Theory of Stochastic Processes
Vol. 29 (45), no. 1, 2025, pp. 9–23

A. A. DOROGOVTSEV AND NAOUFEL SALHI

UNIVERSAL GENERALIZED FUNCTIONALS AND FINITELY ABSOLUTELY CONTINUOUS MEASURES ON BANACH SPACES

To the memory of professor Habib Ouerdiane

In this paper we collect several examples of convergence of functions of random processes to generalized functionals of those processes. We remark that the limit is always finitely absolutely continuous with respect to Wiener measure. We try to unify those examples in terms of convergence of probability measures in Banach spaces. The key notion is the condition of uniform finite absolute continuity.

Introduction

In this article we discuss the positive generalized functionals from the stochastic processes. The idea of generalized functionals is closely related to the investigation of geometric properties of the random processes. Simple but important examples are the local time of Wiener process [19] and the Rice formula [1].

Example 0.1. Let $w(t)$, $t \in [0,1]$ be a standard Wiener process. It is well known [19, p.32] that for any $x \in \mathbb{R}$ there exists the local time which w spends at the infinitesimal small neighborhood of x, i.e.

$$\ell(x) = L_2 - \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^1 \mathbf{1}_{[x-\varepsilon, x+\varepsilon]}\big(w(t)\big)dt.$$

Keeping in mind that in the sense of generalized functions

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \mathbf{1}_{[x-\varepsilon, x+\varepsilon]} = \delta_x$$

where δ_x denotes the Dirac delta function at the point x, one can write formally

$$\ell(x) = \int_0^1 \delta_x\big(w(t)\big)dt.$$

Such expression is very useful for production of new formulas for local time which can then be proved rigorously. For example, occupation formula for bounded measurable function f

$$\begin{aligned}\int_{\mathbb{R}} f(x)\ell(x)dx &= \int_{\mathbb{R}} f(x) \int_0^1 \delta_x\big(w(t)\big)dtdx \\ &= \int_0^1 \Big(\int_{\mathbb{R}} f(x)\delta_x\big(w(t)\big)dx\Big)dt \\ &= \int_0^1 f\big(w(t)\big)dt,\end{aligned}$$

2010 *Mathematics Subject Classification.* 60A10 ; 60G15 ; 60H05.

Key words and phrases. generalized Wiener functions; generalized Gaussian functional of first kind; Gaussian integrator; intersection local time; Itô-Wiener expansion; finite absolute continuity; uniform finite absolute continuity;

or Kac moment formula

$$\begin{aligned}\mathbb{E}\ell(x)^n &= \mathbb{E}\int_0^1 \cdots \int_0^1 \delta_x\big(w(t_1)\big)\cdots\delta_x\big(w(t_n)\big)dt_1\cdots dt_n \\ &= n!\int_{0\leqslant t_1\cdots\leqslant t_n\leqslant 1}\mathbb{E}\Big(\delta_x\big(w(t_1)\big)\cdots\delta_x\big(w(t_n)\big)\Big)dt_1\cdots dt_n \\ &= n!\int_{\Delta_n}\frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{x^2}{2t_1}}\frac{1}{\sqrt{t_1}}\frac{1}{\sqrt{t_2-t_1}}\cdots\frac{1}{\sqrt{t_n-t_{n-1}}}dt_1\cdots dt_n,\end{aligned}$$

where $\Delta_n = \{(t_1,\cdots,t_n) : 0\leqslant t_1\cdots\leqslant t_n\leqslant 1\}$. In this case also it is useful to create a formal definition of $\delta_x\big(w(t)\big)$ and the rules of manipulation with it. Such formal definition was done in [5, 16] and became to be a partial case of the notion of generalized Wiener function.

The same problem arises under the consideration of the number of upcrossings of a stochastic process.

Example 0.2. Suppose that $\xi(t)$, $t\in[0,1]$ is a centered Gaussian process with the smooth covariance. Then, for fixed level c, the expected number of upcrossings of the level c by the process ξ is equal [1, p. 263-264] to the following expression

$$\int_0^1\int_0^{+\infty} xq_t(c,x)dxdt. \tag{1}$$

Here $q_t(x)$ is the joint distribution of $\xi(t)$ and $\xi'(t)$. Formally, expression (1) can be obtained easily. Every upcrossing is associated with the formal expression

$$\delta_c\big(\xi(t)\big)\mathbf{1}_{(0,+\infty)}\big(\xi'(t)\big)\xi'(t) \tag{2}$$

Keeping in mind that the result of action of δ_c on the continuous bounded function is the value of the function at the point c we easily get the Rice formula

$$\int_0^1\mathbb{E}\Big(\delta_c\big(\xi(t)\big)\mathbf{1}_{(0,+\infty)}\big(\xi'(t)\big)\xi'(t)\Big)dt = \int_0^1\int_0^{+\infty} xq_t(c,x)dxdt. \tag{3}$$

Such approach together with the formal expression was proposed by M. Kac [15]. But how to bring the rigorous sense to the expression (2)?

One of the possible applications of the mentioned theory is the study of the geometry of the trajectories of multi-dimensional Brownian motion. Such investigations are related to the mathematical model of free linear polymer [4, 12]. In this model, a trajectory of Wiener process is treated as an instant conformation of the long linear polymer molecule. Then, the excluded volume effect leads to the necessity to study self-intersections. In this way Evans measure arises. Since the theory is useful and well-developed for Wiener process then it is natural to ask about the same constructions and properties for processes different from Wiener process. A lot of attempts were done in this direction (see, for instance, [19, 22, 17]) but here we mention the class of processes introduced by A.A. Dorogovtsev and called Gaussian integrators.

Definition 0.1. [6] A (one dimensional) centered Gaussian process $x(t), t\in[0,1]$ is said to be an integrator if there exists a constant $C>0$ such that for any arbitrary partition $0=t_0<t_1<\cdots<t_n=1$ and real numbers $a_0,\cdots,a_{n-1}$:

$$\mathbb{E}\left[\sum_{k=0}^{n-1}a_k(x(t_{k+1})-x(t_k))\right]^2 \le C\sum_{k=0}^{n-1}a_k^2(t_{k+1}-t_k). \tag{4}$$

For such processes some singular functionals were considered (see, for example, [8, 9]). Recently, in [11] the large deviation principle for the measure which is corresponding to generalized functional of self-intersection was obtained.

