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ASYMPTOTIC BEHAVIOR OF THE SOLUTION TO THE MULTI-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATION WITH INTERACTIONS

This paper explores the asymptotic behavior of solutions to multi-dimensional stochastic differential equations with interactions (SDEWI). By integrating interaction terms reflecting the distribution of all participating particles, SDEWI provides a complex model that captures dynamic changes across a system of particles. Theoretical insights are substantiated by the derivation of conditions under which the solutions exhibit specific asymptotic properties. This work extends previous research by confirming the shift-compactness criterion and establishing conditions for the existence of asymptotic limits, thereby offering a deeper understanding of the interaction dynamics within stochastic systems.

1. Introduction

Stochastic differential equations (SDEs) are widely used to describe the behavior of a single particle in a random media. Andrey A. Dorogovtsev in [2] proposed incorporating the distribution μ_t of all other particles as additional term to SDE's coefficients in order to model the system of interacting particles. These equations, known as stochastic differential equations with interactions (SDEWI), take the form:

$$(1)\qquad \begin{cases} \mathrm{d}x(u,t) & = a(x(u,t),\mu_t)\mathrm{d}t + b(x(u,t),\mu_t)\mathrm{d}W_t \\ x(u,0) & = u, \quad u \in \mathbb{R}^d \\ \mu_t & = \mu_0 \circ x^{-1}(\cdot,t), \quad t \geq 0, \end{cases}$$

where W_t is m-dimensional Brownian motion, $\mu_t = \mu_0 \circ x^{-1}(\,\cdot\,,t)$ is an image of μ_0 under the map $x(\cdot,t) : \mathbb{R}^d \to \mathbb{R}^d$ and $\mu_t \in \mathcal{M}_n$. Here, $\mathcal{M}_n$ is the space of all probability measures μ on $\mathbb{R}^d$ with finite n-th moment. The coefficients in (1) are

$$a : \mathbb{R}^d \times \mathcal{M}_n \to \mathbb{R}^d, \quad b : \mathbb{R}^d \times \mathcal{M}_n \to \mathbb{R}^{d\times m}$$

In this paper, we examine SDEWI characterized by the following structure, where the first summand represents pairwise interactions,

$$(2)\qquad \begin{cases} dx(u,t) & = \int_{\mathbb{R}^d} \phi(x(u,t)-v)\mu_t(\mathrm{d}v)\mathrm{d}t + b(x(u,t),\mu_t)\mathrm{d}W_t \\ x(u,0) & = u \in \mathbb{R}^d \\ \mu_t & = \mu_0 \circ x^{-1}(\cdot,t), \quad t \geq 0 \end{cases}$$

$b(x(u,t),\mu_t)$ is the diagonal matrix $d \times d$ and W_t is d-dimensional Brownian motion. The long-term dynamics of the interacting particles can exhibit various behavior, such as converging to a single point or becoming densely concentrated within a confined area. To illustrate this phenomenon, we will explore a scenario where particles tend to aggregate around the center of mass.

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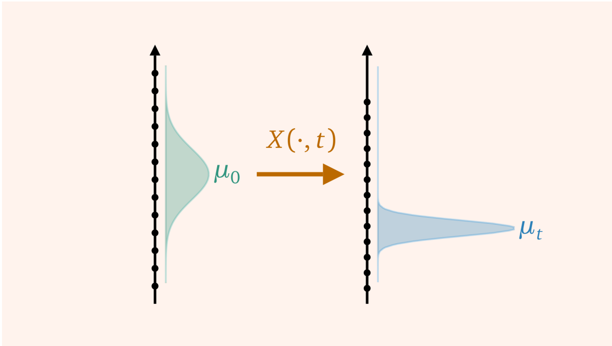

FIGURE 1. The example of an evolution of μ_t by the solution to SDEWI $X(u,t)$

Example 1.1. Consider d-dimensional SDEWI, where A is real-valued $d\times d$ matrix and $\{W_t,\ t\geq 0\}$ is d-dimensional Brownian motion

$$\begin{cases} \mathrm{d}x(u,t) & = A\left(x(u,t) - \int_{\mathbb{R}^d} v\mu_t(\mathrm{d}v)\right)\mathrm{d}t + \mathrm{d}W_t \\ x(u,0) & = u, \quad u\in\mathbb{R}^d \\ \mu_t & = \mu_0\circ x^{-1}(\cdot,t), \quad t\geq 0. \end{cases} \tag{3}$$

