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TIME-VARYING VECTOR RANDOM FIELDS ON THE ARCCOS-QUASI-QUADRATIC METRIC SPACE

An arccos-quasi-quadratic metric is defined on a subset of $\mathbb{R}^{d+1}$ such as a sphere, a ball, an ellipsoidal surface, an ellipsoid, a simplex, a conic surface, or a hyperbolic surface, and the corresponding metric space incorporates several important cases in a unified framework that makes possible for us to study metric-dependent random fields on different metric spaces in a unified manner. Over the arccos-quasi-quadratic metric space, this paper constructs a class of time-varying vector random fields via either spherical harmonics or ultraspherical polynomials, and builds up various parametric and semiparametric covariance matrix structures. The extension problem is discussed as well.

1. Introduction

This paper attempts to develop spatio-temporal vector random fields whose spatial index domains are termed as an arccos-quasi-quadratic metric space recently introduced in [35, 36]. Such a metric space is a subset $\mathbb{D}$ of $\mathbb{R}^{d+1}$ together with a metric (distance function) that is the composition of arccosine and quasi-quadratic functions. More precisely, over $\mathbb{D}$ we define the arccos-quasi-quadratic metric by

$$\vartheta(\mathbf{x}_1,\mathbf{x}_2) = \arccos\left((\mathbf{w}(\mathbf{x}_1))'\Sigma\mathbf{w}(\mathbf{x}_2)\right), \ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \tag{1}$$

where $\mathbf{w}(\mathbf{x})$ is a $(d+1)$-variate function defined on $\mathbb{D}$, Σ is a $(d+1)\times(d+1)$ strictly positive definite matrix, $|(\mathbf{w}(\mathbf{x}_1))'\Sigma\mathbf{w}(\mathbf{x}_2)| \le 1$, $\mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}$, and $(\mathbf{w}(\mathbf{x}_1))'\Sigma\mathbf{w}(\mathbf{x}_2) = 1$ if and only if $\mathbf{x}_1 = \mathbf{x}_2$. Important examples of this type of metric spaces are

(i) the unit sphere $\mathbb{S}^d = \{\mathbf{x} \in \mathbb{R}^{d+1} : \|\mathbf{x}\| = 1\}$, $\vartheta(\mathbf{x}_1,\mathbf{x}_2) = \arccos(\mathbf{x}_1'\mathbf{x}_2)$, $\mathbf{x}_1,\mathbf{x}_2 \in \mathbb{S}^d$, in (1) $\mathbf{w}(\mathbf{x}) = \mathbf{x}$ and $\Sigma = \mathbf{I}_{d+1}$, where $\|\mathbf{x}\|$ represents the Euclidean norm of $\mathbf{x} \in \mathbb{R}^{d+1}$ and $\mathbf{I}_{d+1}$ is a $(d+1)\times(d+1)$ identity matrix;

(ii) an ellipsoidal surface $\mathbb{D} = \{\mathbf{x} \in \mathbb{R}^{d+1} : \mathbf{x}'\Sigma\mathbf{x} = 1\}$, $\vartheta(\mathbf{x}_1,\mathbf{x}_2) = \arccos(\mathbf{x}_1'\Sigma\mathbf{x}_2)$, and $\mathbf{w}(\mathbf{x}) = \mathbf{x}$;

(iii) the unit ball $\mathbb{B}^d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \le 1\}$, with distance function [6, 10]

$$\vartheta(\mathbf{x}_1,\mathbf{x}_2) = \arccos\left(\mathbf{x}_1'\mathbf{x}_2 + \sqrt{1-\|\mathbf{x}_1\|^2}\sqrt{1-\|\mathbf{x}_2\|^2}\right),$$

in (1), $\mathbf{w}(\mathbf{x}) = \begin{pmatrix} \mathbf{x} \\ \sqrt{1-\|\mathbf{x}\|^2} \end{pmatrix}$ and $\Sigma = \mathbf{I}_{d+1}$;

(iv) an ellipsoid $\mathbb{D} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}'\Sigma_0\mathbf{x} \le 1\}$, Σ_0 is a $d\times d$ strictly positive definite matrix,

$$\mathbf{w}(\mathbf{x}) = \begin{pmatrix} \mathbf{x} \\ \sqrt{1-\mathbf{x}'\Sigma_0\mathbf{x}} \end{pmatrix}, \ \Sigma = \begin{pmatrix} \Sigma_0 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix},$$

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and

$$\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos\left(\mathbf{x}_1'\Sigma_0\mathbf{x}_2 + \sqrt{1-\mathbf{x}_1'\Sigma_0\mathbf{x}_1}\sqrt{1-\mathbf{x}_2'\Sigma_0\mathbf{x}_2}\right);$$

(v) the probability simplex

$$\triangle^d = \left\{\mathbf{x} = (x_1, \ldots, x_{d+1})' \in \mathbb{R}^{d+1} : x_1 \geq 0, \ldots, x_{d+1} \geq 0, \sum_{k=1}^{d+1} x_k = 1\right\},$$

with distance function [6, 10, 24]

$$\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos\left(\sum_{k=1}^{d+1}\sqrt{x_{k1}x_{k2}}\right), \ \mathbf{x}_k = (x_{1k}, \ldots, x_{d+1,k})' \in \triangle^d, \ k = 1, 2,$$

in (1), $\mathbf{w}(\mathbf{x}) = (\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_{d+1}})'$ and $\Sigma = \mathbf{I}_{d+1}$;

(vi) a simplex or a corner of the d-cube

$$\mathbb{D} = \left\{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, \sum_{k=1}^{d} x_k \leq 1\right\},$$

$$\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos\left(\sum_{k=1}^{d}\sqrt{x_{k1}x_{k2}} + \sqrt{\left(1-\sum_{k=1}^{d}x_{k1}\right)\left(1-\sum_{k=1}^{d}x_{k2}\right)}\right),$$
$$\mathbf{x}_k = (x_{1k}, \ldots, x_{d,k})' \in \mathbb{D}, \ k = 1, 2,$$

in (1), $\mathbf{w}(\mathbf{x}) = \left(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{1-\sum_{k=1}^{d}x_k}\right)'$ and $\Sigma = \mathbf{I}_{d+1}$;

(vii) the double conic surface

$$\mathbb{D} = \left\{\begin{pmatrix}\mathbf{x}\\ x_{d+1}\end{pmatrix} \in \mathbb{R}^{d+1} : \|\mathbf{x}\| = |x_{d+1}|, \ |x_{d+1}| \leq 1\right\},$$

with distance function [53]

$$\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos\left(\mathbf{x}_1'\mathbf{x}_2 + \sqrt{\left(1-x_{d+1,1}^2\right)\left(1-x_{d+1,2}^2\right)}\right),$$
$$\begin{pmatrix}\mathbf{x}_k\\ x_{d+1,k}\end{pmatrix} \in \mathbb{D}, \ k = 1, 2,$$

in (1), $\mathbf{w}(\mathbf{x}) = \begin{pmatrix}\mathbf{x}\\ \sqrt{1-x_{d+1}^2}\end{pmatrix}$ and $\Sigma = \mathbf{I}_{d+1}$;

(viii) the hyperbolic surface

$$\mathbb{D} = \left\{\begin{pmatrix}\mathbf{x}\\ x_{d+1}\end{pmatrix} \in \mathbb{R}^{d+1} : x_{d+1}^2 - \|\mathbf{x}\|^2 = r^2, \ r \leq x_{d+1} \leq \sqrt{1+r^2}\right\},$$

with distance function [53]

$$\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos\left(\mathbf{x}_1'\mathbf{x}_2 + \sqrt{\left(1+r^2-x_{d+1,1}^2\right)\left(1+r^2-x_{d+1,2}^2\right)}\right),$$
$$\begin{pmatrix}\mathbf{x}_k\\ x_{d+1,k}\end{pmatrix} \in \mathbb{D}, k = 1, 2,$$

where r is a nonnegative constant, $\mathbf{w}(\mathbf{x}) = \begin{pmatrix}\mathbf{x}\\ \sqrt{1+r^2-x_{d+1}^2}\end{pmatrix}$, and $\Sigma = \mathbf{I}_{d+1}$.

Over the spherical metric space $(\mathbb{S}^d, \vartheta)$, spatial or spatio-temporal random fields are extremely well studied [7, 8, 15, 19, 22, 23, 27, 28, 33, 38, 40, 54, 55]. In contrast, just recently vector random fields with the metric-dependent correlation structures on $\mathbb{B}^d$ and $\triangle^d$ are drawn to attention in [9, 26, 34], besides random fields with radial covariance functions on $\mathbb{B}^d$ are considered in [25, 42, 43] when $\mathbb{B}^d$ is endowed with the Euclidean distance. As an original and innovative contribution, this paper places these (non-Euclidean) metric spaces into a unified context and attempts to study metric-dependent random fields on different metric spaces in a unified manner. With a relatively wide coverage, the investigation on the arccos-quasi-quadratic metric space of random fields is expected to offer more theoretical options to practical demands in climatology, cosmology, earth science, and medical imaging, just to name but a few. Complex data in modern data analysis may be described as elements of a metric space that satisfies certain structural conditions and features a probability measure [12]. One of our applied motivations stems from astronomical sciences, particularly the future European Space Agency mission Euclid and Cosmic Microwave Background Stage 4 (CMB-S4) project; see https://cmb-s4.org.CMB-S4.

In Section 3 we construct m-variate second-order random fields on a spatio-temporal domain $\mathbb{D} \times \mathbb{T}$ via two infinite series expressions, one is based on spherical harmonics and the other is in terms of the ultraspherical polynomials, and present an infinite series representation for the covariance matrix function of an m-variate elliptically contoured random field on $\mathbb{D} \times \mathbb{T}$ by means of the ultraspherical polynomials. Many statistical models in cosmological perturbation theory are time-varying isotropic Gaussian random fields on $\mathbb{S}^2$. For instance, the temperature in CMB situation is expanded in spherical harmonics, for an observer sitting in $\mathbf{x}$ at time t,

$$T(\mathbf{x}, t, \mathbf{n}) = \bar{T}(t) \sum_{l=0}^{\infty} \sum_{k=-l}^{l} a_{lk}(\mathbf{x}, t) Y_{lk}(\mathbf{n}), \tag{2}$$

where $\mathbf{n}$ is the direction of observation, and $\{Y_{nj}(\mathbf{x}), \mathbf{x} \in \mathbb{S}^2, j = 0, \pm 1, \ldots, \pm n\}$ is an orthonormal basis of spherical harmonics on $\mathbb{S}^2$; see (3.8) of [13]. But, at least locally, the true Universe is not perfectly homogeneous and isotropic[13, 44]. The advantage of (8) or (13) developed in Section 3 allows for anisotropy, due to many possible selections of $\mathbf{w}(\mathbf{x})$ and Σ over $\mathbb{D}$ in (1).

Section 4 establishes various parametric or semiparametric correlation structures on $\mathbb{D} \times \mathbb{T}$, with three approaches offered in Theorem 3 in terms of completely monotone functions and conditionally negative definite matrices, noticing that the arccos-quasi-quadratic metric (1) is not only a variogram on $\mathbb{D}$ but also a measure definite kernel [35].

The extension problem is considered in Section 5, where a covariance matrix function on $\mathbb{D} \times \mathbb{Z}$ that is metric-dependent on $\mathbb{D}$ and stationary on $\mathbb{Z}$ is extended to one on $\mathbb{D} \times \mathbb{R}$. It essentially provides an effective approach for generating covariance matrix models on $\mathbb{D} \times \mathbb{R}$ from those on $\mathbb{D} \times \mathbb{Z}$. We refer the reader to [46] for the extension problem in a more general context, although it was primarily concerned with positive-definite functions on Euclidean spaces and on groups of lattice points. Examples of stationary covariance functions on $\mathbb{R}$ being extended to isotropic covariance functions in $\mathbb{R}^d$ $(d \geq 2)$ may be found in [29].