In order to define generalized Wiener functions we need to introduce the family of Sobolev spaces $\mathbb{D}^{2,\gamma}$ over Wiener space (see for example [18, 24] for more details). First of all, let $w(t) = (w_1(t), \cdots, w_d(t))$, $t \geqslant 0$, be a d-dimensional Brownian motion and denote by $\sigma(w)$ the σ-field generated by it. It is known that every square integrable Wiener random variable $\eta \in L^2\big(\Omega, \sigma(w), \mathbb{P}\big)$ has an Itô-Wiener expansion [20, p.13] which consists on the L^2-convergent series with orthogonal summands

$$\eta = \sum_{k=0}^{\infty} I_k(f_k)$$

where $I_k(f_k)$ denotes a k-multiple Itô stochastic integral of the deterministic and symmetric square integrable kernel f_k. Now let $\gamma \in \mathbb{R}$. The Sobolev space $\mathbb{D}^{2,\gamma}$ is the completion of the following space :

$$\left\{ \eta = \sum_{k=0}^{n} I_k(f_k) \in L^2\big(\Omega, \sigma(w), \mathbb{P}\big),\, n \in \mathbb{N} \right\}$$

with respect to the norm :

$$\big\|\eta\big\|_{2,\gamma}^2 = \sum_{k=0}^{n} (k+1)^{\gamma}\, \mathbb{E}\, I_k(f_k)^2 \,.$$

If $0 < \gamma_1 < \gamma_2$ then the following inclusions are true

$$\mathbb{D}^{2,\gamma_2} \subset \mathbb{D}^{2,\gamma_1} \subset \mathbb{D}^{2,0} = L^2\big(\Omega, \sigma(w), \mathbb{P}\big) \subset \mathbb{D}^{2,-\gamma_1} \subset \mathbb{D}^{2,-\gamma_2}.$$

Moreover, for any real number γ the space $\mathbb{D}^{2,-\gamma}$ is the dual space of $\mathbb{D}^{2,\gamma}$. When $\gamma < 0$, the elements of $\mathbb{D}^{2,\gamma}$ are called generalized Wiener functionals. The spaces $\mathbb{D}^{2,\pm\infty}$ are defined respectively as projective and inductive limits

$$\mathbb{D}^{2,+\infty} = \cap_{\gamma>0} \mathbb{D}^{2,\gamma} \,,\ \mathbb{D}^{2,-\infty} = \cup_{\gamma>0} \mathbb{D}^{2,-\gamma} \,.$$

Elements of $\mathbb{D}^{2,+\infty}$ are called Wiener test functions.
Without loss of generality, the probability space $\big(\Omega, \sigma(w), \mathbb{P}\big)$ can be replaced by the classical Wiener space

$$\bigg(\mathscr{C}_0\big([0,1], \mathbb{R}^d\big), \mathcal{B}\Big(\mathscr{C}_0\big([0,1], \mathbb{R}^d\big)\Big), \mu_0 \bigg)$$

where

$$\big(\mathscr{C}_0\big([0,1], \mathbb{R}^d\big) = \big\{\omega : [0,1] \to \mathbb{R}^d, \text{ continuous}, \ \omega(0) = 0\big\}$$

is endowed with the Borel σ-field $\mathcal{B}\Big(\mathscr{C}_0\big([0,1], \mathbb{R}^d\big)\Big)$ generated by the supremum norm and with the standard Wiener measure μ_0.
A test function $F \in \mathbb{D}^{2,+\infty}$ is said to be positive if

$$F(\omega) \geqslant 0 \quad \mu_0 - \text{a.e.}$$

A generalized Wiener function $\eta \in \mathbb{D}^{2,-\infty}$ is said to be positive if the bilinear pairing with any positive test function $F \in \mathbb{D}^{2,+\infty}$ is non-negative :

$$(\eta, F) \geqslant 0.$$

A Sugita theorem [24] states that any positive generalized Wiener function can be represented by a measure on the Wiener space.

Theorem 0.1. [24] *If a generalized Wiener functional $\eta \in \mathbb{D}^{2,-\infty}$ is positive then there exists a unique finite positive measure θ on $\mathscr{C}_0\big([0,1], \mathbb{R}^d\big)$ such that*

$$\forall\ F \in \mathcal{F}\mathscr{C}_b^{\infty}\Big(\mathscr{C}_0\big([0,1], \mathbb{R}^d\big)\Big), \quad (\eta, F) = \int_{\mathscr{C}_0\left([0,1], \mathbb{R}^d\right)} F(\omega)\, \theta(d\omega)\,.$$

Here $\mathcal{F}\mathscr{C}_b^\infty\Big(\mathscr{C}_0\big([0,1],\mathbb{R}^d\big)\Big)$ *denotes the set of Wiener test functions* $F \in \mathbb{D}^{2,+\infty}$ *of the form*

$$F(\omega) = f(\phi_1(\omega), \cdots, \phi_n(\omega))$$

where $f \in \mathscr{C}_b^\infty(\mathbb{R}^d)$ *and* $\phi_1, \cdots, \phi_n \in \Big(\mathscr{C}_0\big([0,1],\mathbb{R}^d\big)\Big)^*$.

In view of mentioned works we face the following general problem which is main object of discussions in this paper. Let us formulate it in the abstract form.

Let B be a real separable Banach space with the norm $\|\cdot\|$. All the measures on B are supposed to be defined on the Borel σ-field $\mathcal{B}$. Also, all functions on B are supposed to be Borel measurable. Consider a family Φ_ε, $\varepsilon > 0$ of functions on B such that for every $\varepsilon > 0$, $\Phi_\varepsilon : B \to \mathbb{R}$ has a continuous Frechet derivative and is bounded on B together with its derivative. Suppose that μ is a probability measure on B with all finite moments of the norm and such that finite-dimensional polynomials are dense in $L_2\big(B,\mathcal{B},\mu\big)$. Then, for every $\varepsilon > 0$, Φ_ε has an expansion via orthogonal polynomials

$$\Phi_\varepsilon = \sum_{n=0}^{\infty} I_n^\varepsilon.$$

Suppose that, for every $\varepsilon > 0$, $\Phi_\varepsilon \geqslant 0$ and for every $n \geqslant 0$ there exists a limit

$$I_n^0 = L_2 - \lim_{\varepsilon\to 0} I_n^\varepsilon.$$

Then the formal series

$$\sum_{n=0}^{\infty} I_n^0$$

can be considered as a non-negative generalized functional on the measurable space $\big(B,\mathcal{B},\mu\big)$. In certain cases it can be associated with some new measure ν on $\mathcal{B}$ (as in Sugita theorem). In this article we discuss the following problem. When for the family $\{\Phi_\varepsilon, \varepsilon > 0\}$ there exists a set of probability measures $\mathcal{M}$ such that for every $\mu \in \mathcal{M}$, Φ_ε converges to a generalized functional on $\big(B,\mathcal{B},\mu\big)$? Consider an example of such situation.