The center of mass of the system at time t is defined as

$$m_t = \int_{\mathbb{R}^d} v\mu_t(\mathrm{d}v) = \int_{\mathbb{R}^d} x(v,t)\mu_0(\mathrm{d}v).$$

Thus, the center of mass itself follows a simple Brownian motion:

$$\mathrm{d}m_t = \mathrm{d}W_t.$$

We can now reformulate equation (3) as follows:

$$\mathrm{d}\left(x(u,t) - m_t\right) = A\left(x(u,t) - m_t\right)\mathrm{d}t$$

and obtain the explicit solution :

$$x(u,t) = m_t + e^{At}\left(u - m_0\right) = W_t + e^{At}\left(u - m_0\right) + \mathrm{const}.$$

If we assume that the matrix A satisfies condition $\|e^{At}\|\leq 1$, then the last term on the right-hand side of the equation represents a minor fluctuation around the center of mass. Informally, it can be described that each particle fluctuates around the center of mass and that the center of mass executes a d-dimensional Brownian motion.

In order to characterize the above-mentioned behavior with mathematical rigor we can use the concept introduced in [2] called shift compactness.

Definition 1.1. The set of measures $\{\mu_a : a\in\mathcal{A}\}\subset\mathcal{M}_n$ is called **shift compact** if for every $a\in\mathcal{A}$ there is $u_a\in\mathbb{R}^d$ such that $\{\mu_a - u_a : a\in\mathcal{A}\}$ is a compact set in $\mathcal{M}_n$.

The Theorem 5.4.1 of [2] provides the sufficient criterion for $\{\mu_t;\, t\geq 0\}\in\mathcal{M}_{2n}$ from (2) to be shift compact. This criterion is detailed as follows:

$$\begin{cases} (u-v, \phi(u)-\phi(v)) \leq -\alpha\|u-v\|^2, \quad u,v\in\mathbb{R}^d, \\ b \text{ is Lipschitz continuous with the constant } B \text{ for fixed second argument} \\ \alpha - \frac{1}{2}B^2(2n+1)\geq 0. \\ \mu_0\in\mathcal{M}_{2n+2} \end{cases} \tag{4}$$

One-dimensional stochastic differential equations without drift were considered in [4] in the following form:

$$\begin{cases} \mathrm{d}x(u,t) &= b(x(u,t),\mu_t)\mathrm{d}W_t \\ x(u,0) &= u, \quad u \in \mathbb{R} \\ \mu_t &= \mu_0 \circ x^{-1}(\cdot,t), \quad t \geq 0 \end{cases} \tag{5}$$

Then it was proved that

$$\lim_{t\to\infty} (x(u,t) - x(v,t)) \text{ exists a.e.}$$

for all $u, v \in \mathbb{R}$. The author applied the random time change for $x(u,t) - x(v,t)$ and the bounded martingale part with finite stopping time using the point recurrence of a one-dimensional Brownian motion.

Subsequently, [1] expanded upon the work of [4] by examining two-dimensional SDEWI (2). The author applied Itô's formula to $\|x(u,t) - x(v,t)\|^2$, resulting in a decomposition that includes both drift and martingale terms. The author then established an upper bound for the martingale component using a random time change and the point recurrence of one-dimensional Brownian motion. However, due to the presence of the drift term in Itô's decomposition, new conditions were necessitated, akin to those in (4).

This work further extends above mentioned results to $\mathbb{R}^d$, by affirming the shift-compactness criteria outlined in (4) for SDEWI.

The paper is organized as follows. Section 2 gives the criterion of existence of the limit after subtraction of integral for $\|x(u,t) - x(v,t)\|^{2n+2}$. We start with applying Itô's formula, then we come up with the upper bound for the martingale part. The approach is mostly inspired by [1].

In Section 3, we formulate Theorem 2 about the existence of the limit for the double integral

$$\mathcal{E}_t = \int_{\mathbb{R}^{2d}} \|x(u,t) - x(v,t)\|^{2n+2}\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v),$$

which can be considered as the inner energy of the system .