Some preliminary results are made available in Section 2 for use. Section 6 presents the proofs of lemmas and theorems that are stated in Sections 2-5 respectively. In what follows let $d \geq 2$. Denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{N}_0$ the set of nonnegative integers. The trace of a square matrix $\mathbf{B}$ is denoted by trace$(\mathbf{B})$. For a positive definite matrix $\mathbf{B}$, there is a symmetric matrix $\mathbf{B}^{\frac{1}{2}}$ of the same order of $\mathbf{B}$, which is called its positive definite square root [20], such that $\mathbf{B} = \mathbf{B}^{\frac{1}{2}} \mathbf{B}^{\frac{1}{2}}$.

2. Preliminary results

This section provides some preliminary results for our constructions of second-order time-varying random fields on the arccos-quasi-quadratic metric space $(\mathbb{D}, \vartheta)$. The important building blocks in the next section are spherical harmonics and ultraspherical or Gegenbauers polynomials [2], which are closely related to each other via the addition formula displayed in Lemma 1 below. Lemma 2 is a rephrased version of Lemma 2 in [33], where it illustrates a basis of the set of isotropic and mean square continuous random fields on the spherical metric space $(\mathbb{S}^d, \vartheta)$. Lemma 3 recalls a couple of key connections between a positive definite matrix and a conditionally negative definite matrix, which will be employed in Section 4 to formulate some parametric or semiparmetric covariance matrix models on $\mathbb{D} \times \mathbb{T}$.

Recall that the ultraspherical polynomials [50] possess the exact expressions

$$\begin{aligned} P_0^{(\lambda)}(x) &\equiv 1, \\ P_n^{(\lambda)}(x) &= \tfrac{1}{\Gamma(\lambda)} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \tfrac{\Gamma(n-k+\lambda)}{\Gamma(k+1)\Gamma(n-2k+1)} (2x)^{n-2k}, \ x \in \mathbb{R}, n \in \mathbb{N}, \end{aligned}$$

where λ is a positive constant, $\Gamma(x)$ is Euler's gamma function, and $[x]$ stands for the greatest integer less than or equal to x. In a particular case $\lambda = \frac{1}{2}$, $P_n^{\left(\frac{1}{2}\right)}(x) = P_n(x)$ $(n \in \mathbb{N}_0)$ are the Legendre polynomials, and, when $\lambda = 1$, $P_n^{(1)}(\cos\vartheta) = \frac{\sin((n+1)\vartheta)}{\sin\vartheta}$. Over the interval $[-1, 1]$, $P_n^{(\lambda)}(x)$ is bounded in absolute value, and

$$\left| P_n^{(\lambda)}(x) \right| \le P_n^{(\lambda)}(1) = \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)\Gamma(2\lambda)}, \ x \in [-1,1], \ n \in \mathbb{N}_0.$$

Spherical harmonics are special functions on $\mathbb{S}^d$ $(d \ge 2)$, and form an orthonormal basis, so that each function defined on $\mathbb{S}^d$ can be written as a sum of these spherical harmonics. Using the Gram-Schmidt orthogonalization, it is possible to choose $c_{n,d}$ spherical harmonics of degree n in $d+1$ variables that are orthonormal with respect to the invariant measure on $\mathbb{S}^d$ [2], where

$$c_{n,d} = \frac{(2n+d-1)(n+d-2)!}{n!(d-1)!}, \ n \in \mathbb{N}. \tag{3}$$

Denote the members of this orthonormal basis by $S_{n,j}(\mathbf{x})$, $\mathbf{x} \in \mathbb{S}^d$, $j = 1, \ldots, c_{n,d}$, and, in the particular case $d = 2$, $\{Y_{nj}(\mathbf{x}), \mathbf{x} \in \mathbb{S}^2, j = 0, \pm 1, \ldots, \pm n\}$ is also adopted for CMB data in (2). The orthonormal property is

$$\int_{\mathbb{S}^d} S_{n,j}(\mathbf{x}) S_{k,l}(\mathbf{x}) d\sigma(\mathbf{x}) = \delta_{nk}\delta_{jl}, \ n, k \in \mathbb{N}, \ j \in \{1, \ldots, c_{n,d}\}, \ l \in \{1, \ldots, c_{k,d}\},$$

where $\sigma(\cdot)$ represents the invariant measure on $\mathbb{S}^d$ and δ_{nk} is the Kronecker delta function. Under such an orthonormal basis, spherical harmonics connect with the ultraspherical polynomials through the addition formula; see, for instance, Theorem 9.6.3 of [2]. A rephrased version of the addition formula is displayed as (4) below, in the language of the arccos-quasi-quadratic metric space $(\mathbb{D}, \vartheta)$.

Lemma 1.

$$\beta_n^2 \sum_{j=1}^{c_{n,d}} S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_1)\right) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_2)\right) = P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1, \mathbf{x}_2)), \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \tag{4}$$

where

$$\beta_n = \left(\frac{\varpi_d P_n^{\left(\frac{d-1}{2}\right)}(1)}{c_{n,d}} \right)^{\frac{1}{2}} = \left(\frac{(d-1)\varpi_d}{2n+d-1} \right)^{\frac{1}{2}}, \ n \in \mathbb{N}, \tag{5}$$

and $\varpi_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$ *is the surface area of* $\mathbb{S}^d$.

Similarly, Lemma 2 of [33] is rephrased as follows, although it is not clear whether $\left\{ P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)), \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, n \in \mathbb{N}_0 \right\}$ would be a basis of the set of mean square continuous random fields that are metric-dependent on the arccos-quasi-quadratic metric space $(\mathbb{D}, \vartheta)$.

Lemma 2. *If* $\mathbf{U}$ *is a (d+1)-dimensional random vector uniformly distributed on* $\mathbb{S}^d$ *(*$d \geq 2$*), then, for a fixed* $n \in \mathbb{N}$*,*

$$Z_n(\mathbf{x}) = \frac{\sqrt{\varpi_d}}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)} \left(\mathbf{U}' \Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}) \right), \ \mathbf{x} \in \mathbb{D}, \tag{6}$$

is a random field with mean 0 and covariance function

$$\mathrm{cov}(Z_n(\mathbf{x}_1), Z_n(\mathbf{x}_2)) = P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)), \ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \tag{7}$$

where β_n *is defined in (5). Moreover, for* $i \neq j$*,* $\{Z_i(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ *and* $\{Z_j(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ *are uncorrelated; that is*

$$\mathrm{cov}(Z_i(\mathbf{x}_1), Z_j(\mathbf{x}_2)) = 0, \ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}.$$

For $m \geq 2$, a real $m \times m$ symmetric matrix $\mathbf{\Theta} = (\theta_{ij})_{m\times m}$ is said to be conditionally negative definite (or almost negative definite) [41], if the inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \theta_{ij} \leq 0$$

holds for any $a_k \in \mathbb{R}$ $(k = 1,\ldots,m)$ subject to the condition $\sum\limits_{k=1}^{m} a_k = 0$. Examples of conditionally negative definite matrices are

(i) a matrix with identical entries,
(ii) a matrix with entries $\theta_{ij} = \theta_i + \theta_j$,
(iii) a matrix with entries $\theta_{ij} = \max(\theta_i, \theta_j)$,
(iv) a matrix with entries $\theta_{ij} = |\theta_i - \theta_j|$, where $\theta_1,\ldots,\theta_m$ are real numbers,
(v) a matrix with entries $\theta_{ij} = \|\boldsymbol{\theta}_i - \boldsymbol{\theta}_j\|^2$, where $\boldsymbol{\theta}_i \in \mathbb{R}^m, i = 1,\ldots,m$.

For an $m \times m$ matrix $\mathbf{\Theta} = (\theta_{ij})_{m\times m}$, denote by $\exp(-\mathbf{\Theta})$ an $m \times m$ matrix whose ij-th entry is $\exp(-\theta_{ij})$ $i,j = 1,\ldots,m$. The crucial connection between a conditionally negative definite matrix and a positive definite matrix is released in the following lemma, for whose proof we refer the reader to Corollary 2.1 of [41] or Theorems 4.1.3 and 4.1.7 of [3].

Lemma 3. *A real* $m \times m$ *symmetric matrix* $\mathbf{\Theta} = (\theta_{ij})_{m\times m}$ *is conditionally negative definite if and only if one of the following conditions holds:*

(i) *There exist* $a_k \in \mathbb{R}$ *and* $\boldsymbol{\theta}_k \in \mathbb{R}^m$ *(*$k = 1,\ldots,m$*) such that*

$$\theta_{ij} = a_i + a_j + \|\boldsymbol{\theta}_i - \boldsymbol{\theta}_j\|^2, \ i,j = 1,\ldots,m.$$

(ii) *There exist* $a_k \in \mathbb{R}$ *(*$k = 1,\ldots,m$*) and an* $m \times m$ *positive definite matrix* $\mathbf{B} = (b_{ij})_{m\times m}$ *such that*

$$\theta_{ij} = a_i + a_j - b_{ij}, \ i,j = 1,\ldots,m.$$

(iii) $\exp(-\lambda\mathbf{\Theta}) = (\exp(-\lambda\theta_{ij}))_{m\times m}$ *is an* $m \times m$ *positive definite matrix for every nonnegative constant* λ.

3. Time-varying vector random fields on arccos-quasi-quadratic metric space

Let $\mathbb{D}\times\mathbb{T}$ be a spatio-temporal index domain, where $(\mathbb{D},\vartheta)$ is an arccos-quasi-quadratic metric space with the metric (1), and $\mathbb{T}$ is a temporal domain such as $\mathbb{R}$ or $\mathbb{Z}$. This section introduces second-order vector random fields on $\mathbb{D}\times\mathbb{T}$ whose covariance matrix functions depend on the metric $\vartheta(\mathbf{x}_1,\mathbf{x}_2)$ and the time variable as well, via two constructions in Theorems 1 and 2 respectively.

Denote by $\mathbf{I}_m$ an $m\times m$ identity matrix. For a sequence of $m\times m$ matrices $\{\mathbf{B}_n, n\in\mathbb{N}_0\}$, the series $\sum\limits_{n=0}^{\infty}\mathbf{B}_n$ is said to be (entry-by-entry) convergent, if each of its entries is convergent. As an example, the convergence of $\sum\limits_{n=0}^{\infty}\mathbf{B}_nP_n^{\left(\frac{d-1}{2}\right)}(1)$ is equivalent to that of $\sum\limits_{n=0}^{\infty}n^{d-2}\mathbf{B}_n$, because of $P_n^{\left(\frac{d-1}{2}\right)}(1)=\frac{\Gamma(n+d-1)}{\Gamma(n+1)\Gamma(d-1)}\sim\frac{n^{d-2}}{\Gamma(d-1)}, n\to\infty$.

An m-variate Gaussian random field on $\mathbb{D}\times\mathbb{T}$ is constructed in Theorem 1 via an infinite series expression in terms of spherical harmonics, with an infinite series representation for its covariance matrix function by means of the ultraspherical polynomials.