Example 0.3. Let $B = \mathscr{C}_0\big([0,1],\mathbb{R}\big)$ be the 1-dimensional Wiener space. Define the family Φ_ε as follows :

$$\forall\, f \in \mathscr{C}_0\big([0,1],\mathbb{R}\big), \quad \Phi_\varepsilon(f) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{f(1)^2}{2\varepsilon}}.$$

Now consider as a measure μ the standard Wiener measure μ_0 on $\mathscr{C}_0\big([0,1],\mathbb{R}\big)$. Then, (see, for example, [10]) Φ_ε converges when $\varepsilon \to 0^+$ to a positive generalized functional which has an Itô-Wiener expansion

$$\Phi_0(f) = \sum_{n=0}^{\infty} \frac{1}{n!\sqrt{2\pi}} H_n(0) H_n\big(f(1)\big)$$

where H_n denote the Hermite polynomials

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \Big(\frac{d}{dx}\Big)^n e^{-\frac{x^2}{2}}.$$

If we denote by $w(t), t \in [0,1]$, the one-dimensional Wiener process then, for any Wiener test function $F \in \mathbb{D}^{2,+\infty}$, we have

$$\begin{aligned}
(\Phi_0, F) &= \lim_{\varepsilon\to 0} \big(\Phi_\varepsilon, F\big)_{L_2\left(B,\mu_0\right)} \\
&= \lim_{\varepsilon\to 0} \mathbb{E}\Big[\Phi_\varepsilon(w)F(w)\Big] \\
&= \lim_{\varepsilon\to 0} \mathbb{E}\Big[p_\varepsilon(w(1))\mathbb{E}\big[F(w)|w(1)\big]\Big] \\
&= \frac{1}{\sqrt{2\pi}}\mathbb{E}\big[F(w)|w(1)=0\big].
\end{aligned}$$

Here p_ε is the density of the centered normal distribution with variance ε. From this point we deduce that the generalized functional Φ_0 corresponds to the distribution of the Brownian bridge. From other side, define μ as a distribution in $\mathscr{C}_0\big([0,1],\mathbb{R}\big)$ of the random process

$$\eta(t) = t.\xi,\ t \in [0,1]$$

where ξ is a standard Gaussian random variable. Then, again, Φ_ε converges when $\varepsilon \to 0^+$ to a generalized functional with the same chaotic expansion but with another measure representation via Sugita theorem. Now it is a probability measure concentrated on the one function $f \equiv 0$. Actually, it can be checked that Φ_ε converges to a generalized functional for a very wide set of measures μ. In this sense, Φ_ε is universal.

In this paper we will discuss the conditions for universality of the different families $\{\Phi_\varepsilon, \varepsilon > 0\}$. To do this we use the notion of finite absolute continuity of measures on the Banach space introduced in [7]. This notion defines connection between the weak moments of two probability measures.

Definition 0.2. [7] A finite measure ν with weak moments of arbitrary order on the Banach space B is finitely absolutely continuous with respect to the probability measure μ (this fact is denoted $\nu <<_0 \mu$) if for any $n \in \mathbb{N}$ there exists a constant $c_n > 0$ such that for any polynomial $P_n : \mathbb{R}^n \to \mathbb{R}$ with degree at most n and any $\phi_1, \cdots, \phi_n \in B^*$ we have

$$\Big|\int_B P_n(\phi(\omega), \cdots, \phi_n(\omega))d\nu(\omega)\Big| \leqslant c_n\Big(\int_B P_n^2(\phi_1(\omega), \cdots, \phi_n(\omega))d\mu(\omega)\Big)^{\frac{1}{2}}.$$

Note that if μ is a Gaussian measure on B and ν is a measure related to a positive generalized functional on $\big(B, \mathcal{B}, \mu\big)$ then, evidently,

$$\nu <<_0 \mu.$$

This observation leads to the following idea. It seems that uniform condition of finite absolute continuity

$$\Phi_\varepsilon\mu <<_0 \mu,\ \varepsilon > 0$$

where $\Phi_\varepsilon\mu$ denotes the measure which has the density Φ_ε with respect to μ, will guarantee existence of the generalized functional Φ_0 on $\big(B, \mathcal{B}, \mu\big)$ which is a limit of Φ_ε when ε tends to 0. We present the corresponding example related to the functional counting self-intersections.

Example 0.4. Let $w(t), t \in [0,1]$ be a d-dimensional Brownian motion and let $u \in \mathbb{R}^d \setminus \{0\}$. Consider the following formal expression

$$\rho(u) = \int_{\Delta_2} \delta_u\big(w(t_2) - w(t_1)\big)dt_1dt_2. \tag{5}$$

In [13] a rigorous meaning was associated to $\rho(u)$ as follows. First we define the Dirac δ-function δ_u as a limit of the family $\{p^d_\varepsilon(.-u), \varepsilon>0\}$ where

$$p^d_\varepsilon(x)=\frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}}\exp\Big(-\frac{\|x\|^2}{2\varepsilon}\Big),\quad \varepsilon>0,\, x\in\mathbb{R}^d.$$

(p^1_ε will simply be denoted p_ε.)
Consequently, $\rho(u)$ should be understood as

$$\rho(u)=\lim_{\varepsilon\to 0}\int_{\Delta_2}p^d_\varepsilon\big(w(t_2)-w(t_1)-u\big)dt_1dt_2. \tag{6}$$

Denote by Φ_ε the integral in the right hand side of (6). It was proved (see [13]) that Φ_ε has an Itô-Wiener expansion given by

$$\Phi_\varepsilon=\sum_{k=0}^{\infty}\sum_{n_1+\cdots+n_d=k}\int_{\Delta_2}\prod_{1\leqslant j\leqslant d}\Big\{\frac{1}{n_j!}H_{n_j}\Big(\frac{w_j(t_2)-w_j(t_1)}{\sqrt{t_2-t_1+\varepsilon}}\Big)H_{n_j}\Big(\frac{u_j}{\sqrt{t_2-t_1+\varepsilon}}\Big)\Big\} \times p^d_{t_2-t_1+\varepsilon}(u)dt_1dt_2. \tag{7}$$

By taking the limit when $\varepsilon\to 0$ in each term of (7) we obtain the formal Itô-Wiener expansion of $\rho(u)$

$$\rho(u)=\sum_{k=0}^{\infty}\sum_{n_1+\cdots+n_d=k}\int_{\Delta_2}\prod_{1\leqslant j\leqslant d}\Big\{\frac{1}{n_j!}H_{n_j}\Big(\frac{w_j(t_2)-w_j(t_1)}{\sqrt{t_2-t_1}}\Big)H_{n_j}\Big(\frac{u_j}{\sqrt{t_2-t_1}}\Big)\Big\} \times p^d_{t_2-t_1}(u)dt_1dt_2. \tag{8}$$

It was proved in [13] that the formal series $\rho(u)$ is in fact an element of the Sobolev spaces $\mathbb{D}^{2,\gamma}$ such that

$$\gamma<\frac{4-d}{2}$$

and that Φ_ε converges, when $\varepsilon\to 0$, to $\rho(u)$ in each of those spaces.

Therefore, if $d\geqslant 4$ then the intersection local time formally defined by $\rho(u)$ is a positive generalized Wiener functional, and not a square integrable random variable as in the case $d\leqslant 3$. Moreover, using Sugita theorem, (see [10]), $\rho(u)$ can be represented by a measure on the Wiener space $\mathscr{C}_0\big([0,1],\mathbb{R}^d\big)$.

Correspondingly to the above mentioned arguments, the article is divided into three parts. The first part contains necessary definitions and facts about finite absolute continuity, mostly from [7, 21]. Second part contains a proof of convergence of Φ_ε under uniform finite absolute continuity. The last part contains concrete examples of universal families Φ_ε, $\varepsilon>0$.