2. Asymptotic behavior of the distance in $\mathbb{R}^d$

Applying Itô's formula [5, Th.18.18] for the solution of Equation (2) with the diagonal diffusion matrix $b(x,\mu)$ we can decompose the difference norm into the martingale and drift parts:

$$\begin{aligned} \|x(u,t) - x(v,t)\|^{2n+2} = \|u - v\|^{2n+2} + \\ + \int_0^t \Phi(x(u,s), x(v,s), \mu_s)\mathrm{d}s + \\ + \sum_{k=1}^{d} \int_0^t P_k(x(u,s), x(v,s), \mu_s) dW_k(s) \end{aligned} \tag{6}$$

The drift term Φ in (6) is derived from Itô's lemma. Let us introduce the shorthand notation $y = x(u,s) - x(v,s)$ and $\Delta b = b(x(u,s),\mu_s) - b(x(v,s),\mu_s)$.

The total drift term is the sum:

$$\Phi(x(u,s), x(v,s), \mu_s) = \Phi_a + \Phi_{b,1} + \Phi_{b,2} \tag{7}$$

where the components are defined as follows.

The first component, Φ_a, arises from the drift coefficient $a(x,\mu) = \int \phi(x-z)\mu(dz)$:

$$\Phi_a = (2n+2)\|y\|^{2n} \int_{\mathbb{R}^d} (y, \phi(x(u,s) - z) - \phi(x(v,s) - z))\, \mu_s(\mathrm{d}z). \tag{8}$$

The next two components, $\Phi_{b,1}$ and $\Phi_{b,2}$, arise from the Itô's term in Itô's lemma. The first of these is:

$$\Phi_{b,1} = (n+1)\|y\|^{2n}\|\Delta b\|^2, \tag{9}$$

and the second diffusion component is:

$$\Phi_{b,2} = 2n(n+1)\|y\|^{2n-2}\sum_{k=1}^{d} y_k^2 \cdot (\Delta b_k)^2. \tag{10}$$

The martingale part for all $k \in \{1, 2, \dots, d\}$,

$$P_k(x(u,s), x(v,s), \mu_s) = 2(n+1)\|y\|^{2n} y_k \Delta b_k. \tag{11}$$

Now we can formulate the statement about the asymptotic behavior of the difference norm

Theorem 2.1. *Consider SDEWI (2) with conditions on coefficients given by (4). Then for all $u, v \in \mathbb{R}^d$ the following limit exists a.s.*

$$\lim_{t\to\infty}\left(\|x(u,t) - x(v,t)\|^{2n+2} - \int_0^t \Phi(x(v,s), x(u,s), \mu_s)ds\right) \ a.e. \tag{12}$$

where functions Φ and P_k, for all $k \in \{1, 2, \dots, d\}$ are defined in (6).

Proof. At the beginning let us consider the sum

$$\sum_{k=1}^{d}\int_0^t P_k(x(u,s), x(v,s), \mu_s)\mathrm{d}W_k(s) \tag{13}$$

Let's define

$$I_k(t) = \int_0^t P_k(x(v,s), x(u,s), \mu_s)\mathrm{d}W_k(s), \quad k \in \{1, 2, \dots, d\}$$

$I_1, I_2, \dots, I_d$ are mutual uncorrelated continuous martingales, being Itô integrals with respect to independent Brownian motions $w_1, w_2, \dots, w_d$ with quadratic variation

$$\langle \sum_{k=1}^{d} I_k \rangle_t = \sum_{k=1}^{d}\int_0^t P_k^2(x(v,s), x(u,s), \mu_s)\mathrm{d}s.$$

Then we can replace the sum (13) with some time-changed Brownian motion w [5, Th.19.4]

$$\begin{aligned} \|x(u,t) - x(v,t)\|^{2n+2} &- \int_0^t \Phi(x(v,s), x(u,s), \mu_s)\mathrm{d}s \\ &= \|u-v\|^{2n+2} + \sum_{k=1}^{d} I_k(t) \\ &= \|u-v\|^{2n+2} + w\left(\int_0^t \sum_{k=1}^{d} P_k^2(x(v,s), x(u,s), \mu_s))ds\right), \end{aligned} \tag{14}$$

