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LIMIT BEHAVIOR OF MEASURE-VALUED SOLUTIONS TO NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH INTERACTION

The classification of critical points of solutions to deterministic differential equation with interaction is proposed. The critical points of the type "A" and of the type "B" are introduced. The large time behavior of measure-valued solutions to such equations in the neighbourhoods of critical points of both types is studied.

1. Introduction

Stochastic flows were studied intensively since the works by K. Ito [7], I.I. Gihman, A.V. Skorohod [6]. In [8] K. Ito considered such flows as random analogues of flows generated by ordinary differential equations. Stochastic flows generated by stochastic differential equations were investigated in details by H. Kunita [9].

In the paper [1] by A. A. Dorogovtsev a new class of stochastic differential equation with interaction

$$\begin{cases} dx(u,t) = a(x(u,t),\mu_t,t)dt + \int\limits_{\mathbb{R}^d} b(x(u,t),\mu_t,t,q)W(dt,dq) \\ x(u,0) = u,\ \mu_t = \mu_0 \circ x(\cdot,t)^{-1} \end{cases}$$

was introduced. Here W is a Brownian sheet that plays the role of a random medium in which the particles that form the stochastic flow move, μ_0 is a probability measure, that is initial distribution of mass of particles, $x(u,\cdot)$ is the trajectory of the particle, that left the point u at zero time, μ_t characterize the distribution of mass of particles at time t.

In monograph [5] by A.A. Dorogovtsev properties of stochastic flows, generated by SDE with interaction, have been obtained. The limit behavior of solutions to SDE with interaction in one-dimensional case have been studied in [4]. In [2] the existence of intermittency phenomena with dissipative coefficients has been proved by showing uniform convergence of their Lyapunov exponents.

Ordinary differential equations with interaction are deterministic cases of SDE with interaction. They can be used by describing processes that appear by changing speeds of movements of particles along trajectories of ordinary differential equation by the next way.

Consider differential equation

$$\begin{cases} d\xi(u,t) = f\left(\xi(u,t)\right)dt \\ \xi(u,0) = u, \end{cases} \tag{1}$$

where $f \in C_b^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$.

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Let $\mathfrak{M}$ be the set of all probability measures on $\mathfrak{B}\left(\mathbb{R}^d\right)$, $\mu_0 \in \mathfrak{M}$. Consider the next functional

$$I_s^t = \int\limits_s^t \int\limits_{\mathbb{R}^d} \varphi(u)\mu_\tau(du)d\tau,\ 0 \le s \le t < +\infty, \quad \varphi \in C_b\left(\mathbb{R}^d\right), \quad \mu_t = \mu_0 \circ (\xi(\cdot,t))^{-1},\ t \ge 0.$$

and time

$$\tau(t) = \inf\{r \ge 0, I_0^r = t\}. \tag{2}$$

Then the process

$$\nu_t = \mu_{\tau(t)},\ t \ge 0$$

can be considered like description of movement of particles along the trajectories $\xi(u,\cdot)$ with the velocity, that depends on the distribution of the mass of all particles. It occurs that this model is related to ODE with interaction.

Consider

$$x(u,t) = \xi(u, \tau(t)).$$

Then from (1) and (2) we have

$$\begin{cases} dx(u,t) = f\left(x(u,t)\right) \left(\int\limits_{\mathbb{R}^d} \varphi(v)\nu_t(dv) \right)^{-1} dt \\ x(u,0) = u, \\ \nu_t = \nu_0 \circ x(\cdot,t)^{-1}, \quad u \in \mathbb{R}^d,\ t \ge 0. \end{cases}$$

The work is devoted to the study of the asymptotic behavior of measure-valued solutions to nonlinear ODE with interaction, that contains this case.

2. Main results

Consider the equation

$$\begin{cases} dx(u,t) = \int\limits_{\mathbb{R}^d} f\left(x(u,t), v\right) \mu_t(dv)dt \\ x(u,0) = u, \\ \mu_t = \mu_0 \circ x(\cdot,t)^{-1}, \end{cases} \tag{3}$$

where $f \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^{d\times d})$, initial measure μ_0 satisfies the condition $\int\limits_{\mathbb{R}^d} ||v||\mu_0(dv) < +\infty$.

Definition 2.1. We call the point $x_0 \in \mathbb{R}^d$ a critical point of the type "A" of the system (3) if

$$f(x_0, \cdot) \equiv 0. \tag{4}$$

If x_0 is a critical point of the type "A" , then $x(x_0, t) \equiv x_0$ satisfy (3).

Definition 2.2. We call the point $x_0 \in \mathbb{R}^d$ a critical point of the type "B" of the system (3) if

$$f(\cdot, x_0) \equiv 0. \tag{5}$$

The next example shows the situation for a linear case.