1. Survey about finite absolute continuity and polynomially non degenerate measures

We recall here some definitions, examples and statements introduced and analysed mainly by A.A. Dorogovtsev in [7].
We consider a Banach space B equipped with a probability measure μ and satisfying the assumptions presented in the introduction. For every $n\in\mathbb{N}$ let $\mathcal{P}_n$ be the set of all polynomials of degree less or equal to n defined on B. Denote by $\overline{\mathcal{P}_n}$ its closure in $L_2\big(B,\mu\big)$. Define K_n as the orthogonal complement of $\overline{\mathcal{P}_n}$ in $\overline{\mathcal{P}_{n+1}}$. Since the set of all finite dimensional polynomials is dense in $L_2\big(B,\mu\big)$ then the following orthogonal decomposition holds

$$L_2\big(B,\mu\big)=\bigoplus_{n=0}^{\infty}K_n.$$

Denote by J_n the orthogonal projection in $L_2(B,\mu)$ onto K_n. Denote by $H_{n,s}$ the space of n-linear continuous symmetric forms on B. If we denote by H_2 the space K_1 then the space $H_{n,s}$ can be identified with the symmetric part of the tensor power $H_2^{\otimes n}$. Denote by $\|\cdot\|_n$ the associated norm.

Definition 1.1. [7] A measure μ on B is called polynomially non-degenerate if there exist sequences $(c_n)_n$ and $(C_n)_n$ of positive numbers such that, for any finite dimensional n-linear symmetric continuous form A_n on B, the following inequality holds

$$c_n\|A_n\|_n^2 \leqslant \int_B \Big(J_nA_n\Big)^2(\omega)d\mu(\omega) \leqslant C_n\|A_n\|_n^2.$$

Every Gaussian measure on B is polynomially non-degenrate. Besides, for every $n\in\mathbb{N}$, the constants c_n and C_n in Definition 1.1 become equal to $n!$. The following results present more examples of polynomially non-degenerate measures.

Lemma 1.1. [7] *Suppose a measure μ is polynomially non-degenerate and a measure ν is such that $\mu\sim\nu$ and the following conditions are satisfied*

(1) $0<\operatorname{ess\,inf}\dfrac{d\nu}{d\mu}\leqslant \operatorname{ess\,sup}\dfrac{d\nu}{d\mu}<\infty$
(2) *the mean value of ν is equal to* 0.

Then ν is polynomially non-degenerate.

Lemma 1.2. [7] *Suppose that B is a real separable Hilbert space and μ is a Gaussian measure on B with mean value zero and the non degenerate correlation operator whose eigenvalues $\{\lambda_n, n\geqslant 0\}$ are such that*

$$\sum_{n=1}^{\infty}\lambda_n\log^2(n)<\infty.$$

Then the measure ν, obtained from μ by restriction to the ball $B(0,r)$ of center 0 and radius r and by normalization, is polynomially non-degenerate.

Now we focus on the notion of finite absolute continuity presented in Definition 0.2. In the remainder of this section we assume that the reference measure μ is polynomially non-degenerate. One of the main examples of finite absolutely continuous measures is provided by the following result.

Theorem 1.1. [21] *Let $\eta\in\mathbb{D}^{2,-\infty}$ be a positive generalized Wiener function and ν be the measure on the Wiener space $\mathscr{C}_0([0,1],\mathbb{R}^d)$ associated to η. Then ν is finitely absolutely continuous with respect to the Wiener measure μ_0.*

Finite absolute continuity may imply absolute continuity in certain cases.

Example 1.1. [7] The sequence $(c_n)_{n\geqslant 0}$ in definition 0.2 can be chosen bounded if, and only if, $\nu<<\mu$ and

$$\frac{d\nu}{d\mu}\in L_2(B,\mu).$$

Lemma 1.3. [7] *Suppose that ν and μ are Gaussian measures in a real separable Hilbert space B that have the same correlation operator S and mean values 0 and h, respectively. If $\nu<<_0\mu$ then $\nu<<\mu$.*

If a measure ν is finitely absolutely continuous with respect to μ then it is possible to obtain a chaotic expansion of ν with respect to μ in the sense precised by the following theorem.

Theorem 1.2. [7] *Let ν be a probability measure on B which is finitely absolutely continuous with respect to μ. Then there exists a sequence of kernels $A_n \in H_{n,s}$ such that for any polynomial Q defined on B the following equality holds*

$$\int_B Q(\omega)\nu(d\omega) = \sum_{n=0}^{\infty} \int_B Q(\omega)A_n(\omega)\mu(d\omega).$$

2. Existence of the generalized functional

The aim of this section is to illustrate how the finite absolute continuity can guarantee the weak compactness of the set of finite measures. The reason behind such kind of statements is the following. Let $\{\Phi_\varepsilon, \varepsilon > 0\}$ be a family of non-negative functions defined on the Banach space B and let μ be a probability measure on B. Assume that the measures $\Phi_\varepsilon\mu$ (Φ_ε is now a density with respect to μ) are finitely absolutely continuous with respect to μ and for any polynomial G on B there exists a limit

$$\lim_{\varepsilon\to 0} \int_B G(u)\Phi_\varepsilon(u)\mu(du).$$

Then, the weak compactness of $\Phi_\varepsilon, \varepsilon > 0$, can guarantee that Φ_ε converges when $\varepsilon \to 0$ to some measure which is a positive generalized functional from μ. The same can happen for another probability measure $\widetilde{\mu}$. Then we will get another measure (positive genarlized functional from $\widetilde{\mu}$).This will emphasize the universality of the family $\Phi_\varepsilon, \varepsilon > 0$ (it produces positive generalized functionals from different measures). But firstly we have to guarantee weak compactness. It occurs that the condition of uniform finite absolute continuity is enough for that in some cases. The corresponding statements are presented in this section.
We consider two cases : when B is a Hilbert space and when B is the Wiener space $\mathscr{C}([0,1], \mathbb{R}^d)$. Such choice is enough for our purposes. Let us start from the Hilbert space. Suppose that μ is polynomially non-degenerate centered probability measure on the Hilbert space H such that

$$\forall\, h \in H \; : \; \int_H (h,u)^2\mu(du) > 0.$$

Theorem 2.1. *Suppose that the probability measure μ on H has finite strong moments of any order and let the family of finite measures $\{\mu_\alpha,\, \alpha \in \Theta\}$ on H satisfies conditions*

A1. *for every $\alpha \in \Theta$, μ_α has all weak moments*
A2. *for every $\alpha \in \Theta$, $\mu_\alpha <<_0 \mu$ with the constants $\{c_n^\alpha,\, n \geqslant 0\}$*
A3. *for every $n \geqslant 0$*

$$\sup_{\alpha\in\Theta} c_n^\alpha := c_n < \infty.$$

Then the family $\{\mu_\alpha,\, \alpha \in \Theta\}$ is weakly compact.