Theorem 1. *Assume that $\{\mathbf{V}_{nj}(t), t\in\mathbb{T}\}$ is an m-variate Gaussian stochastic process with mean function $\mathrm{E}\mathbf{V}_{nj}(t)\equiv\mathbf{0}$ and covariance matrix function*

$$\mathrm{cov}(\mathbf{V}_{nj}(t_1),\mathbf{V}_{nj}(t_2))=\mathbf{B}_n(t_1,t_2),\ \textit{for each } n\in\mathbb{N}, j\in\{1,\ldots,c_{n,d}\},$$

$\{\mathbf{V}_0(t), t\in\mathbb{T}\}$ is an m-variate Gaussian stochastic process with mean $\mathbf{0}$ and covariance matrix function $\mathbf{B}_0(t_1,t_2)$, and that $\{\mathbf{V}_0(t), t\in\mathbb{T}\}$ and $\{\mathbf{V}_{nj}(t), t\in\mathbb{T}\}$, $n\in\mathbb{N}, j\in\{1,\ldots,c_{n,d}\}$, are independent. If $\sum\limits_{n=1}^{\infty}\mathbf{B}_n(t,t)P_n^{\left(\frac{d-1}{2}\right)}(1)$ converges for every $t\in\mathbb{T}$, then

$$\mathbf{Z}(\mathbf{x};t)=\mathbf{V}_0(t)+\sum_{n=1}^{\infty}\beta_n\sum_{j=1}^{c_{n,d}}\mathbf{V}_{nj}(t)S_{n,j}\left(\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x})\right),\ \mathbf{x}\in\mathbb{D},\ t\in\mathbb{T},\tag{8}$$

is an m-variate Gaussian random field on $\mathbb{D}\times\mathbb{T}$, with mean $\mathbf{0}$ and covariance matrix function

$$\begin{aligned}\mathrm{cov}\left(\mathbf{Z}(\mathbf{x}_1;t_1),\mathbf{Z}(\mathbf{x}_2;t_2)\right) &= \sum_{n=0}^{\infty}\mathbf{B}_n(t_1,t_2)P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)),\\ &\qquad \mathbf{x}_1,\mathbf{x}_2\in\mathbb{D},\ t_1,t_2\in\mathbb{T},\end{aligned}\tag{9}$$

where $c_{n,d}$ and $\{\beta_n, n\in\mathbb{N}\}$ are defined by (3) and (5), respectively.

In particular, for a fixed $t_0\in\mathbb{T}$, $\{\mathbf{Z}(\mathbf{x};t_0),\mathbf{x}\in\mathbb{D}\}$ is a purely spatial random field on $\mathbb{D}$ with covariance matrix function $\sum\limits_{n=0}^{\infty}\mathbf{B}_n(t_0,t_0)P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2))$, $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{D}$. Observing that $\mathbf{B}_n(t_0,t_0)$ $(n\in\mathbb{N}_0)$ are positive definite matrices and rewriting them simply as $\mathbf{B}_n$, it leads to the following corollary; see Theorem 3.1 of [36].

Corollary 1.1. *For a sequence $\{\mathbf{B}_n, n\in\mathbb{N}_0\}$ of $m\times m$ positive definite matrices, if $\sum\limits_{n=1}^{\infty}\mathbf{B}_nP_n^{\left(\frac{d-1}{2}\right)}(1)$ converges, then there exists an m-variate Gaussian random field on $\mathbb{D}$ with metric-dependent covariance matrix function $\sum\limits_{n=0}^{\infty}\mathbf{B}_nP_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2))$, $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{D}$.*

The next corollary is obtained, when $\{\mathbf{V}_0(t), t\in\mathbb{T}\}$ and $\{\mathbf{V}_{nj}(t), t\in\mathbb{T}\}$ in (8) are assumed to be stationary on $\mathbb{T}$.

Corollary 1.2 *Under the assumptions of Theorem 1, let* $\{\mathbf{V}_0(t), t \in \mathbb{T}\}$ *and* $\{\mathbf{V}_{nj}(t), t \in \mathbb{T}\}$ *be stationary with*

$$\begin{aligned}&\mathrm{cov}(\mathbf{V}_0(t_1), \mathbf{V}_0(t_2)) = \mathbf{B}_0(t_1 - t_2),\\ &\mathrm{cov}(\mathbf{V}_{nj}(t_1), \mathbf{V}_{nj}(t_2)) = \mathbf{B}_n(t_1 - t_2),\ t_1, t_2 \in \mathbb{T},\ n \in \mathbb{N}, j \in \{1, \dots, c_{n,d}\}.\end{aligned}$$

If $\sum_{n=1}^{\infty} \mathbf{B}_n(0) P_n^{\left(\frac{d-1}{2}\right)}(1)$ *converges, then* (8) *is an* m*-variate Gaussian random field on* $\mathbb{D} \times \mathbb{T}$*, and its covariance matrix function*

$$\begin{aligned}\mathrm{cov}(\mathbf{Z}(\mathbf{x}_1; t_1), \mathbf{Z}(\mathbf{x}_2; t_2)) &= \sum_{n=0}^{\infty} \mathbf{B}_n(t_1 - t_2) P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1, \mathbf{x}_2)),\\ &\qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D},\ t_1, t_2 \in \mathbb{T},\end{aligned} \tag{10}$$

is metric-dependent on $\mathbb{D}$ *and stationary on* $\mathbb{T}$*. Its temporal margin,* $\{\mathbf{Z}(\mathbf{x}_0; t), t \in \mathbb{T}\}$*, is an* m*-variate stationary Gaussian stochastic process on* $\mathbb{T}$*, with mean* $\mathbf{0}$ *and covariance matrix function* $\sum_{n=0}^{\infty} \mathbf{B}_n(t_1 - t_2) P_n^{\left(\frac{d-1}{2}\right)}(1)$*,* $t_1, t_2 \in \mathbb{T}$*, where* $\mathbf{x}_0 \in \mathbb{D}$ *is a fixed point.*

Example 1. In (8) let's choose $\{\mathbf{V}_0(t), t \in \mathbb{Z}\}$ and $\{\mathbf{V}_{nj}(t), t \in \mathbb{T}\}$ as m-variate stationary linear processes [45],

$$\begin{aligned}\mathbf{V}_0(t) &= \sum_{k=0}^{\infty} \mathbf{\Psi}_k \boldsymbol{\varepsilon}_0(t-k),\\ \mathbf{V}_{nj}(t) &= \sum_{k=0}^{\infty} \mathbf{\Psi}_k \boldsymbol{\varepsilon}_{nj}(t-k),\ t \in \mathbf{Z},\ n \in \mathbb{N},\ j \in \{1, \dots, c_{n,d}\},\end{aligned}$$

with mean $\mathbf{0}$ and covariance matrix function

$$\mathbf{B}_n(l) = \begin{cases} \sum_{k=0}^{\infty} \mathbf{\Psi}_k \mathbf{B}_n \mathbf{\Psi}'_{k+l}, & l \in \mathbb{N}_0,\\ (\mathbf{B}_n(-l))', & -l \in \mathbb{N},\end{cases}$$

where $\{\mathbf{\Psi}_k, k \in \mathbb{N}_0\}$ is a sequence of $m \times m$ matrices and $\sum_{k=0}^{\infty} (\mathrm{trace}(\mathbf{\Psi}'_k \mathbf{\Psi}_k))^{\frac{1}{2}} < \infty$, and $\{\boldsymbol{\varepsilon}_0(t), t \in \mathbb{Z}\}$ and $\{\boldsymbol{\varepsilon}_{nj}(t), t \in \mathbb{Z}\}$ are m-variate Gaussian white noise with mean $\mathbf{0}$ and covariance matrix function

$$\mathrm{cov}(\boldsymbol{\varepsilon}_0(t_1), \boldsymbol{\varepsilon}_0(t_2)) = \begin{cases} \mathbf{B}_0, & t_1 = t_2,\\ \mathbf{0}, & t_1 \neq t_2,\ t_1, t_2 \in \mathbb{Z},\end{cases}$$

$$\mathrm{cov}(\boldsymbol{\varepsilon}_{nj}(t_1), \boldsymbol{\varepsilon}_{nj}(t_2)) = \begin{cases} \mathbf{B}_n, & t_1 = t_2,\\ \mathbf{0}, & t_1 \neq t_2,\ t_1, t_2 \in \mathbb{Z}.\end{cases}$$

Under the convergent assumption of $\sum_{n=1}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(1)$, (10) becomes

$$\mathrm{cov}(\mathbf{Z}(\mathbf{x}_1; t_1), \mathbf{Z}(\mathbf{x}_2; t_2)) = \begin{cases} \sum_{k=0}^{\infty} \mathbf{\Psi}_k \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_{k+l}, & t_2 - t_1 = l,\\ & \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, l \in \mathbb{N}_0,\\ \sum_{k=0}^{\infty} \mathbf{\Psi}_{k-l} \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_k, & t_2 - t_1 = -l,\end{cases} \tag{11}$$

where $\mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1, \mathbf{x}_2))$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}$, is an $m \times m$ covariance matrix function on $\mathbb{D}$ by Corollary 1.1.

In particular, if $\mathbf{\Psi}_k = \mathbf{0}$ $(k \ge q+1)$, then $\{\mathbf{V}_0(t), t \in \mathbb{Z}\}$ and $\{\mathbf{V}_{nj}(t), t \in \mathbb{Z}\}$ are m-variate moving average time series of order $q \ge 1$, with covariance matrix function

$$\mathbf{B}_n(l) = \begin{cases} \sum\limits_{h=0}^{q-l} \mathbf{\Psi}_h \mathbf{B}_n \mathbf{\Psi}'_{h+l}, & l = 0, 1, \ldots, q, \\ \sum\limits_{h=0}^{q+l} \mathbf{\Psi}_{h-l} \mathbf{B}_n \mathbf{\Psi}'_h, & l = -1, \ldots, -q, \\ \mathbf{0}, & |l| > q, \ l \in \mathbb{Z}. \end{cases}$$

In this case, (11) reduces to

$$\mathrm{cov}(\mathbf{Z}(\mathbf{x}_1; t), \mathbf{Z}(\mathbf{x}_2; t+l)) = \begin{cases} \sum\limits_{h=0}^{q-l} \mathbf{\Psi}_h \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_{h+l}, & l = 0, 1, \ldots, q, \\ \sum\limits_{h=0}^{q+l} \mathbf{\Psi}_{h-l} \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_h, & l = -1, \ldots, -q, \\ \mathbf{0}, & |l| > q, \ l \in \mathbb{Z}, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}. \end{cases}$$

In another particular case where $\{\mathbf{V}_0(t), t \in \mathbb{Z}\}$ and $\{\mathbf{V}_{nj}(t), t \in \mathbb{Z}\}$ are m-variate first-order autoregressive time series, with covariance matrix function

$$\mathbf{B}_n(l) = \begin{cases} \mathbf{B}_n + \mathbf{\Phi} \mathbf{B}_n(0) \mathbf{\Phi}', & l = 0, \\ \mathbf{B}_n(0) (\mathbf{\Phi}')^l, & l \in \mathbb{N}, \\ \mathbf{\Phi}^{-l} \mathbf{B}_n(0), & -l \in \mathbb{N}, \end{cases}$$

where $\mathbf{\Phi}$ is an $m \times m$ matrix with $\mathrm{trace}(\mathbf{\Phi}'\mathbf{\Phi}) < 1$ and $\mathbf{B}_n(0)$ satisfies the equation $\mathbf{B}_n(0) = \mathbf{B}_n + \mathbf{\Phi} \mathbf{B}_n(0) \mathbf{\Phi}'$, whose solution can be derived by use of the vectorizing operation [45], (11) reduces to

$$\mathrm{cov}(\mathbf{Z}(\mathbf{x}_1; t), \mathbf{Z}(\mathbf{x}_2; t+l)) = \begin{cases} \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) + \mathbf{\Phi} \mathbf{C}_1(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Phi}', & l = 0, \\ \mathbf{C}_1(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) (\mathbf{\Phi}')^l, & l \in \mathbb{N}, \\ \mathbf{\Phi}^{-l} \mathbf{C}_1(\vartheta(\mathbf{x}_1, \mathbf{x}_2)), & -l \in \mathbb{N}, \\ & t \in \mathbb{Z}, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \end{cases}$$

where $\mathbf{C}_1(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) = \sum\limits_{n=0}^{\infty} \mathbf{B}_n(0) P_n^{(\frac{d-1}{2})}(\cos \vartheta(\mathbf{x}_1, \mathbf{x}_2)), \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}$.

While Gaussian random fields are widely adopted for modelling spatial or spatio-temporal data, it is well documented that many cosmological, geological, informational, environmental, physical, and biological systems are highly complex, non-Gaussian, and exhibit non-linear patterns of spatial or spatio-temporal connectivity [11, 18, 44]. Containing the Gaussian case as a particular case, elliptically contoured (or spherically invariant) random fields [21, 30, 56] enjoy the following important properties:

(i) It is well known that all mean square estimation and predicition problems for Gaussian random fields have linear solutions and that Gaussian random fields are closed under linear operations. These two properties do not uniquely characterize the Gaussian one, but they do characterize the class of second-order elliptically contoured random fields, as is observed in [21, 51].