Example 2.1. Consider the equation

$$\begin{cases} dx(u,t) = \left(A \int\limits_{\mathbb{R}^d} v\mu_t(dv) \right) dt \\ x(u,0) = u, \\ \mu_t = \mu_0 \circ x(\cdot,t)^{-1}, \end{cases} \tag{6}$$

where $A \in \mathbb{R}^{d\times d}$, initial measure μ_0 is such, that $\int\limits_{\mathbb{R}^d} ||v||\mu_0(dv) < +\infty$.

This equation has a unique critical point zero and it is of the type "B".

Let $m_t = \int\limits_{\mathbb{R}^d} v\mu_t(dv)$. Then from (6) it follows that

$$m_t = e^{At} m_0.$$

In such a case from (6) we get

$$x(u,t) = e^{At} m_0 + u - m_0.$$

Then if all eigenvalues of the matrix A have negative real parts, we have

$$\lim_{t\to\infty} x(u,t) = u - m_0. \tag{7}$$

By the help of obtained asymptotic behavior of x we can investigate the asymptotic behavior of μ_t.

For $\mu, \nu \in \mathfrak{M}$ let us define $C(\mu,\nu)$ as a set of all probability measures on $\mathfrak{B}\left(\mathbb{R}^d \times \mathbb{R}^d\right)$, that has μ, ν as their marginal distributions.

We call the Wasserstain ditance of zero order on $\mathfrak{M}$ the metric

$$\gamma_0(\mu,\nu) = \inf_{C(\mu,\nu)} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{||u-v||}{1+||u-v||}\kappa(du,dv).$$

In the example

$$\gamma_0\left(\mu_t, \mu_0 - m_0\right) \le \int\limits_{\mathbb{R}} \frac{\|x(u,t) - u - m_0\|}{1 + \|x(u,t) - u - m_0\|}\mu_0\left(du\right).$$

Using (7) from this we have

$$\lim_{t\to\infty} \gamma_0\left(\mu_t, \mu_0 - m_0\right) = 0.$$

Thus, in this case the set of measures $\{\mu_t, t \ge 0\}$ has exactly one limit point $\mu_0 - m_0$.

Now let us consider general case of equation (3).

Let us introduce some notations firstly. Let $A \in \mathbb{R}^{d\times d}$ and all eigenvalues $\lambda_1, ..., \lambda_d$ of the matrix A have negative real parts. Then we can take $\alpha > 0$ such that

$$Re(\lambda_i) < -\alpha < 0 \quad \forall i = 1, ..., d.$$

Because columns of the matrix e^{At} are the elements of fundamental system of solutions to respective linear system of ODE, we can choose $K > 0$ such that

$$\left\|e^{At}\right\| < Ke^{-\alpha t} \quad \text{for all } t \ge 0. \tag{8}$$

We will denote the constant $\frac{\alpha}{K}$ as $M(A)$, and the constant K as $K(A)$.

Let $x_0 \in \mathbb{R}^d$ be a critical point of the type "A" or of the type "B" of the system (3). Then we can write the function f in the next way

$$f(x_1,x_2) = J_{f(x,x_0)}(x_0)(x_1 - x_0) + J_{f(x_0,x)}(x_0)(x_2 - x_0) + \sum_{k=1}^{2} R_k(x_1,x_2)(x_k - x_0), \tag{9}$$

where $J_{f(x_0,x)}$ is the Jacobian matrix of the function $f(x_0,x)$ and $J_{f(x,x_0)}$ is the Jacobian matrix of the function $f(x,x_0)$.

Let $x_0 \in \mathbb{R}^d$ be a critical point of the type "A" or of the type "B". Let us introduce the next notations

$$\widetilde{x}(u,t) = x(u,t) - x_0, \quad \widetilde{m}_t = \int\limits_{\mathbb{R}} v\mu_t(dv) - x_0. \tag{10}$$

The following theorem describes the limit behavior of the first moment of μ_t.

Theorem 2.1. *Let $x_0 \in \mathbb{R}^d$ is a critical point of the type "A" or of the type "B" of the system (3), all eigenvalues of the matrix $J_{f(x_0,x)}(x_0) + J_{f(x,x_0)}(x_0)$ have negative real parts and for every $k \in \{1, 2\}$*

$$\sup_{x_1,x_2 \in \mathbb{R}^d} \|R_k(x_1,x_2)\| = M_k < \frac{1}{2} M\left(J_{f(x_0,x)}(x_0) + J_{f(x,x_0)}(x_0)\right). \tag{11}$$