Proof. Since μ is polynomially non-degenerate then, according to Definition 1.1, there exists a constant $C_2 > 0$ such that for any continuous linear form $\varphi \in H^*$ we have (by taking $A_2(x_1, x_2) = \varphi(x_1)\varphi(x_2)$)

$$\int_H \varphi(u)^4\mu(du) \leqslant C_2\Big(\int_H \varphi(u)^2\mu(du)\Big)^2.$$

In other words,

$$\forall\, h \in H, \quad \int_H (h,u)^4\mu(du) \leqslant C_2\Big(\int_H (h,u)^2\mu(du)\Big)^2.$$

Let S be the covariance operator of the measure μ. Denote by $\{e_k;\ k \geqslant 1\}$ the orthonormal eigenbasis of S. Then, for every $\alpha \in \Theta$

$$\sum_{k=1}^{\infty} \int_H (e_k, u)^2 \mu_\alpha(du) \leqslant c_2^\alpha \sum_{k=1}^{\infty} \sqrt{\int_H (e_k, u)^4 \mu(du)} \leqslant C_2 c_2^\alpha \sum_{k=1}^{\infty} \int_H (e_k, u)^2 \mu(du). \tag{9}$$

Taking into account that μ has finite strong moment of order 2, it can be checked that

$$\sup_{\alpha \in \Theta} \sum_{k=n}^{\infty} \int_H (e_k, u)^2 \mu_\alpha(du) \xrightarrow[n \to \infty]{} 0. \tag{10}$$

It is known (see [2]) that (9) and (10) are sufficient for the weak compactness of $\{\mu_\alpha,\ \alpha \in \Theta\}$.
Theorem is proved. □

Remark 2.1. As it can be seen from the proof, only second and fourth moments were used but we formulate the theorem in a frame of our main considerations.

Now consider the space $B = \mathscr{C}\big([0,1], \mathbb{R}^d\big)$. Suppose that the measure μ is centered and polynomially non-degenerate.

Theorem 2.2. *Suppose that the family $\{\mu_\alpha,\ \alpha \in \Theta\}$ of finite measures on B satisfies conditions A1, A2 and A3 of Theorem 2.1 and the probability measure μ on B satisfies the condition*

$$\exists\, c > 0 \quad \exists\, \gamma > 0 \quad \forall\, t_1, t_2 \in [0,1] \qquad \int_B \big\|u(t_2) - u(t_1)\big\|^2 \mu(du) \leqslant c\big|t_2 - t_1\big|^\gamma. \tag{11}$$

Then the family $\{\mu_\alpha,\ \alpha \in \Theta\}$ is weakly compact in B.

Proof. Similarly to the previous proof it can be checked that for an arbitrary $m \geqslant 1$ there exists $C_m > 0$ such that for every $\varphi \in B^*$

$$\int_B \varphi(u)^{2m} \mu(du) \leqslant C_m \left(\int_B \varphi(u)^2 \mu(du) \right)^m.$$

Then it follows from the condition of the theorem that for some $m_0 \geqslant 1$

$$\int_B \big\|u(t_2) - u(t_1)\big\|^{2m_0} \mu(du) \leqslant C_{m_0} c^{m_0} \big|t_2 - t_1\big|^{1+\beta}$$

where $\beta > 0$. Now from uniform finite absolute continuity one can conclude that

$$\sup_{\alpha \in \Theta} \int_B \big\|u(0)\big\|^2 \mu_\alpha(du) < \infty \tag{12}$$

and for some $D > 0$

$$\forall\, t_1, t_2 \in [0,1] \quad \sup_{\alpha \in \Theta} \int_B \big\|u(t_2) - u(t_1)\big\|^{2m_0} \mu_\alpha(du) \leqslant D\big|t_2 - t_1\big|^{1+\beta}. \tag{13}$$

It is known (see [3, p.82]) that (12) and (13) are sufficient for weak compactness of $\{\mu_\alpha,\ \alpha \in \Theta\}$.
Theorem is proved. □

Theorem 2.3. *Suppose that the probability measure μ and the family of finite measures $\big\{\Phi_\varepsilon \mu, \varepsilon > 0\big\}$ on B satisfy conditions of Theorem 2.1 or 2.2 and, in addition, for any finite dimensional polynomial G there exists a limit*

$$\lim_{\varepsilon \to 0} \int_B G(u) \Phi_\varepsilon(u) \mu(du).$$

Then, all partial limits of $\Phi_\varepsilon \mu$ when $\varepsilon \to 0$ can be treated as positive generalized functionals from the measure μ and have the same formal orthogonal expansion with respect to μ.

3. Examples of universal families

In this section we consider concrete approximating families and their limits. Let us start from the generalized local time.

Example 3.1. Suppose that $\xi(t)$, $t \in [0,1]$ is a centered Gaussian process such that

(1) $\forall\, t \in [0,1],\ \mathbb{E}\xi^2(t) = \sigma^2(t) > 0$

(2) $\exists\, c > 0,\ \exists\, \gamma > 0,\ \forall\, t_1, t_2 \in [0,1]\ :\ \mathbb{E}\big|\xi(t_2) - \xi(t_1)\big|^2 \leqslant c\big|t_2 - t_1\big|^\gamma.$

Note that under these conditions ξ has a continuous modification. Consequently, the distribution μ of ξ is a centered Gaussian measure in $\mathscr{C}\big([0,1],\mathbb{R}\big)$. We recall that every Gaussian measure is polynomially non-degenerate and has finite strong moments of any order. Moreover, condition (11) in Theorem 2.2 follows immediately from the assumption on the process ξ. Now consider for $\varepsilon > 0$ the functional on $\mathscr{C}\big([0,1],\mathbb{R}\big)$ defined by

$$\Phi_\varepsilon(f) = \int_0^1 p_\varepsilon\big(f(t)\big)dt$$

where p_ε is the density of the centered normal distribution with variance ε.
For every $\varepsilon > 0$ denote by μ_ε the measure on $\mathscr{C}\big([0,1],\mathbb{R}\big)$ which has the density Φ_ε with respect to μ.
Let us check that the measures μ_ε are uniformly finitely absolutely continuous with respect to μ.
Assumption A1 is satisfied because Φ_ε is bounded and μ has finite strong moments of any order.
Let us check assumptions A2 and A3. Fix $n \in \mathbb{N}$ and consider a finite dimensional polynomial P_n of degree n on $\mathscr{C}\big([0,1],\mathbb{R}\big)$. Then

$$\begin{aligned}
\Big|\int_{\mathscr{C}\big([0,1],\mathbb{R}\big)} P_n(f)\mu_\varepsilon(df)\Big| &= \Big|\int_{\mathscr{C}\big([0,1],\mathbb{R}\big)} \Phi_\varepsilon(f)P_n(f)\mu(df)\Big| \\
&= \Big|\mathbb{E}\big(\Phi_\varepsilon(\xi)P_n(\xi)\big)\Big| \\
&= \Big|\mathbb{E}\int_0^1 p_\varepsilon\big(\xi(t)\big)dt\, P_n(\xi)\Big| \\
&= \Big|\int_0^1 \mathbb{E}\Big[p_\varepsilon\big(\xi(t)\big)\,\mathbb{E}\big(P_n(\xi)\big|\xi(t)\big)\Big]dt\Big| \\
&\leqslant \int_0^1 \mathbb{E}\Big[p_\varepsilon\big(\xi(t)\big)\,\Big|\mathbb{E}\big(P_n(\xi)\big|\xi(t)\big)\Big|\Big]dt.
\end{aligned}$$