(ii) An elliptically contoured random field may or may not have first-order moments, such as a Student's t or stable one.

(iii) Among all second-order random fields, the class of second-order elliptically contoured random fields is one of the largest, if not the largest, classes that allow for any given correlation structure.

An elliptically contoured random field is essentially a scale mixture of Gaussian random fields, and it is also termed as a type G random field if it is infinitely divisible [39, 47]. Examples of elliptically contoured random fields include Student's t, Cauchy, power-law, exponential power, hyperbolic, hyperbolic cosine ratio, hyperbolic sine ratio, hyperbolic secant, Laplace, logistic, variance Gamma, normal inverse Gaussian, α-stable, K-differenced, K-combined, Linnik, and Mittag-Leffler ones.

Denote the right-hand side function of (9) by $\mathbf{C}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2)$. Being an $m \times m$ covariance matrix function, it certainly satisfies the following two properties:

(i) $(\mathbf{C}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2))' = \mathbf{C}(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1)$, $\mathbf{x}_k \in \mathbb{D}, t_k \in \mathbb{T}$, $k = 1, 2$;

(ii) the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{a}_i' \mathbf{C}(\mathbf{x}_i, t_i; \mathbf{x}_j, t_j) \mathbf{a}_j \geq 0$$

holds for every $n \in \mathbb{N}$, arbitrary $\mathbf{a}_k \in \mathbb{R}^m$, $\mathbf{x}_k \in \mathbb{D}$, and $t_k \in \mathbb{T}$, $k = 1, \ldots, n$.

Such a function can be adopted as the covariance matrix function of an m-variate elliptically contoured random field on $\mathbb{D} \times \mathbb{T}$ and is stated in Corollary 1.3, by Theorem 8 of [30].

Corollary 1.3. *There is an m-variate elliptically contoured random field on $\mathbb{D} \times \mathbb{T}$ with covariance matrix function*

$$\begin{aligned} \operatorname{cov}(\mathbf{Z}(\mathbf{x}_1; t_1), \mathbf{Z}(\mathbf{x}_2; t_2)) &= \sum_{n=0}^{\infty} \mathbf{B}_n(t_1, t_2) P_n^{\left(\frac{d-1}{2}\right)}(\cos \vartheta(\mathbf{x}_1, \mathbf{x}_2)), \\ &\qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t_1, t_2 \in \mathbb{T}. \end{aligned} \tag{12}$$

One of the benefits of the infinite series expansion (8) is for simulation. To get an implementable algorithm, the series expansion has of course to be truncated, so that the simulation algorithm is delivered from the truncation of the infinite series expansion at some finite order. It would be of interest to derive the rate of convergence of such approximations, in terms of some bounds of $\mathbf{B}_n(t_1, t_2)$ for large n, say.

Another infinite series expansion is presented in the next theorem, which is established by [33] in a particular case where $\mathbb{D} = \mathbb{S}^d$. More interestingly, the series representation (15) provides an answer for the open question raised in [33] over the spherical metric space $(\mathbb{S}^d, \vartheta)$.

Theorem 2. *Suppose that $\{\mathbf{V}_n(t), t \in \mathbb{T}\}$ is an m-variate second-order stochastic process with* $\mathrm{E}\mathbf{V}_n(t) \equiv \mathbf{0}$ *and* $\operatorname{cov}(\mathbf{V}_n(t_1), \mathbf{V}_n(t_2)) = \mathbf{B}_n(t_1, t_2)$ *for each fixed $n \in \mathbb{N}_0$, $\mathbf{U}$ is a $(d+1)$-variate random vector uniformly distributed on $\mathbb{S}^d$, and that $\mathbf{U}$, $\{\mathbf{V}_n(t), t \in \mathbb{T}\}$, $n \in \mathbb{N}_0$, are independent. Let $\{\beta_n, n \in \mathbb{N}\}$ be defined by* (5).

If $\sum_{n=1}^{\infty} \mathbf{B}_n(t,t) P_n^{\left(\frac{d-1}{2}\right)}(1)$ converges for every $t \in \mathbb{T}$, then

(i)

$$\mathbf{Z}(\mathbf{x}; t) = \mathbf{V}_0(t) + \sqrt{\varpi_d} \sum_{n=1}^{\infty} \frac{\mathbf{V}_n(t)}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)} \left(\mathbf{U}' \Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}) \right), \ \mathbf{x} \in \mathbb{D}, \ t \in \mathbb{T}, \tag{13}$$

is an m-variate random field on $\mathbb{D} \times \mathbb{T}$, with mean $\mathbf{0}$ *and covariance matrix function*

$$\begin{aligned} \operatorname{cov}(\mathbf{Z}(\mathbf{x}_1; t_1), \mathbf{Z}(\mathbf{x}_2; t_2)) &= \sum_{n=0}^{\infty} \mathbf{B}_n(t_1, t_2) P_n^{\left(\frac{d-1}{2}\right)}(\cos \vartheta(\mathbf{x}_1, \mathbf{x}_2)), \\ &\qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t_1, t_2 \in \mathbb{T}. \end{aligned} \tag{14}$$

(ii)

$$\begin{aligned} \mathbf{Z}(\mathbf{x}; t) = \mathbf{V}_0(t) + \sqrt{\varpi_d} \sum_{n=1}^{\infty} \mathbf{V}_n(t_1) \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\left(\beta_{k,n}^{\left(\frac{d-1}{2}\right)}\right)^{\frac{1}{2}}}{\beta_{n-2k}} P_{n-2k}^{\left(\frac{d-1}{2}\right)} \left(\mathbf{U}' \Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}) \right), \\ \mathbf{x} \in \mathbb{D}, \ t \in \mathbb{T}, \end{aligned} \tag{15}$$

is an m-variate random field on $\mathbb{D} \times \mathbb{T}$, with mean $\mathbf{0}$ and covariance matrix function

$$\begin{aligned} \operatorname{cov}(\mathbf{Z}(\mathbf{x}_1;t_1),\mathbf{Z}(\mathbf{x}_2;t_2)) &= \sum_{n=0}^{\infty} \mathbf{B}_n(t_1,t_2)(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2))^n, \\ & \qquad \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D},\ t_1,t_2 \in \mathbb{T}, \end{aligned} \tag{16}$$

where

$$\beta_{k,n}^{\left(\frac{d-1}{2}\right)} = \frac{n!\left(n-2k+\frac{d-1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{2^n k!\Gamma\left(n-k+\frac{d+1}{2}\right)},\ k=0,1,\ldots,\left[\frac{n}{2}\right].$$

A key ingredient or building block in (13) or (15) is $\mathbf{U}$, a random vector uniformly distributed on $\mathbb{S}^d$. A uniform distribution on $\mathbb{S}^d$ can be easily simulated via standard normal distributions [37], since $\left(\frac{Y_1}{\sqrt{\sum_{k=1}^{d+1} Y_k^2}},\ldots,\frac{Y_{d+1}}{\sqrt{\sum_{k=1}^{d+1} Y_k^2}}\right)$ is uniformly distributed on $\mathbb{S}^d$, if $Y_1,\ldots,Y_{d+1}$ are independent and identically distributed standard normal random variables, according to Theorem 2.3 of [14].

Both the series representations (8) and (13) are useful for modeling and simulation. The simulation algorithm via the truncation of (8) might be less efficient than (13) since, in order to reach the same level of accuracy, one needs a $\left(\sum_{n=0}^{\ell} c_{n,d}+1\right)$-term truncation of the series representation (8), in contrast to an $(\ell+1)$-term (13) that significantly reduces the computational burden. On the other hand, the advantage of (8) is that its finite-dimensional distributions are clearly Gaussian.

It is not sure whether (14) is a general form for $m \times m$ covariance matrix functions on $\mathbb{D} \times \mathbb{T}$ that are metric-dependent over $\mathbb{D}$, even in a particular case $\mathbb{D} = \mathbb{S}^d$ [33]. It may be quite difficult to identify whether a certain function is of the form (9), simply because the expression at its right-hand side may be too complicated to derive. This calls for efficient methods to construct parametric or semiparametric spatio-temporal correlation structures on $\mathbb{D} \times \mathbb{T}$. Some constructing approaches are offered in Section 4.

4. Parametric or semiparametric covariance matrix models

This section illustrates some parametric or semiparametric spatio-temporal correlation structures on $\mathbb{D} \times \mathbb{T}$. Three constructing approaches are offered in Theorem 3 using two ingredients or building blocks: a completely monotone function and a conditionally negative definite matrix, besides the arccos-quasi-quadratic metric $\vartheta(\mathbf{x}_1,\mathbf{x}_2)$ that is a variogram on $\mathbb{D}$ according to Theorem 3 of [34]. While an infinite series expression like (9) may not be available, the existence of such a random field is ensured by Theorem 8 of [30].

By definition, a nonnegative and continuous function $\ell(x)$ is completely monotone on $[0,\infty)$, if it possesses derivatives of all orders and

$$(-1)^n \frac{d^n}{dx^n}\ell(x) \geq 0,\ x>0,\ n\in\mathbb{N}.$$

Theorem 3. *Assume that $\ell(x)$ is a completely monotone function on $[0,\infty)$, $g(x)$ is a strictly positive function on $[0,\infty)$ with a completely monotone derivative, and that $\Theta = (\theta_{ij})_{m\times m}$ is an $m\times m$ conditionally negative definite matrix with nonnegative entries. Then*

(i) *there is an m-variate elliptically contoured random field on* $\mathbb{D}\times\mathbb{R}$ *with direct/cross covariance functions*

$$C_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2) = (g(\vartheta(\mathbf{x}_1,\mathbf{x}_2))+\theta_{ij})^{-\frac{1}{2}}\,\ell\left(\tfrac{(t_1-t_2)^2}{g(\vartheta(\mathbf{x}_1,\mathbf{x}_2))+\theta_{ij}}\right), \quad \mathbf{x}_1,\mathbf{x}_2\in\mathbb{D},\ t_1,t_2\in\mathbb{R},\ i,j=1,\ldots,m; \tag{17}$$

(ii) *there is an m-variate elliptically contoured random field on* $\mathbb{D}\times\mathbb{R}$ *with direct/cross covariance functions*

$$C_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2) = (g(\vartheta(\mathbf{x}_1,\mathbf{x}_2))+\theta_{ij})^{-\frac{1}{2}}\,\ell\left((g(\vartheta(\mathbf{x}_1,\mathbf{x}_2))+\theta_{ij})^{-\frac{1}{2}}\ |t_1-t_2|\right), \quad \mathbf{x}_1,\mathbf{x}_2\in\mathbb{D},\ t_1,t_2\in\mathbb{R},\ i,j=1,\ldots,m; \tag{18}$$

(iii) *there is an m-variate elliptically contoured random field on* $\mathbb{D}\times\mathbb{T}$ *with direct/cross covariance functions*

$$C_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2) = \ell(\vartheta(\mathbf{x}_1,\mathbf{x}_2)+\gamma(t_1,t_2)+\theta_{ij}), \quad \mathbf{x}_1,\mathbf{x}_2\in\mathbb{D},\ t_1,t_2\in\mathbb{T},\ i,j=1,\ldots,m, \tag{19}$$

provided that $\gamma(t_1,t_2)$ *is a variogram on* $\mathbb{T}$.

There is a rich source of completely monotone functions on $[0,\infty)$. For instance, the Laplace transform of every nonnegative random variable is completely monotone.