Applying conditions (4) to (7) we get the upper bound

$$\begin{aligned} \int_0^t \Phi(x(u,s), x(v,s), \mu_s)\mathrm{d}s \le \big(-\alpha(2n+2) + B^2(n+1)(2n+1)\big) \\ \times \int_0^t \|x(u,t) - x(v,t)\|^{2n+2}\mathrm{d}s \end{aligned} \tag{15}$$

As constants α, B satisfy the last condition in (4), we have established the non-positivity of drift term

$$\int_0^t \Phi(x(u,s),x(v,s),\mu_s)\mathrm{d}s \le 0$$

and then the following inequality takes place for all $t \ge 0$,

$$\|x(u,t)-x(v,t)\|^{2n+2} - \int_0^t \Phi(x(u,s),x(v,s),\mu_s)ds \ge 0.$$

The functions Φ and x are continuous and, for $u \neq 0$, we have that $|x(u,0)| = |u| > 0$. So for $t > 0$, $u \neq 0$, t small enough, and taking into account that the integral on the right-hand side is non-positive, we can conclude that for all $t \ge 0$

$$\|x(u,t)-x(v,t)\|^{2n+2} - \int_0^t \Phi(x(v,s),x(u,s),\mu_s)\mathrm{d}s \ge 0$$

The one-dimensional Brownian motion w is point recurrent and we obtain almost surely

$$\sum_{k=1}^d \int_0^t P_k^2(x(v,s),\ x(u,s),\ \mu_s)\mathrm{d}s \le \tau_{u,v} < \infty. \tag{16}$$

Finally, the limit does exist almost surely

$$\lim_{t\to\infty}\left(\|x(u,t)-x(v,t)\|^{2n+2} - \int_0^t \Phi(x(v,s),x(u,s),\mu_s)\mathrm{d}s\right) =$$
$$= \|u-v\|^{2n+2} + w\left(\int_0^\infty \sum_{k=1}^d P_k^2(x(v,s),x(u,s),\mu_s)\mathrm{d}s\right).$$

The Theorem is proved. □

3. Asymptotic behavior of 'internal' energy

Also, we would like to investigate the double integral for $x(u,t)$ from SDEWI (2)

$$\mathcal{E}_t = \int_{\mathbb{R}^{2d}} \|x(u,t)-x(v,t)\|^{2n+2}\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v) \tag{17}$$

which we will refer to as 'internal' energy of the system.

Repeating steps from the previous chapter, we can apply Itô's formula for $\mathcal{E}_t$. Then (6) will become

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \|x(u,t)-x(v,t)\|^{2n+2}\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v) &= \int_{\mathbb{R}^{2d}} \|u-v\|^{2n+2}\mu_0(du)\mu_0(dv)\\ &+ \int_{\mathbb{R}^{2d}}\int_0^t \Phi(x(u,s),x(v,s),\mu_s)\mathrm{d}s\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v)\\ &+ \sum_{k=1}^d \int_{\mathbb{R}^{2d}}\int_0^t P_k(x(u,s),x(v,s),\mu_s)\mathrm{d}W_k(s)\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v)\end{aligned} \tag{18}$$

Now we can formulate the statement about the asymptotic behavior of $\mathcal{E}_t$

Theorem 3.1. *Consider SDEWI (2) with restrictions on coefficients given by (4). Then for all $u, v \in \mathbb{R}^d$ the following limit exists a.e.*

$$\lim_{t\to\infty}\left(\mathcal{E}_t - \int_{\mathbb{R}^{2d}}\int_0^t \Phi(x(u,s),x(v,s),\mu_s)\mathrm{d}s\mu_0(\mathrm{d}u)\mu_0(\mathrm{d}v)\right) \tag{19}$$

where functions Φ is defined in (18).

Proof. Using the same arguments as in the proof for Theorem 1, but each term is integrated in $\mathbb{R}^{2d}$. The presence of integral does not affect (15) and we get

$$\int_{\mathbb{R}^{2d}} \int_0^t \Phi(x(u,s), x(v,s), \mu_s) \mathrm{d}s \mu_0(\mathrm{d}u) \mu_0(\mathrm{d}v) \le 0$$

All inequalities hold true because we integrate over non-negative measure. □

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