The conditional expectation in the last expression can be written

$$\mathbb{E}\big(P_n(\xi)\big|\xi(t)\big) = Q_n(\xi(t), t)$$

where

$$Q_n(x,t) = \mathbb{E}\Big(P_n(\xi)\Big|\xi(t) = x\Big)$$

is a polynomial of degree not greater than n. Thus,

$$\mathbb{E}\Big[p_\varepsilon\big(\xi(t)\big)\,\Big|\mathbb{E}\big(P_n(\xi)\big|\xi(t)\big)\Big|\Big] = \int_{\mathbb{R}} p_\varepsilon(x)\big|Q_n(x,t)\big|p_{\sigma^2(t)}(x)dx.$$

Using Newton-Leibniz formula, Cauchy inequality and integration by parts formula it can be checked that, when

$$\min_{t\in[0,1]} \sigma^2(t) > 0,$$

and for any $x \in \mathbb{R}$,

$$\begin{aligned}
\Big|p_{\sigma^2(t)}(x)\,Q_n(x,t)\Big| &= \left|\int_{-\infty}^{x}\Big(\frac{-z}{\sigma^2(t)}Q_n(z,t)+\frac{d}{dz}Q_n(z,t)\Big)p_{\sigma^2(t)}(z)dz\right| \\
&\leqslant \left|\int_{-\infty}^{x}\frac{-z}{\sigma^2(t)}Q_n(z,t)p_{\sigma^2(t)}(z)dz\right| + \left|\int_{-\infty}^{x}\frac{d}{dz}\Big(Q_n(z,t)\Big)p_{\sigma^2(t)}(z)dz\right| \\
&\leqslant c_1\sqrt{\int_{\mathbb{R}}Q_n^2(z,t)p_{\sigma^2(t)}(z)dz} + \sqrt{\int_{\mathbb{R}}\Big(\frac{d}{dz}\big(Q_n(z,t)\big)\Big)^2 p_{\sigma^2(t)}(z)dz} \\
&\leqslant c_n\sqrt{\int_{\mathbb{R}}Q_n^2(z,t)p_{\sigma^2(t)}(z)dz} \\
&= c_n\sqrt{\mathbb{E}\Big[\Big(\mathbb{E}\big(P_n(\xi)\big|\xi(t)\big)\Big)^2\Big]} \\
&= c_n\sqrt{\mathbb{E}\Big(P_n^2(\xi)\Big)}.
\end{aligned}$$

It follows that

$$\Big|\int_{\mathscr{C}\left([0,1],\mathbb{R}\right)}P_n(f)\mu_\varepsilon(df)\Big| \leqslant c_n\sqrt{\int_{\mathscr{C}\left([0,1],\mathbb{R}\right)}P_n^2(f)\mu(df)}.$$

Since the constant c_n does not depend on ε then assumptions A2 and A3 are satisfied. According to Theorem 2.2 the family $\{\mu_\varepsilon,\,\varepsilon>0\}$ is weakly compact in $\mathscr{C}\big([0,1],\mathbb{R}\big)$.

Example 3.2. Let $w(t)=(w_1(t),\cdots,w_d(t))$ be a d-dimensional Brownian motion and A an invertible bounded linear operator in the Hilbert space $L_2([0,1])$. Define a multi-dimensional Gaussian integrator by

$$X(t)=\big(X_1(t),\cdots,X_d(t)\big),\quad X_j(t)=\int_0^1\big(A\mathbf{1}_{[0,t]}\big)(s)dw_j(s),\ j=1,\cdots,d \tag{14}$$

The distribution μ of X is a centered Gaussian measure in $B=\mathscr{C}_0\big([0,1],\mathbb{R}^d\big)$. Moreover, for any $0\leqslant t_1<t_2\leqslant 1$ we have

$$\int_B\big\|u(t_2)-u(t_1)\big\|^2\mu(du)=\mathbb{E}\|X(t_2)-X(t_1)\|^2=d\|A\mathbf{1}_{[t_1,t_2]}\|^2\leqslant d\|A\|^2(t_2-t_1).$$

Now fix $u\in\mathbb{R}^d\setminus\{0\}$ and consider the functional defined on B by

$$F_\varepsilon(f)=\int_0^1 p_\varepsilon^d\big(f(t)-u\big)dt,\ \varepsilon>0.$$

Let us check that the measures μ_ε defined on $\mathscr{C}_0\big([0,1],\mathbb{R}^d\big)$ by

$$\frac{d\mu_\varepsilon}{d\mu}=F_\varepsilon$$

are uniformly finitely absolutely continuous with respect to μ.
Assumption A1 is satisfied because Φ_ε is bounded and the Gaussian measure μ has finite strong moments of any order.
Now consider a finite dimensional polynomial P_n of degree n on B. Similarly to Example 3.1 one can check that

$$\Big|\int_B P_n(f)\mu_\varepsilon(df)\Big| = \Big|\int_B F_\varepsilon(f)P_n(f)\mu(df)\Big| \leqslant \int_0^1\mathbb{E}\Big[p_\varepsilon^d\big(X(t)-u\big)\,\Big|\mathbb{E}\big(P_n(X)\big|X(t)\big)\Big|\Big]dt.$$

The conditional expectation can be written

$$\mathbb{E}\big(P_n(X)\big|X(t)\big) = Q_n(X(t), t)$$

where

$$Q_n(x,t) = \mathbb{E}\Big(P_n(X)\Big|X(t) = x\Big)$$

is a polynomial of degree not greater than n. Therefore,

$$\begin{aligned}\mathbb{E}\Big[p^d_\varepsilon\big(X(t)-u\big)\,\Big|\mathbb{E}\big(P_n(X)\big|X(t)\big)\Big|\Big] &= \int_{\mathbb{R}^d} p^d_\varepsilon(x-u)\big|Q_n(x,t)\big|p^d_{\sigma^2(t)}(x)dx \\ &= p^d_\varepsilon * \Big|p^d_{\sigma^2(t)}\cdot Q_n(\cdot,t)\Big|(u)\end{aligned}$$

where $\sigma^2(t) = \|A\mathbf{1}_{[0,t]}\|^2$ is the variance of $X_j(t)$ for any $j \in \{1,\cdots,d\}$.
Since A is invertible then there exist $0 < m < M$ such that

$$\sqrt{m}\sqrt{t} \leqslant \sigma(t) \leqslant \sqrt{M}\sqrt{t} \tag{15}$$

Let us recall that for $\sigma^2 > 0$ the Hilbert space $L_2\Big(\mathbb{R}^d, p^d_{\sigma^2}(x)dx\Big)$ has an orthonormal basis :

$$R_{n_1,\cdots,n_d}(x) = \sigma^{n_1+\cdots+n_d}\prod_{j=1}^d H_{n_j}\Big(\frac{x_j}{\sigma}\Big),\ n_1,\cdots,n_d \in \mathbb{N}$$