Example 2. Some parametric examples of completely monotone functions $\ell(x)$ on $[0,\infty)$ are

(i) $\frac{(1+\alpha_1 x^\nu)^{\kappa_1}}{(1+\alpha_2 x^\nu)^{\kappa_2}}$, where $0\le\alpha_1\le\alpha_2$, $0<\kappa_1\le\kappa_2$, and $\nu\in(0,1]$;
(ii) $(x^\nu+\alpha_2)^\kappa-(x^\nu+\alpha_1)^\kappa$, where $0<\alpha_1<\alpha_2$, $\nu\in(0,1]$, and $\kappa\in(0,1]$;
(iii) $\ln(x^\nu+\alpha_2)-\ln(x^\nu+\alpha_1)$, where $0<\alpha_1<\alpha_2$ and $\nu\in(0,1]$;
(iv) $\exp(-\alpha x^\nu)$, where $\alpha>0$ and $0<\nu\le 1$;
(v) $\exp(-\alpha x^\nu)(1+\exp(-\alpha x^\nu))^{-2}$, where $\alpha>0$ and $0<\nu\le1$;
(vi) $\left(\frac{\alpha\sqrt{x}}{\sinh(\alpha\sqrt{x})}\right)^\nu$, where $\alpha>0$ and $\nu>0$;
(vii) $\left(\frac{\tanh(\alpha\sqrt{x})}{\alpha\sqrt{x}}\right)^\nu$, where $\alpha>0$ and $\nu>0$;
(viii) $\left(\frac{\cosh(\alpha_1\sqrt{x})}{\cosh(\alpha_2\sqrt{x})}\right)^\nu$, where $0\le\alpha_1<\alpha_2$ and $\nu>0$;
(ix) $\left(\frac{\sinh(\alpha_1\sqrt{x})}{\sinh(\alpha_2\sqrt{x})}\right)^\nu$, where $0<\alpha_1<\alpha_2$ and $\nu>0$;
(x) $x^{\frac{\nu}{2}}K_\nu(\alpha\sqrt{x})$, where $\alpha>0$, $\nu>0$, and $K_\nu(x)$ is the modified Bessel function of the second kind of order ν;
(xi) $1-\left(\frac{\alpha_2-\alpha_1)x^\nu}{1+\alpha_2 x^\nu}\right)^\kappa$, where $0\le\alpha_1<\alpha_2$ or $0<\alpha_1\le\alpha_2$, $\nu\in(0,1]$, and $\kappa\in(0,1]$;
(xii) the Mittag-Leffler function [48]

$$E_{\alpha,\beta}(-x)=\sum_{n=0}^\infty\frac{(-x)^n}{\Gamma(\alpha n+\beta)},$$

where $0<\alpha\le1$ and $\beta\ge\alpha$.

Next we display three examples of parametric or semiparametric covariance matrix models on $\mathbb{D}\times\mathbb{T}$.

Example 3. Let $\beta>\frac{\sqrt{2}}{2}$ be a constant and let $\nu_k(t)\in(0,1)$ $(k=1,\ldots,m)$ be positive-valued functions on $\mathbb{T}$. Then

$$C_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2)=\frac{\Gamma\left(1-\frac{\nu_i(t)+\nu_j(t)}{2}\right)}{\nu_i(t)+\nu_j(t)}\left\{\beta^{\nu_i(t)+\nu_j(t)}-\left(\sin\frac{\vartheta(\mathbf{x}_1,\mathbf{x}_2)}{2}\right)^{\nu_i(t)+\nu_j(t)}\right\}, \quad \mathbf{x}_1,\mathbf{x}_2\in\mathbb{D},\ t_1,t_2\in\mathbb{T}, i,j=1,\ldots,m,$$

form an $m\times m$ covariance matrix function on $\mathbb{D}\times\mathbb{T}$. To see this, we apply the identity

$$x^\nu=\frac{\nu}{\Gamma(1-\nu)}\int_0^\infty\frac{1-e^{-xu}}{u^{1+\nu}}du,\ x\ge0,\ 0<\nu<1,$$

to rewrite $C_{ij}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2)$ as

$$\begin{aligned}
& C_{ij}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2) \\
= \quad & \frac{\Gamma\left(1-\frac{\nu_i(t_1)+\nu_j(t_2)}{2}\right)}{\alpha_i+\alpha_j} \left\{ \beta^{\nu_i(t_1)+\nu_j(t_2)} - \left(\frac{1-\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)}{2}\right)^{\frac{\nu_i(t_1)+\nu_j(t_2)}{2}} \right\} \\
= \quad & \frac{1}{2}\int_0^\infty \frac{1-\exp\left(-\beta^2 u\right)-\left[1-\exp\left(-\frac{1-\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)}{2}u\right)\right]}{u^{1+\frac{\nu_i(t_1)+\nu_j(t_2)}{2}}} du \\
= \quad & \frac{1}{2}\int_0^\infty \frac{\exp\left(-\frac{u}{2}\right)\exp\left(\frac{\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)}{2}u\right)-\exp\left(-\beta^2 u\right)}{u^{1+\frac{\nu_i(t_1)+\nu_j(t_2)}{2}}} du \\
= \quad & \frac{1}{2}\int_0^\infty \left(\exp\left(-\frac{u}{2}\right)\sum_{k=1}^\infty \frac{u^k\cos^k\vartheta(\mathbf{x}_1,\mathbf{x}_2)}{2^k k!} + \exp\left(-\frac{u}{2}\right) - \exp\left(-\beta^2 u\right)\right) \frac{du}{u^{1+\frac{\nu_i(t_1)+\nu_j(t_2)}{2}}} \\
& \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, t_1, t_2 \in \mathbb{T}, \ i,j = 1, \ldots, m.
\end{aligned}$$

They are direct/cross covariance functions of an m-variate elliptically contoured random field on $\mathbb{D}$ by Theorems 3 and 4 of [31], since an $m \times m$ matrix function with entries $u^{-1-\frac{\nu_i(t_1)+\nu_j(t_2)}{2}}$ is obviously a temporal covariance function on $\mathbb{T}$, and

$$\exp\left(-\frac{u}{2}\right)\sum_{k=1}^\infty \frac{u^k\cos^k\vartheta(\mathbf{x}_1,\mathbf{x}_2)}{2^k k!} + \exp\left(-\frac{u}{2}\right) - \exp\left(-\beta^2 u\right)$$

is a covariance function on $\mathbb{D}$ by Theorem 2, for each fixed $u > 0$.

Example 4. Suppose that $\Theta = (\theta_{ij})_{m\times m}$ is an $m \times m$ conditionally negative definite matrix and all its entries are positive, $\nu_k(t)$ $(k = 1, \ldots, m)$ are positive functions on $\mathbb{T}$, and $0 \le \alpha_1 < \alpha_2$. By Theorems 3 and 4 of [31], there exists an m-variate elliptically contoured random field on $\mathbb{D} \times \mathbb{T}$ with direct/cross covariance functions

$$\begin{aligned}
C_{ij}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2) \quad = \quad & \Gamma(\nu_i(t_1)+\nu_j(t_2)) \left\{ (\vartheta(\mathbf{x}_1,\mathbf{x}_2)+\theta_{ij}+\alpha_1)^{-\nu_i(t_1)-\nu_j(t_2)} \right. \\
& \left. -(\vartheta(\mathbf{x}_1,\mathbf{x}_2)+\theta_{ij}+\alpha_2)^{-\nu_i(t_1)-\nu_j(t_2)} \right\}, \\
& \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t_1, t_2 \in \mathbb{R}, \ i,j = 1, \ldots, m,
\end{aligned}$$

since

$$\begin{aligned}
C_{ij}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2) = \int_0^\infty \exp(-\vartheta(\mathbf{x}_1,\mathbf{x}_2)u)u^{\nu_i(t_1)+\nu_j(t_2)-1}\exp(-\theta_{ij}u)\left(e^{-\alpha_1 u}-e^{-\alpha_2 u}\right) du, \\
\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t_1, t_2 \in \mathbb{R}, \ i,j = 1, \ldots, m,
\end{aligned}$$

where $\exp(-\vartheta(\mathbf{x}_1,\mathbf{x}_2)u)$ is a covariance function on $\mathbb{D}$ since $\vartheta(\mathbf{x}_1,\mathbf{x}_2)$ is a variogram, an $m \times m$ matrix function with entries $u^{\nu_i(t_1)+\nu_j(t_2)-1}$ is a covariance matrix function on $\mathbb{T}$, and an $m \times m$ matrix with entries $\exp(-\theta_{ij}u)$ is positive definite by Lemma 3, for each fixed $u > 0$.

Example 5. If α is a positive constant, and $\Theta = (\theta_{ij})_{m\times m}$ is an $m \times m$ conditionally negative definite matrix and all its entries are positive, then there exists an m-variate elliptically contoured random field on $\mathbb{D} \times \mathbb{T}$ with direct/cross covariance functions

$$\begin{aligned}
C_{ij}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2) \quad = \quad & \exp(-\alpha|t_1-t_2|)\,\mathrm{Erfc}\left(\sqrt{\vartheta(\mathbf{x}_1,\mathbf{x}_2)+\theta_{ij}} - \frac{\alpha|t_1-t_2|}{2\sqrt{\vartheta(\mathbf{x}_1,\mathbf{x}_2)+\theta_{ij}}}\right) \\
& + \exp(\alpha|t_1-t_2|)\,\mathrm{Erfc}\left(\sqrt{\vartheta(\mathbf{x}_1,\mathbf{x}_2)+\theta_{ij}} + \frac{\alpha|t_1-t_2|}{2\sqrt{\vartheta(\mathbf{x}_1,\mathbf{x}_2)+\theta_{ij}}}\right), \\
& \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t_1, t_2 \in \mathbb{R}, \ i,j = 1, \ldots, m,
\end{aligned}$$

by Theorems 3 and 4 of [31]. To see this, we apply the identity (see page 15 of [4])

$$\begin{aligned}
& \frac{\pi}{2}\int_0^\infty \cos(u\omega)\exp\left(-(1+\omega^2)\beta\right)\frac{d\omega}{1+\omega^2} \\
= \quad & \exp(-u)\,\mathrm{Erfc}\left(\sqrt{\beta}-\frac{u}{2\sqrt{\beta}}\right) + \exp(u)\,\mathrm{Erfc}\left(\sqrt{\beta}+\frac{u}{2\sqrt{\beta}}\right), \ u \in \mathbb{R}, \beta \ge 0,
\end{aligned}$$

to rewrite $C_{ij}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2)$ as

$$
\begin{aligned}
C_{ij}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2) = \frac{\pi}{2} \int_0^\infty & \cos(\alpha(t_1 - t_2)\omega) \exp\left(-\vartheta(\mathbf{x}_1, \mathbf{x}_2)(1+\omega^2)\right) \\
& \times \exp\left(-\theta_{ij}(1+\omega^2)\right) \frac{d\omega}{1+\omega^2}, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t_1, t_2 \in \mathbb{R}, \ i, j = 1, \ldots, m,
\end{aligned}
$$

where $\cos(\alpha(t_1 - t_2)\omega)$ is a covariance function on $\mathbb{R}$, $\exp\left(-\vartheta(\mathbf{x}_1, \mathbf{x}_2)(1+\omega^2)\right)$ is a covariance function on $\mathbb{D}$, and an $m \times m$ matrix with entries $\exp\left(-\theta_{ij}(1+\omega^2)\right)$ is positive definite by Lemma 3, for each fixed $\omega \geq 0$.

5. The extension problem

While there are rich sources of multivariate time series models on $\mathbb{Z}$ for use [45], it is often of interest to extend their index domain from $\mathbb{Z}$ to $\mathbb{R}$. Given an $m \times m$ covariance matrix function on $\mathbb{D} \times \mathbb{Z}$ that is metric-dependent on $\mathbb{D}$ and stationary on $\mathbb{Z}$, $\mathbf{C}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t)$ say, the extension problem addressed in this section is: is it possible to extend its index domain from $\mathbb{D} \times \mathbb{Z}$ to $\mathbb{D} \times \mathbb{R}$ so that $\mathbf{C}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t)$ is an $m \times m$ covariance matrix function on $\mathbb{D} \times \mathbb{R}$ that is metric-dependent on $\mathbb{D}$ and stationary on $\mathbb{R}$? An answer is given in Theorem 4, under the assumption that $\mathbf{C}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t)$ is of the form (10) where $\mathbb{T} = \mathbb{Z}$.

