-\frac{2p}{1-\varepsilon}} e^{-\frac{x^2}{2\sigma^2}}\, dx.$$

This integral converge because $2 - \frac{2p}{1-\varepsilon} > -1$ and equals to $C_3\sigma^{3-\frac{2p}{1-\varepsilon}}$. Then finally

$$\mathbb{E}(c(\gamma_1,\gamma_2))^p \le C_1C_2C_3 \int_0^{2\pi}\int_0^{2\pi} \sigma^{-\frac{2p}{1-\varepsilon}}\, dt\, ds.$$

As circles θ_1 and θ_2 are non intersecting $\sigma(t,s) \neq 0$ and subintegral function is continuous thus the following integral converge. This finishes the proof. □

Corollary 3.3. *For any* $p \in [0, \frac{3}{2})$

$$\mathbb{E}|link(\gamma_1,\gamma_2)|^p \textit{ is finite}$$

Proof. From formulas (2.1) and (2.2) it follows that

$$|link(\gamma_1,\gamma_2)| \le c(\gamma_1,\gamma_2).$$

□

Theorem 3.4. *Let $\mathcal{E}$ be a total energy of a random link γ. Then for any $p \in [0, \frac{1}{4})$*

$$\mathbb{E}\mathcal{E}^p \textit{is finite}$$

Proof. By formula (2.7) and Holder inequality

$$\mathbb{E}\mathcal{E}^p \le C_1\big(\mathbb{E}\mathcal{E}(\gamma_1,\gamma_1)^p + \mathbb{E}\mathcal{E}(\gamma_2,\gamma_2)^p + \frac{1}{2^p}\mathbb{E}\mathcal{E}(\gamma_1,\gamma_2)^p\big).$$

By the proof of Theorem 3.3

$$\mathbb{E}\Big(\int_0^{2\pi}\int_0^{2\pi} \frac{||\dot{\gamma_1}(t)|| \cdot ||\dot{\gamma_2}(s)||}{||\gamma_1(t) - \gamma_2(s)||^2}\, dt\, ds\Big)^p < \infty \text{ for } 0 \le p < \frac{3}{2}.$$

Then $\mathbb{E}\mathcal{E}(\gamma_1,\gamma_2)^p < \infty$. Suppose that $i \in \{1,2\}$. By formula (2.6)

$$\mathbb{E}\mathcal{E}(\gamma_i,\gamma_i)^p \le \mathbb{E}\Big(\int_0^{2\pi}\int_0^{2\pi} \frac{||\dot{\gamma_i}(t)|| \cdot ||\dot{\gamma_i}(s)||}{||\gamma_i(t) - \gamma_i(s)||^2}\, dt\, ds\Big)^p.$$

By Holder inequality

$$\mathbb{E}\Big(\int_0^{2\pi}\int_0^{2\pi}\frac{||\dot{\gamma_i}(t)||\cdot||\dot{\gamma_i}(s)||}{||\gamma_i(t)-\gamma_i(s)||^2}\,dt\,ds\Big)^p\leq C_2\int_0^{2\pi}\int_0^{2\pi}\mathbb{E}\frac{||\dot{\gamma_i}(t)||^p\cdot||\dot{\gamma_i}(s)||^p}{||\gamma_i(t)-\gamma_i(s)||^{2p}}\,dt\,ds.$$

Suppose that $\varepsilon>0$ is such that $p<\frac{1}{4}(1-\varepsilon)$. Using the same arguments as in the proof of Theorem 3.3 we get that

$$\mathbb{E}\frac{||\dot{\gamma_i}(t)||^p\cdot||\dot{\gamma_i}(s)||^p}{||\gamma_i(t)-\gamma_i(s)||^{2p}}\leq C_3\sigma(t,s)^{-\frac{2p}{1-\varepsilon}},$$

where

$$\sigma(t,s)=2-2e^{-\frac{1}{2}||\theta_i(t)-\theta_i(s)||^2}=2-2e^{-2\sin^2(\frac{t-s}{2})}.$$

Thus

$$\mathbb{E}\mathcal{E}(\gamma_i,\gamma_i)^p\leq\tilde{C}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{(1-\exp(-2\sin^2(\frac{t-s}{2})))^{\frac{2p}{1-\varepsilon}}}\,dt\,ds.$$

For $\delta>0$ we can split the integral in a following way

$$\int_0^{2\pi}\int_0^{2\pi}=\iint_{|t-s|\geq\delta}+\iint_{|t-s|<\delta}.$$

First integral obviously converge(subintegral function is continuous). Notice, that

$$\Big(1-\exp\big(-2\sin^2(\frac{t-s}{2})\big)\Big)^{-\frac{2p}{1-\varepsilon}}\sim 2^{\frac{2p}{1-\varepsilon}}\cdot(t-s)^{-\frac{4p}{1-\varepsilon}},t-s\to 0.$$

At the same time $\frac{4p}{1-\varepsilon}<1$ due to choose of ε. Then for some small δ the second integral converge due to comparison test. Theorem is proved. ☐

Corollary 3.4. *For any* $p\in[0,\frac{1}{4})$

$$\mathbb{E}c(\gamma_1,\gamma_1)^p \text{ is finite}$$

Proof. From formula (2.2) it follows that

$$c(\gamma_1,\gamma_1)\leq\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{||\dot{\gamma_1}(t)||\cdot||\dot{\gamma_1}(s)||}{||\gamma_1(t)-\gamma_1(s)||^2}\,dt\,ds.$$

By the proof of Theorem 3.4 integral at right side of the equation have moments of orders $[0,\frac{1}{4})$. ☐

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