where H_j, $j \geqslant 0$, are Hermite polynomials. Consequently,

$$Q_n(x,t) = \sum_{n_1+\cdots+n_d\leqslant n} \alpha_{n_1,\cdots,n_d}(t)R_{n_1,\cdots,n_d}(x)$$

and therefore

$$\begin{aligned}\Big|p^d_{\sigma^2(t)}(x)\cdot Q_n(x,t)\Big| &= \Big|p^d_{\sigma^2(t)}(x)\sum_{n_1+\cdots+n_d\leqslant n}\alpha_{n_1,\cdots,n_d}(t)R_{n_1,\cdots,n_d}(x)\Big| \\ &\leqslant p^d_{\sigma^2(t)}(x)\sqrt{\sum_{n_1+\cdots+n_d\leqslant n}\alpha_{n_1,\cdots,n_d}(t)^2}\sqrt{\sum_{n_1+\cdots+n_d\leqslant n}R_{n_1,\cdots,n_d}(x)^2} \\ &= p^d_{\sigma^2(t)}(x)\Big\|Q_n(\cdot,t)\Big\|_{L_2\big(\mathbb{R}^d,p^d_{\sigma^2(t)}(x)dx\big)}\sqrt{\sum_{n_1+\cdots+n_d\leqslant n}R_{n_1,\cdots,n_d}(x)^2} \\ &= \sqrt{\mathbb{E}\Big[\Big(\mathbb{E}\big(P_n(X)\big|X(t)\big)\Big)^2\Big]}\,p^d_{\sigma^2(t)}(x)\sqrt{\sum_{n_1+\cdots+n_d\leqslant n}R_{n_1,\cdots,n_d}(x)^2} \\ &\leqslant \big(\max\{\sqrt{M},1\}\big)^n\sqrt{\mathbb{E}\Big(P^2_n(X)\Big)}\,p^d_{\sigma^2(t)}(x)\sqrt{\sum_{n_1+\cdots+n_d\leqslant n}\prod_{j=1}^d H^2_{n_j}\Big(\frac{x_j}{\sigma(t)}\Big)}.\end{aligned}$$

Now we use the following inequality (see e.g. [8])

$$\forall\, n\in\mathbb{N}\quad \exists\, a_n > 0\quad \forall\, x\in\mathbb{R}\quad |H_n(x)| \leqslant a_n e^{\frac{x^2}{4}} \tag{16}$$

Consequently, there exists $C_n > 0$ such that

$$\Big|p^d_{\sigma^2(t)}(x)\cdot Q_n(x,t)\Big| \leqslant C_n\sqrt{\mathbb{E}\Big(P^2_n(X)\Big)}\,p^d_{2\sigma^2(t)}(x).$$

Therefore,

$$\Big|\int_B P_n(f)\mu_\varepsilon(df)\Big| \leqslant C_n\sqrt{\mathbb{E}\Big(P_n^2(X)\Big)}\int_0^1 p_\varepsilon^d * p_{2\sigma^2(t)}^d(u)dt$$
$$= C_n\sqrt{\mathbb{E}\Big(P_n^2(X)\Big)}\int_0^1 p_{\varepsilon+2\sigma^2(t)}^d(u)dt.$$

One can check that there exists a constant $C(u)>0$ such that for any $t\in[0,1]$,

$$p_{\varepsilon+2\sigma^2(t)}^d(u)\leqslant C(u).$$

Finally, there exists a constant $c_n = C_nC(u)$ which does not depend on ε and such that

$$\Big|\int_B P_n(f)\mu_\varepsilon(df)\Big| \leqslant c_n\sqrt{\mathbb{E}\Big(P_n^2(X)\Big)}.$$

Thus, assumptions A2 and A3 are satisfied.
According to Theorem 2.2 the family $\{\mu_\varepsilon,\,\varepsilon>0\}$ is weakly compact in $\mathscr{C}\big([0,1],\mathbb{R}^d\big)$.
In addition, for any finite dimensional polynomial P_n on $\mathscr{C}\big([0,1],\mathbb{R}^d\big)$ we have, using the previous computations,

$$\int_B F_\varepsilon(f)P_n(f)\mu(df) = \int_0^1 p_\varepsilon^d * \Big(p_{\sigma^2(t)}^d\cdot Q_n(\cdot,t)\Big)(u)dt$$
$$\xrightarrow[\varepsilon\to 0]{} \int_0^1 p_{\sigma^2(t)}^d(u)Q_n(u,t)dt.$$

Therefore, we can apply Theorem 2.3. As a consequence of this, one can say that the local time of the multidimensional integrator (14) at any point different from the origin exists as a generalized function and admits an expansion into a series of multiple stochastic integrals with respect to the integrator itself.

Example 3.3. We continue with the multidimensional Gaussian integrator defined by (14). Let μ be the distribution of X in the Banach space $B=\mathscr{C}_0\big([0,1],\mathbb{R}^d\big)$. Fix $u\in\mathbb{R}^d\setminus\{0\}$ and consider the functional defined on B by

$$G_\varepsilon(f)=\int_{\Delta_2} p_\varepsilon^d\big(f(t)-f(s)-u\big)dsdt,\ \varepsilon>0.$$

Let us check that the measures μ_ε defined on B by

$$\frac{d\mu_\varepsilon}{d\mu}=G_\varepsilon$$

are uniformly finitely absolutely continuous with respect to μ. Consider a finite dimensional polynomial P_n of degree n on B. Similarly to Example 3.2 one can check that

$$\Big|\int_B G_\varepsilon(f)P_n(f)\mu(df)\Big| \leqslant \int_{\Delta_2}\mathbb{E}\Big[p_\varepsilon^d\big(X(t)-X(s)-u\big)\,\Big|\mathbb{E}\big(P_n(X)\big|X(t)-X(s)\big)\Big|\Big]dsdt$$
$$\leqslant \int_{\Delta_2} p_\varepsilon^d * \Big|p_{\sigma^2(s,t)}^d\cdot Q_n(\cdot,s,t)\Big|(u)dsdt$$

where

$$Q_n(x,s,t)=\mathbb{E}\Big(P_n(X)\Big|X(t)-X(s)=x\Big)$$

is a polynomial of degree not greater than n, and

$$\sigma(s,t)=\|A\mathbf{1}_{[s,t]}\|.$$

Since A is invertible then there exist $0<m<M$ such that

$$\sqrt{m}\sqrt{t-s}\leqslant\sigma(s,t)\leqslant\sqrt{M}\sqrt{t-s} \tag{17}$$

Using again the orhtonormal basis of $L_2\Big(\mathbb{R}^d, p^d_{\sigma^2}(x)dx\Big)$ one can find a constant $C_n > 0$ such that

$$\Big|p^d_{\sigma^2(s,t)}(x)\cdot Q_n(x,s,t)\Big| \leqslant C_n\sqrt{\mathbb{E}\Big(P_n^2(X)\Big)}\; p^d_{2\sigma^2(s,t)}(x).$$