Theorem 4. *Suppose that $\mathbf{B}_n(t)$ is an $m \times m$ stationary covariance matrix function on $\mathbb{Z}$, for each $n \in \mathbb{N}_0$. If $\sum_{n=1}^{\infty} \mathbf{B}_n(0) P_n^{\left(\frac{d-1}{2}\right)}(1)$ converges and*

$$
\mathbf{C}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t) = \sum_{n=0}^{\infty} \mathbf{B}_n(t) P_n^{\left(\frac{d-1}{2}\right)}(\cos \vartheta(\mathbf{x}_1, \mathbf{x}_2)), \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t \in \mathbb{Z}, \tag{20}
$$

is an $m \times m$ covariance matrix function on $\mathbb{D} \times \mathbb{Z}$ that is stationary on $\mathbb{Z}$, then there exists an $m \times m$ covariance matrix function $\tilde{\mathbf{C}}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t)$ on $\mathbb{D} \times \mathbb{R}$ that is metric-dependent on $\mathbb{D}$ and stationary on $\mathbb{R}$, which is identical to $\mathbf{C}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t)$ over $\mathbb{D} \times \mathbb{Z}$, i.e.,

$$
\tilde{\mathbf{C}}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t) = \mathbf{C}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t), \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t \in \mathbb{Z}. \tag{21}
$$

There might be several approaches to verify the existence of $\tilde{\mathbf{C}}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t)$ on $\mathbb{D} \times \mathbb{R}$ with the desired properties. The idea here is to precisely construct a sequence $\{\tilde{\mathbf{B}}_n(t), n \in \mathbb{N}_0\}$ of $m \times m$ stationary covariance matrix functions on $\mathbb{R}$ satisfying

$$
\tilde{\mathbf{B}}_n(t) = \mathbf{B}_n(t), \ t \in \mathbb{Z}, \ n \in \mathbb{N}_0, \tag{22}
$$

and then to formulate $\tilde{\mathbf{C}}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t)$ as follows:

$$
\tilde{\mathbf{C}}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t) = \sum_{n=0}^{\infty} \tilde{\mathbf{B}}_n(t) P_n^{\frac{d-1}{2}}(\cos \vartheta(\mathbf{x}_1, \mathbf{x}_2)), \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \ t_1, t_2 \in \mathbb{R},
$$

which is well-defined since $\sum_{n=1}^{\infty} \tilde{\mathbf{B}}_n(0) P_n^{\frac{d-1}{2}}(1) = \sum_{n=1}^{\infty} \mathbf{B}_n(0) P_n^{\frac{d-1}{2}}(1) < \infty$. Clearly, (21) holds for such a construction of $\tilde{\mathbf{C}}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t)$. For each $n \in \mathbb{N}_0$, one of possible formulations of $\{\tilde{\mathbf{B}}_n(t), t \in \mathbb{R}$, is

$$
\tilde{\mathbf{B}}_n(t) = (l+1-t)^\nu \mathbf{B}_n(l) + (t-l)^\nu \mathbf{B}_n(l+1), \ l \leq t \leq l+1, \ l \in \mathbb{Z}, \ t \in \mathbb{R}, \tag{23}
$$

which plainly enjoys the property (22), where ν is a positive constant. This actually provides an efficient method for constructing covariance matrix models on $\mathbb{D} \times \mathbb{R}$ based on those on $\mathbb{D} \times \mathbb{Z}$.

Example 6. In (23) let $\mathbf{B}_n(l)$ be the covariance matrix function of an m-variate moving average time series of order $q \geq 1$ (see Example 1),

$$\mathbf{B}_n(l) = \begin{cases} \sum_{h=0}^{q-l} \mathbf{\Psi}_h \mathbf{B}_n \mathbf{\Psi}'_{h+l}, & l = 0, 1, \ldots, q, \\ \sum_{h=0}^{q+l} \mathbf{\Psi}_{h-l} \mathbf{B}_n \mathbf{\Psi}'_h, & l = -1, \ldots, -q, \\ \mathbf{0}, & |l| > q, \ l \in \mathbb{Z}, \end{cases}$$

it yields

$$\tilde{\mathbf{B}}_n(t) = \begin{cases} (q+1+t)^\nu \mathbf{\Psi}_q \mathbf{B}_n \mathbf{\Psi}'_0, & -q-1 \leq t \leq -q, \\ (l+1-t)^\nu \sum_{h=0}^{q+l} \mathbf{\Psi}_{h-l} \mathbf{B}_n \mathbf{\Psi}'_h & \\ \qquad +(t-l)^\nu \sum_{h=0}^{q+l+1} \mathbf{\Psi}_{h-l} \mathbf{B}_n \mathbf{\Psi}'_h, & \\ & l \leq t \leq l+1, \ l = -q+1, \ldots, -1, \\ (l+1-t)^\nu \sum_{h=0}^{q-l} \mathbf{\Psi}_h \mathbf{B}_n \mathbf{\Psi}'_{h+l} & \\ \qquad +(t-l)^\nu \sum_{h=0}^{q-l-1} \mathbf{\Psi}_h \mathbf{B}_n \mathbf{\Psi}'_{h+l+1}, & \\ & l \leq t \leq l+1, \ l = 0, 1, \ldots, q-1, \\ (q+1-t)^\nu \mathbf{\Psi}_0 \mathbf{B}_n \mathbf{\Psi}'_q, & q \leq t \leq q+1, \\ \mathbf{0}, & |t| > q+1, \end{cases}$$

which may be regarded as the covariance matrix function of an m-variate continuous-time moving average process of order q, for each $n \in \mathbb{N}_0$. The resulting covariance matrix function on $\mathbb{D} \times \mathbb{R}$ is

$$\tilde{\mathbf{C}}(\vartheta(\mathbf{x}_1, \mathbf{x}_2); t) = \begin{cases} (q+1+t)^\nu \mathbf{\Psi}_q \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_0, & -q-1 \leq t \leq -q, \\ (l+1-t)^\nu \sum_{h=0}^{q+l} \mathbf{\Psi}_{h-l} \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_h & \\ \qquad +(t-l)^\nu \sum_{h=0}^{q+l+1} \mathbf{\Psi}_{h-l} \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_h, & \\ & l \leq t \leq l+1, \ l = -q+1, \ldots, -1, \\ (l+1-t)^\nu \sum_{h=0}^{q-l} \mathbf{\Psi}_h \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_{h+l} & \\ \qquad +(t-l)^\nu \sum_{h=0}^{q-l-1} \mathbf{\Psi}_h \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_{h+l+1}, & \\ & l \leq t \leq l+1, \ l = 0, 1, \ldots, q-1, \\ (q+1-t)^\nu \mathbf{\Psi}_0 \mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) \mathbf{\Psi}'_q, & q \leq t \leq q+1, \\ \mathbf{0}, & |t| > q+1, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \end{cases}$$

where $\mathbf{C}_0(\vartheta(\mathbf{x}_1, \mathbf{x}_2)) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(\cos \vartheta(\mathbf{x}_1, \mathbf{x}_2)), \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}$, is an $m \times m$ covariance matrix function on $\mathbb{D}$ by Corollary 1.

6. Proof

6.1. **Proof of Lemma 1.** Over $\mathbb{S}^d$ the addition formula connecting spherical harmonics with the ultraspherical polynomials reads

$$\beta_n^2 \sum_{j=1}^{c_{n,d}} S_{n,j}(\mathbf{x}_1) S_{n,j}(\mathbf{x}_2) = P_n^{\left(\frac{d-1}{2}\right)}(\mathbf{x}_1'\mathbf{x}_2), \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d; \tag{24}$$

see, for instance, Theorem 9.6.3 of [2]. If $\mathbf{x}_k \in \mathbb{D}$ $(k=1,2)$, then, by the definition of the arccos-quasi-quadratic metric,

$$\|\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_k)\|^2 = (\mathbf{w}(\mathbf{x}_k))' \Sigma \mathbf{w}(\mathbf{x}_k) = 1,$$

i.e., $\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_k) \in \mathbb{S}^d$. Substituting $\mathbf{x}_k$ in (24) by $\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_k)$ $(k=1,2)$ results in (4).

6.2. **Proof of Lemma 2.** According to Lemma 2 of [33], the random field

$$\left\{ \frac{\sqrt{\varpi_d}}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}(\mathbf{U}'\mathbf{x}), \mathbf{x} \in \mathbb{S}^d \right\}$$

has mean 0 and covariance function $P_n^{\left(\frac{d-1}{2}\right)}(\mathbf{x}_1'\mathbf{x}_2)$, and, for $i \neq j$,

$$\left\{ \frac{\sqrt{\varpi_d}}{\beta_i} P_i^{\left(\frac{d-1}{2}\right)}(\mathbf{U}'\mathbf{x}), \mathbf{x} \in \mathbb{S}^d \right\} \text{ and } \left\{ \frac{\sqrt{\varpi_d}}{\beta_j} P_j^{\left(\frac{d-1}{2}\right)}(\mathbf{U}'\mathbf{x}), \mathbf{x} \in \mathbb{S}^d \right\}$$

are uncorrelated. Observing that $\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_k) \in \mathbb{S}^d$ whenever $\mathbf{x}_k \in \mathbb{D}$ $(k=1,2)$, we obtain

$$\begin{aligned}
&\operatorname{cov}(Z_n(\mathbf{x}_1), Z_n(\mathbf{x}_2)) \\
&\quad = \operatorname{cov}\left(\frac{\sqrt{\varpi_d}}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_1)\right), \frac{\sqrt{\varpi_d}}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_2)\right)\right) \\
&\quad = P_n^{\left(\frac{d-1}{2}\right)}\left(\left(\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_1)\right)'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_2)\right) = P_n^{\left(\frac{d-1}{2}\right)}(\cos(\vartheta(\mathbf{x}_1,\mathbf{x}_2)), \ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}.
\end{aligned}$$

Clearly, $\mathrm{E}Z_n(\mathbf{x}) = 0, \mathbf{x} \in \mathbb{D}$, and, for $i \neq j$, $\operatorname{cov}(Z_i(\mathbf{x}_1), Z_j(\mathbf{x}_2)) = 0, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}$.

6.3. **Proof of Theorem 1.** The (entry-by-entry) convergence of $\sum_{n=1}^{\infty} \mathbf{B}_n(t,t)P_n^{\left(\frac{d-1}{2}\right)}(1)$ implies that $\sum_{n=1}^{\infty} \mathbf{B}_n(t_1,t_2)P_n^{\left(\frac{d-1}{2}\right)}(1)$ is absolutely convergent (entry-by-entry) for all $t_k \in \mathbb{T}$, $k=1,2$. To see this, let $\mathbf{V}_{nj}(t) = (V_{nj,1}(t), \ldots, V_{nj,m}(t))', t \in \mathbb{T}, n \in \mathbb{N}, j \in \{1,\ldots,c_{n,d}\}$. It follows from the Cauchy-Schwarz inequality and the inequality of arithmetic and geometric means that

$$\begin{aligned}
|b_{ik,n}(t_1,t_2)| &= |\operatorname{cov}(V_{nj,i}(t_1), V_{nj,k}(t_2))| \\
&\le \sqrt{\operatorname{var}(V_{nj,i}(t_1))\operatorname{var}(V_{nj,k}(t_2)))} \\
&\le \tfrac{\operatorname{var}(V_{nj,i}(t_1)) + \operatorname{var}(V_{nj,k}(t_2)))}{2} \\
&= \tfrac{b_{ii,n}(t_1,t_1) + b_{kk,n}(t_2,t_2)}{2}, \ t_1,t_2 \in \mathbb{T}, \ i,k = 1,\ldots,m,
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{n=0}^{\infty} |b_{ik,n}(t_1,t_2)| P_n^{\left(\frac{d-1}{2}\right)}(1) \le \frac{1}{2}\sum_{n=0}^{\infty} |b_{ii,n}(t_1,t_2)| P_n^{\left(\frac{d-1}{2}\right)}(1) \\
+ \frac{1}{2}\sum_{n=0}^{\infty} |b_{kk,n}(t_1,t_2)| P_n^{\left(\frac{d-1}{2}\right)}(1) < \infty,
\end{aligned}$$

since $P_n^{\left(\frac{d-1}{2}\right)}(1) > 0$.

Under the convergent assumption of $\sum\limits_{n=1}^{\infty} \mathbf{B}_n(t,t) P_n^{\left(\frac{d-1}{2}\right)}(1)$ for every $t \in \mathbb{T}$, the infinite series at the right-hand side of (8) is convergent in mean square over $\mathbf{x} \in \mathbb{D}$ and $t \times \mathbb{T}$, since the independent assumption among $\mathbf{V}_{nj}(t)$s implies

$$\begin{aligned}
& \mathrm{E}\left(\sum_{n=n_1}^{n_1+n_2} \beta_n \sum_{j=1}^{c_{n,d}} \mathbf{V}_{nj}(t) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x})\right)\right)\left(\sum_{k=n_1}^{n_1+n_2} \beta_k \sum_{i=1}^{c_{k,d}} \mathbf{V}_{ki}(t) S_{k,i}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x})\right)\right)' \\
= & \sum_{n=n_1}^{n_1+n_2} \sum_{k=n_1}^{n_1+n_2} \beta_n \beta_k \sum_{j=1}^{c_{n,d}} \sum_{i=1}^{c_{k,d}} \mathrm{E}\left\{\mathbf{V}_{nj}(t)(\mathbf{V}_{ki}(t))'\right\} S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x})\right) S_{k,i}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x})\right) \\
= & \sum_{n=n_1}^{n_1+n_2} \mathbf{B}_n(t,t) \beta_n^2 \sum_{j=1}^{c_{n,d}} S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x})\right) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x})\right) \\
= & \sum_{n=n_1}^{n_1+n_2} \mathbf{B}_n(t,t) P_n^{\left(\frac{d-1}{2}\right)}(1) \\
\to & \ \mathbf{0}, \ n_1, n_2 \to \infty,
\end{aligned}$$

where the third equality is due to Lemma 1. Obviously, the mean vector function $\mathrm{E}\mathbf{Z}(\mathbf{x};t)$ is identical to $\mathbf{0}$. It follows from Lemma 1 that

$$\begin{aligned}
\operatorname{cov}(\mathbf{Z}(\mathbf{x}_1;t_1), \mathbf{Z}(\mathbf{x}_2;t_2)) = \mathrm{E} & \left(\mathbf{V}_0(t_1) + \sum_{n=1}^{\infty} \beta_n \sum_{j=1}^{c_{n,d}} \mathbf{V}_{nj}(t_1) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_1)\right)\right) \\
& \times \left(\mathbf{V}_0(t_2) + \sum_{k=1}^{\infty} \beta_k \sum_{i=1}^{c_{k,d}} \mathbf{V}_{ki}(t_2) S_{k,i}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_2)\right)\right)' \\
& = \mathrm{E}\left\{\mathbf{V}_0(t_1)(\mathbf{V}_0(t_2))'\right\} + \\
\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \beta_n \beta_k \sum_{j=1}^{c_{n,d}} \sum_{i=1}^{c_{k,d}} & \mathrm{E}\left\{\mathbf{V}_{nj}(t_1)(\mathbf{V}_{ki}(t_2))'\right\} S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_1)\right) S_{k,i}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_2)\right) \\
& = \mathbf{B}_0(t_1,t_2) + \sum_{n=1}^{\infty} \mathbf{B}_n(t_1,t_2) \beta_n^2 \sum_{j=1}^{c_{n,d}} S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_1)\right) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_2)\right) \\
& = \mathbf{B}_0(t_1,t_2) + \sum_{n=1}^{\infty} \mathbf{B}_n(t_1,t_2) P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)), \ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{T}.
\end{aligned}$$

To show that $\{\mathbf{Z}(\mathbf{x};t), \mathbf{x} \in \mathbb{D}, t \in \mathbb{T}\}$ is an m-variate Gaussian random field, we just need to take a look at its finite-dimensional characteristic functions. In fact, for every $l \in \mathbb{N}$ and arbitrary $\mathbf{x}_k \in \mathbb{D}$ and $t_k \in \mathbb{T}$ $(k = 1, \ldots, l)$, the characteristic function of an lm-variate random vector $(\mathbf{Z}'(\mathbf{x}_1;t_1), \ldots, \mathbf{Z}'(\mathbf{x}_l;t_l))'$ is

$$\begin{aligned}
& \mathrm{E}\exp\left(\imath \sum_{k=1}^{l} \boldsymbol{\omega}_k' \mathbf{Z}(\mathbf{x}_k;t_k)\right) \\
& = \mathrm{E}\exp\left(\imath \sum_{k=1}^{l} \boldsymbol{\omega}_k' \mathbf{V}_0(t_k)\right) \mathrm{E}\exp\left(\imath \sum_{n=1}^{\infty} \beta_n \sum_{j=1}^{c_{n,d}} \sum_{k=1}^{l} \boldsymbol{\omega}_k' \mathbf{V}_{nj}(t_k) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_k)\right)\right) \\
& = \mathrm{E}\exp\left(\imath \sum_{k=1}^{l} \boldsymbol{\omega}_k' \mathbf{V}_0(t_k)\right) \prod_{n=1}^{\infty} \prod_{j=1}^{c_{n,d}} \mathrm{E}\exp\left(\imath \beta_n \sum_{k=1}^{l} \boldsymbol{\omega}_k' \mathbf{V}_{nj}(t_k) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_k)\right)\right) \\
& \qquad = \exp\left(-\frac{1}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \operatorname{cov}(\mathbf{V}_0(t_k), \mathbf{V}_0(t_{k'})) \boldsymbol{\omega}_{k'}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{n=1}^{\infty} \prod_{j=1}^{c_{n,d}} \exp\left(-\frac{\beta_n^2}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \operatorname{cov}(\mathbf{V}_{nj}(t_k) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_k)\right), \right.\\
&\qquad \left. \mathbf{V}_{nj}(t_{k'}) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_{k'})\right) \boldsymbol{\omega}_{k'} \right)\\
&= \exp\left(-\frac{1}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \mathbf{B}_0(t_k, t_{k'}) \boldsymbol{\omega}_{k'} \right)\\
&\times \prod_{n=1}^{\infty} \prod_{j=1}^{c_{n,d}} \exp\left(-\frac{\beta_n^2}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \mathbf{B}_n(t_k, t_{k'}) \boldsymbol{\omega}_{k'} S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_k)\right) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_{k'})\right) \right)\\
&= \exp\left(-\frac{1}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \mathbf{B}_0(t_k, t_{k'}) \boldsymbol{\omega}_{k'} \right)\\
&\times \prod_{n=1}^{\infty} \exp\left(-\frac{1}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \mathbf{B}_n(t_k, t_{k'}) \boldsymbol{\omega}_{k'} \beta_n^2 \sum_{j=1}^{c_{n,d}} S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_k)\right) S_{n,j}\left(\Sigma^{\frac{1}{2}} \mathbf{w}(\mathbf{x}_{k'})\right) \right)\\
&= \exp\left(-\frac{1}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \mathbf{B}_0(t_k, t_{k'}) \boldsymbol{\omega}_{k'} \right)\\
&\times \prod_{n=1}^{\infty} \exp\left(-\frac{1}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \mathbf{B}_n(t_k, t_{k'}) \boldsymbol{\omega}_{k'} P_n^{\frac{d-1}{2}}(\cos\vartheta(\mathbf{x}_k, \mathbf{x}_{k'})) \right)\\
&= \exp\left\{ -\frac{1}{2} \sum_{k=1}^{l} \sum_{k'=1}^{l} \boldsymbol{\omega}_k' \left[\mathbf{B}_0(t_k, t_{k'}) + \sum_{n=1}^{\infty} \mathbf{B}_n(t_k, t_{k'}) P_n^{\frac{d-1}{2}}(\cos\vartheta(\mathbf{x}_k, \mathbf{x}_{k'})) \right] \boldsymbol{\omega}_{k'} \right\}\\
&\qquad \boldsymbol{\omega}_k \in \mathbb{R}^m,\ k = 1, \ldots, m,
\end{aligned}
$$

where the sixth equality follows from Lemma 1, and $\imath$ represents the imaginary unit.

6.4. **Proof of Theorem 2.** (i) Similar to in the proof of Theorem 1, it can be verified that the (entry-by-entry) convergence of

$$\sum_{n=0}^{\infty} \mathbf{B}_n(t,t) P_n^{\left(\frac{d-1}{2}\right)}(1)$$

implies that

$$\sum_{n=0}^{\infty} \mathbf{B}_n(t_1,t_2) P_n^{\left(\frac{d-1}{2}\right)}(1)$$

is absolutely convergent (entry-by-entry) for all $t_k \in \mathbb{T}$, $k = 1, 2$. The right-hand side series of (14) converges in mean square, since $\mathbf{U}$, $\{\mathbf{V}_n(t), t \in \mathbb{T}\}$, $n \in \mathbb{N}_0$ are independent, and

$$
\begin{aligned}
&\varpi_d \mathrm{E}\left(\sum_{n=n_1}^{n_1+n_2} \frac{\mathbf{V}_n(t)}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x})\right) \right) \left(\sum_{k=n_1}^{n_1+n_2} \frac{\mathbf{V}_k(t)}{\beta_k} P_k^{\left(\mathbf{U}'\frac{d-1}{2}\right)}\left(\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x})\right) \right)'\\
&= \sum_{n=n_1}^{n_1+n_2} \sum_{k=n_1}^{n_1+n_2} \mathrm{E}\left\{\mathbf{V}_n(t)(\mathbf{V}_k(t))'\right\}\\
&\times \mathrm{E}\left\{ \frac{\sqrt{\varpi_d}}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x})\right) \frac{\sqrt{\varpi_d}}{\beta_k} P_k^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x})\right) \right\}\\
&= \sum_{n=n_1}^{n_1+n_2} \mathrm{E}\left\{\mathbf{V}_n(t)(\mathbf{V}_n(t))'\right\} \mathrm{E}\left\{ \frac{\sqrt{\varpi_d}}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x})\right) \frac{\sqrt{\varpi_d}}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x})\right) \right\}
\end{aligned}
$$

$$= \sum_{n=n_1}^{n_1+n_2} \mathbf{B}_n(t,t) P_n^{\left(\frac{d-1}{2}\right)}(1) \to \mathbf{0}, \;\; n_1, n_2 \to \infty,$$

where the third equality is obtained from Lemma 2. The mean function of $\{\mathbf{Z}(\mathbf{x};t), \mathbf{x} \in \mathbb{D}, t \in \mathbb{T}\}$ is clearly identical to $\mathbf{0}$, and its covariance matrix function is

$$\begin{aligned}
&\operatorname{cov}(\mathbf{Z}(\mathbf{x}_1;t_1), \mathbf{Z}(\mathbf{x}_2;t_2)) \\
&= \mathrm{E}\left\{\left(\mathbf{V}_0(t_1) + \sqrt{\varpi_d}\sum_{n=1}^{\infty}\frac{\mathbf{V}_n(t_1)}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_1)\right)\right)\right. \\
&\quad \left.\times \left(\mathbf{V}_0(t_2) + \sqrt{\varpi_d}\sum_{l=1}^{\infty}\frac{\mathbf{V}_l(t_2)}{\beta_l} P_l^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_2)\right)\right)'\right\} \\
&= \mathrm{E}\left\{\mathbf{V}_0(t_1)(\mathbf{V}_0(t_2))'\right\} + \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\mathrm{E}\left\{\mathbf{V}_n(t_1)(\mathbf{V}_l(t_2))'\right\} \\
&\quad \times \mathrm{E}\left\{\frac{\sqrt{\varpi_d}}{\beta_n} P_n^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_1)\right)\frac{\sqrt{\varpi_d}}{\beta_l} P_l^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_2)\right)\right\} \\
&= \mathbf{B}_0(t_1,t_2) + \sum_{n=1}^{\infty}\mathbf{B}_n(t_1,t_2) P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)), \;\; \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \;\; t_1,t_2 \in \mathbb{T},
\end{aligned}$$

where the last equality is due to Lemma 2 and the assumption $\mathrm{E}\left\{\mathbf{V}_n(t_1)(\mathbf{V}_l(t_2))'\right\} = \mathbf{0}$ $(n \neq l)$.