Thus,

$$\begin{aligned}\Big|\int_B G_\varepsilon(f)P_n(f)\mu(df)\Big| &\leqslant C_n\sqrt{\mathbb{E}\Big(P_n^2(X)\Big)}\int_{\Delta_2} p^d_\varepsilon * p^d_{2\sigma^2(s,t)}(u)dsdt\\ &= C_n\sqrt{\mathbb{E}\Big(P_n^2(X)\Big)}\int_{\Delta_2} p^d_{\varepsilon+2\sigma^2(s,t)}(u)dsdt\\ &\leqslant c(u)C_n\sqrt{\mathbb{E}\Big(P_n^2(X)\Big)}.\end{aligned}$$

Since the constant $c(u)C_n$ does not depend on ε then we deduce the uniform finite absolute continuity of the family $\{\mu_\varepsilon,\, \varepsilon > 0\}$ with respect to μ.
According to Theorem 2.2 the family $\{\mu_\varepsilon,\, \varepsilon > 0\}$ is weakly compact in $\mathscr{C}\big([0,1],\mathbb{R}^d\big)$.
In addition, for any finite dimensional polynomial P_n on $\mathscr{C}\big([0,1],\mathbb{R}^d\big)$ we have, using the previous computations,

$$\begin{aligned}\int_B G_\varepsilon(f)P_n(f)\mu(df) &= \int_{\Delta_2} p^d_\varepsilon * \Big(p^d_{\sigma^2(s,t)}\cdot Q_n(\cdot,s,t)\Big)(u)dsdt\\ &\xrightarrow[\varepsilon\to 0]{} \int_{\Delta_2} p^d_{\sigma^2(s,t)}(u)Q_n(u,s,t)dsdt.\end{aligned}$$

Therefore, we can apply Theorem 2.3. From this point, one can deduce that the intersection local time of the multidimensional integrator (14), formally defined by,

$$\rho_X(u) = \int_{\Delta_2}\delta_u\big(X(t)-X(s)\big)dsdt = \lim_{\varepsilon\to 0} G_\varepsilon(X),\; u\in\mathbb{R}^d\setminus\{0\}$$

is a generalized function from X and admits a chaos expansion with respect to X.

Acknowledgment.

The authors are very grateful to professor Andrey Pilipenko for his careful reading of this paper and the valuable comments that were vary useful for the improvement of this work.

References

1. R.J. Adler and J.E. Taylor, *Random fields and geometry*, Springer, 2007. https://doi.org/10.1007/978-0-387-48116-6
2. A. Araujo and E. Giné, *The central limit theorem for real and Banach valued random variables*, Wiley, New York, 1980.
3. P. Billingsley, *Convergence of probability measures*, John Wiley & Sons, 1999. https://doi.org/10.1002/9780470316962
4. E. Bolthausen, E. Perkins and A. Van der Vaart, *Lectures on Probability Theory and Statistics*, Springer, 2002. https://doi.org/10.1007/b93152
5. M. De Faria, T. Hida, L. Streit and H. Watanabe, *Intersection local times as generalized white noise functionals*, Acta Applicandae Mathematicae **46** (1997), 351–362. https://doi.org/10.1023/a:1005782030567
6. A.A. Dorogovtsev, *Stochastic integration and one class of Gaussian random processes*, Ukr. Math. Journal **50** (1998), no. 4, 495–505. https://doi.org/10.1007/bf02487387
7. A.A. Dorogovtsev, *Measurable functionals and finitely absolutely continuous measures on Banach spaces*, Ukrainian Math. J. **52** (2000), no. 9, 1366–1379. https://doi.org/10.1023/a:1010371817382
8. A.A. Dorogovtsev, O.L. Izyumtseva, G. Ryabov and N. Salhi, *Clark formula for the local time of one class of Gaussian processes*, Communication on Stochastic Analysis **10** (2016), no. 2, 195–217. https://doi.org/10.31390/cosa.10.2.05

9. A.A. Dorogovtsev, O.L. Izyumtseva and N. Salhi, *Clark representation for self-intersection local times of Gaussian integrators*, Ukrainian Math. Journal **70** (2019), no. 12, 1829–1860. https://doi.org/10.1007/s11253-019-01613-y
10. A.A. Dorogovtsev and N. Salhi, *Refinements of asymptotics at zero of Brownian self-intersection local times*, Infin. Dimens. Anal. Quantum Probab. Relat. Top, **27**, no. 2 (2024), 2350018 (24 pages). https://doi.org/10.1142/s0219025723500182
11. A.A. Dorogovtsev and N. Salhi, *Large deviation principle for generalised multiple intersection local times of multidimensional Brownian motion*, arXiv:2406.07173 [math.PR].
12. F.D. Hollander, *Random Polymers*, Springer, 2009.
13. P. Imkeller, V. Perez-Abreu and J. Vives, *Chaos expansions of double intersection local time of Brownian motion in* $\mathbb{R}^d$ *and renormalization*, Stochastic processes and their applications **56** (1995), 1–34. https://doi.org/10.1016/0304-4149(94)00041-q
14. O.L. Izyumtseva, *Moments estimates for local times of a class of Gaussian processes*, Communications on Stochastic Analysis **10** (2016), no.1, 97–116. https://doi.org/10.31390/cosa.10.1.07
15. M. Kac, *On the average number of real roots of a random algebraic equation*, Bull. Amer. Math. Soc. **49** (1943), 314–320. https://doi.org/10.1090/s0002-9904-1943-07912-8
16. H.-H. Kuo, *Donsker's delta function as a generalized Brownian functional and its application*, Theory and Application of Random Fields. Lecture Notes in Control and Information Sciences **49** (1983), Springer, Berlin, Heidelberg. https://doi.org/10.1007/bfb0044690
17. J.F. Le Gall, *Temps locaux d'intersection et points multiples des processus de Lévy*, Séminaire de Probabilités (Strasbourg) **21** (1987), 341–374. https://doi.org/10.1007/bfb0077645
18. P. Malliavin, *Stochastic Analysis*, Springer, 1997, XII+347 pp. https://doi.org/10.1007/978-3-642-15074-6
19. M.B. Markus and J. Rosen, *Markov processes, Gaussian processes and local times*, Cambridge University Press, 2006.
20. D. Nualart, *The Malliavin Calculus and Related Topics*, Springer-Verlag Berlin Heidelberg 2006. https://doi.org/10.1007/3-540-28329-3
21. G.V. Riabov, *Finite absolute continuity on an abstract Wiener space*, Theory Stoch. Process. **17** (2011), no. 1, 100–108.
22. J. Rosen, *A local time approach to the Self-intersections of Brownian paths in space*, Communications in Math. Physics **88** (1983), no. 3, 327–338. https://doi.org/10.1007/bf01213212
23. B. Simon, *The* $P(\varphi)_2$ *Euclidian (Quantum) Field Theory*, Princeton University Press, 1974.
24. H. Sugita, *Positive generalized Wiener functions and potential theory over abstract Wiener spaces*, Osaka J. Math. **25** (1988), 665–696.

Institute of Mathematics, National Academy of Sciences of Ukraine, Ukraine
E-mail address: andrey.dorogovtsev@gmail.com

Laboratory of Stochastic Analysis and Applications, Faculty of Sciences of Tunis, University of Tunis El Manar, Tunisia
E-mail address: salhi.naoufel@gmail.com