(ii) It suffices to verify that the covariance matrix function of (15) equals (16), while the rest is analogous to the Part (i). In fact, we have

$$\begin{aligned}
&\operatorname{cov}(\mathbf{Z}(\mathbf{x}_1;t_1), \mathbf{Z}(\mathbf{x}_2;t_2)) \\
&= \mathrm{E}\left\{\left(\mathbf{V}_0(t_1) + \sqrt{\varpi_d}\sum_{n=1}^{\infty}\mathbf{V}_n(t_1)\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\left(\beta_{k,n}^{\left(\frac{d-1}{2}\right)}\right)^{\frac{1}{2}}}{\beta_{n-2k}} P_{n-2k}^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_1)\right)\right)\right. \\
&\quad \left.\times \left(\mathbf{V}_0(t_2) + \sqrt{\varpi_d}\sum_{l=1}^{\infty}\mathbf{V}_l(t_2)\sum_{j=0}^{\left[\frac{l}{2}\right]}\frac{\left(\beta_{j,l}^{\left(\frac{d-1}{2}\right)}\right)^{\frac{1}{2}}}{\beta_{l-2j}} P_{l-2j}^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_2)\right)\right)'\right\} \\
&= \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\mathrm{E}\left\{\mathbf{V}_n(t_1)(\mathbf{V}_l(t_2))'\right\}\sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{l}{2}\right]}\left(\beta_{k,n}^{\left(\frac{d-1}{2}\right)}\beta_{j,l}^{\left(\frac{d-1}{2}\right)}\right)^{\frac{1}{2}} \\
&\quad \times \mathrm{E}\left\{\frac{\sqrt{\varpi_d}}{\beta_{n-2k}} P_{n-2k}^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_1)\right)\frac{\sqrt{\varpi_d}}{\beta_{l-2j}} P_{l-2j}^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_2)\right)\right\} \\
&\quad + \mathrm{E}\left\{\mathbf{V}_0(t_1)(\mathbf{V}_0(t_2))'\right\} = \mathbf{B}_0(t_1,t_2) + \sum_{n=1}^{\infty}\mathrm{E}\left\{\mathbf{V}_n(t_1)(\mathbf{V}_l(t_2))'\right\}\sum_{k=0}^{\left[\frac{n}{2}\right]}\beta_{k,n}^{\left(\frac{d-1}{2}\right)} \\
&\quad \times \mathrm{E}\left\{\frac{\sqrt{\varpi_d}}{\beta_{n-2k}} P_{n-2k}^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_1)\right)\frac{\sqrt{\varpi_d}}{\beta_{n-2k}} P_{n-2k}^{\left(\frac{d-1}{2}\right)}\left(\mathbf{U}'\Sigma^{\frac{1}{2}}\mathbf{w}(\mathbf{x}_2)\right)\right\} \\
&= \mathbf{B}_0(t_1,t_2) + \sum_{n=1}^{\infty}\mathbf{B}_n(t_1,t_2)\sum_{k=0}^{\left[\frac{n}{2}\right]}\beta_{k,n}^{\left(\frac{d-1}{2}\right)} P_{n-2k}^{\left(\frac{d-1}{2}\right)}(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2))
\end{aligned}$$

$$= \mathbf{B}_0(t_1,t_2) + \sum_{n=1}^{\infty} \mathbf{B}_n(t_1,t_2) \left(\cos\vartheta(\mathbf{x}_1,\mathbf{x}_2)\right)^n, \ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{T},$$

where the third equality follows from Lemma 2 and $\mathrm{E}\left\{\mathbf{V}_n(t_1)(\mathbf{V}_l(t_2))'\right\} = \mathbf{0}$ $(n \neq l)$, the fourth one is from Lemma 2, and the last one is from Lemma 3 of [33].

6.5. **Proof of Theorem 3.** (i) Since $\ell(x)$ is completely monotone on $[0,\infty)$, it possesses an integral representation

$$\ell(x) = \int_0^{\infty} \exp(-xu)dF(u), \ x \geq 0, \tag{25}$$

by Bernstein's theorem [52], where $F(u)$ is a bounded and nondecreasing function on $[0,\infty)$ such that the integral converges for all $x \geq 0$. Consequently, we are able to rewrite (17) as

$$\begin{aligned} C_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2) &= \int_0^{\infty} \tilde{C}_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2;u)\, dF(u), \\ & \quad \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{R}, \ i,j = 1,\ldots,m, \end{aligned}$$

where

$$\begin{aligned} \tilde{C}_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2;u) = \left(g(\vartheta(\mathbf{x}_1,\mathbf{x}_2)) + \theta_{ij}\right)^{-\frac{1}{2}} \exp\left(-\frac{(t_1-t_2)^2}{g(\vartheta(\mathbf{x}_1,\mathbf{x}_2))+\theta_{ij}} u\right), \\ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{R}, \ u \geq 0, \ i,j = 1,\ldots,m. \end{aligned}$$

By Theorem 4 of [31], it suffices to verify that $\tilde{C}_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2;u), i,j = 1,\ldots,m$, formulate an $m \times m$ covariance matrix function on $\mathbb{D} \times \mathbb{R}$, for every fixed $u \geq 0$. This is true by Theorems 3 and 4 of [31], since they can be rewritten as

$$\begin{aligned} & \tilde{C}_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2;u) \\ & = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \cos((t_1-t_2)\sqrt{u}\omega) \exp\left(-\frac{\omega^2}{4} g(\vartheta(\mathbf{x}_1,\mathbf{x}_2))\right) \exp\left(-\frac{\omega^2}{4}\theta_{ij}\right) d\omega, \\ & \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{R}, \ i,j = 1,\ldots,m, \end{aligned}$$

where $\cos((t_1-t_2)\sqrt{u}\omega)$ is a covariance function on $\mathbb{R}$, $\exp\left(-\frac{\omega^2}{4} g(\vartheta(\mathbf{x}_1,\mathbf{x}_2))\right)$ is a covariance function on $\mathbb{D}$ since $g(x)$ has a completely monotone derivative and $\vartheta(\mathbf{x}_1,\mathbf{x}_2))$ is a variogram on $\mathbb{D}$, and the matrix with entries $\exp\left(-\frac{\omega^2}{4}\theta_{ij}\right)$ is positive definite by Lemma 3, for each fixed $\omega \geq 0$.

(ii) According to the integral representation (25) of $\ell(x)$, (18) can be rewritten as

$$\begin{aligned} C_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2) = \int_0^{\infty} \tilde{C}_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2;u)\, dF(u), \\ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{R}, \ i,j = 1,\ldots,m, \end{aligned}$$

where

$$\begin{aligned} & \tilde{C}_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2;u) \\ = \ & \left(g(\vartheta(\mathbf{x}_1,\mathbf{x}_2)) + \theta_{ij}\right)^{-\frac{1}{2}} \exp\left(-\left(g(\vartheta(\mathbf{x}_1,\mathbf{x}_2)) + \theta_{ij}\right)^{-\frac{1}{2}} |t_1-t_2| u\right), \\ & \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{R}, \ u \geq 0, \ i,j = 1,\ldots,m, \end{aligned}$$

and, moreover,

$$\begin{aligned} & \tilde{C}_{ij}(\mathbf{x}_1,\mathbf{x}_2;t_1,t_2;u) \\ & = \frac{2}{\pi} \int_0^{\infty} \cos((t_1-t_2)\sqrt{u}\omega) \frac{d\omega}{g(\vartheta(\mathbf{x}_1,\mathbf{x}_2))+\theta_{ij}+\omega^2} \\ & \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{R}, \ i,j = 1,\ldots,m. \end{aligned}$$

They form an $m \times m$ covariance matrix function on $\mathbb{D} \times \mathbb{R}$ by Theorems 3 and 4 of [31], since $\cos((t_1-t_2)\sqrt{u}\omega)$ is a covariance function on $\mathbb{R}$ and an $m \times m$ matrix function with entries

$$\left(g(\vartheta(\mathbf{x}_1,\mathbf{x}_2)) + \theta_{ij} + \omega^2\right)^{-1}, \ \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{D}, \ t_1,t_2 \in \mathbb{R}, \ i,j = 1,\ldots,m,$$

is a covariance matrix function on $\mathbb{D}$, for each fixed $u \geq 0$.

(iii) It follows from Theorem 2 of [1], since $\vartheta(\mathbf{x}_1, \mathbf{x}_2) + \gamma(t_1, t_2)$ is conditionally negative definite on $\mathbb{D} \times \mathbb{T}$ due to the fact that $\vartheta(\mathbf{x}_1, \mathbf{x}_2)$ is conditionally negative definite on $\mathbb{D}$ [34] and the assumption on $\gamma(t_1, t_2)$.

6.6. **Proof of Theorem 4.** It is sufficient to verify that $\tilde{\mathbf{B}}_n(t)$ defined by (23) is an $m \times m$ stationary covariance matrix functions on $\mathbb{R}$, for each $n \in \mathbb{N}_0$. Notice that $\tilde{\mathbf{B}}_n(t)$ is equivalent to

$$\tilde{\mathbf{B}}_n(t) = \sum_{l=-\infty}^{\infty} \mathbf{B}_n(l) C_\nu(t-l), \ t \in \mathbb{R},$$

where

$$C_\nu(x) = \left(\max(1-|x|, 0)\right)^\nu, \ x \in \mathbb{R}, \tag{26}$$

is a stationary covariance function on $\mathbb{R}$ with a spectral density function $\mathcal{F}_{C_\nu}(\omega)$. Given $\alpha \in (0,1)$, we define

$$\tilde{\mathbf{B}}_n(t;\alpha) = \sum_{l=-\infty}^{\infty} \alpha^{|l|} \mathbf{B}_n(l) C_\nu(t-l), \ t \in \mathbb{R}.$$

Observe that $\alpha^{|l|}$ $(l \in \mathbb{Z})$ is a stationary covariance function on $\mathbb{Z}$. Its product with $\mathbf{B}_n(l)$, $\alpha^{|l|}\mathbf{B}_n(l), l \in \mathbb{Z}$, is certainly an $m \times m$ stationary covariance function on $\mathbb{Z}$, with a spectral density matrix function $\sum\limits_{l=-\infty}^{\infty} \alpha^{|l|} \mathbf{B}_n(l) \exp(\imath l\omega), \omega \in \mathbb{R}$, which is positive definite for each fixed $\omega \in \mathbb{R}$. Consequently, the Fourier transform of $\tilde{\mathbf{B}}_n(t;\alpha)$ exists, and

$$\begin{aligned}
&\int_{\mathbb{R}} \tilde{\mathbf{B}}_n(t;\alpha) \exp(\imath t\omega) dt \\
= &\sum_{l=-\infty}^{\infty} \alpha^{|l|} \mathbf{B}_n(l) \int_{\mathbb{R}} C_\nu(t-l) \exp(\imath t\omega) dt \\
= &\sum_{l=-\infty}^{\infty} \alpha^{|l|} \mathbf{B}_n(l) \int_{\mathbb{R}} g_0(y) \exp(\imath(y+l)\omega) dy \ (\text{let } y = t-l) \\
= &\sum_{l=-\infty}^{\infty} \alpha^{|l|} \mathbf{B}_n(l) \exp(\imath l\omega) \mathcal{F}_{C_\nu}(\omega), \ \omega \in \mathbb{R},
\end{aligned}$$

is a positive definite matrix for each fixed $\omega \in \mathbb{R}$. Hence, $\tilde{\mathbf{B}}_n(t;\alpha)$ is an $m \times m$ stationary covariance function on $\mathbb{R}$ [19]. So is $\lim\limits_{\alpha \to 1_-} \tilde{\mathbf{B}}_n(t;\alpha) = \tilde{\mathbf{B}}_n(t), t \in \mathbb{R}$.

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