Then

$$\lim_{t\to\infty} \int\limits_{\mathbb{R}^d} v\mu_t(dv) = x_0.$$

Proof. Using representation (9) and notation (10) from (3) we get

$$\begin{aligned} d\widetilde{x}(u,t) &= \left(J_{f(x,x_0)}(x_0)\widetilde{x}(u,t) + J_{f(x_0,x)}(x_0)\widetilde{m}_t\right) dt+ \\ &+ \int\limits_{R^d} \left(R_1\left(x(u,t),v\right)\widetilde{x}(u,t) + R_2\left(x(u,t),v\right)v\right)\mu_t(dv)dt. \end{aligned} \tag{12}$$

Let

$$A = J_{f(x,x_0)}(x_0) + J_{f(x_0,x)}(x_0).$$

Then $\widetilde{m}_t$ is a solution to the equation

$$d\widetilde{m}_t = A\widetilde{m}_t dt + \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \left(R_1\left(u,v\right)\left(u-x_0\right) + R_2\left(u,v\right)\left(v-x_0\right)\right)\mu_t(dv)\mu_t(du)dt. \tag{13}$$

Because all eigenvalues $\lambda_1, ..., \lambda_d$ of the matrix A have negative real part, we can take $\alpha > 0$ such that

$$Re(\lambda_i) < -\alpha < 0 \quad \forall i = 1, ..., d.$$

and $K > 0$ such that

$$\left\|e^{At}\right\| < Ke^{-\alpha t} \quad \text{for all } t \geq 0,$$

where $M(A) = \frac{\alpha}{K}$.

We have from (13), that

(14)

$$\widetilde{m}_t = m_0 e^{At} + \int\limits_0^t e^{A(t-s)} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \left(R_1\left(u,v\right)\left(u-x_0\right) + R_2\left(u,v\right)\left(v-x_0\right)\right)\mu_s(dv)\mu_s(du)ds.$$

Using (8) and (11) it follows from (14)

$$\|\widetilde{m}_t\| \leq Ke^{-\alpha t}\|m_0\| + \int\limits_0^t Ke^{-\alpha(t-s)} 2M \left\|\widetilde{m}_s\right\| ds,$$

where $M = \max\{M_1, M_2\}$.

Then

$$e^{\alpha t}\left\|\widetilde{m}_t\right\| \leq K\|m_0\| + \int\limits_0^t 2MKe^{\alpha s}\left\|\widetilde{m}_s\right\| ds.$$

From this inequality by Gronwall-Bellman lemma we get, that $\widetilde{m}_t$ satisfies the estimation

$$\|\widetilde{m}_t\| \leq K\|m_0\|e^{(-\alpha+2MK)t} \quad \forall t \geq 0. \tag{15}$$

Then, because $M < M(A)$ we have the inequality $-\alpha + 2MK < 0$ and therefore for every m_0 we have

$$\lim_{t\to\infty} \widetilde{m}_t = 0. \tag{16}$$

By (10) the statement of the theorem follows from this.

□

Let us notice, that it follows from (16) and (15), that system (13) is globally stable (look, for example, [10]) if conditions of Theorem 2.1 are fulfilled. Then by such conditions the system can have only one critical point.

The following theorems contain results about asymptotic behavior of μ_t.

Theorem 2.2. *Let $x_0 \in \mathbb{R}^d$ be a critical point of the type "B" of the system (3), all eigenvalues of the matrix $J_{f(x_0,x)}(x_0)$ have negative real parts, the function $R_1(x_1,x_2) \equiv 0$, R_2 depends only on x_2 and*

$$\sup_{x_2 \in \mathbb{R}^d} \|R_2(x_2)\| = M < M\left(J_{f(x_0,x)}(x_0)\right). \tag{17}$$

Then in Wasserstain distance

$$\lim_{t\to\infty} \mu_t = \mu_0 - \int_{\mathbb{R}^d} v\mu_0(dv) + x_0.$$

Proof. Because x_0 is a critical point of the type "B" of the system (3) using (9) we get

$$f(x_1,x_2) = J_{f(x_0,x)}(x_0)(x_2 - x_0) + R_2(x_2)(x_2 - x_0). \tag{18}$$

Then by (18) and (10) we have that $\widetilde{m}_t$ is the solution to the equation

$$d\widetilde{m}_t = J_{f(x_0,x)}(x_0)\widetilde{m}_t dt + \int_{\mathbb{R}^d} R_2(v)(v - x_0)\mu_t(dv)dt. \tag{19}$$

Because all eigenvalues of the matrix $J_{f(x_0,x)}(x_0)$ have negative real parts it follows from theorem 2.1 that m_t satisfies the estimation (15).

We have from (19)

$$\widetilde{x}(u,t) = u - x_0 + \widetilde{m}_t - \widetilde{m}_0. \tag{20}$$

Then by (15), it follows from (20) that

$$\lim_{t\to\infty} x(u,t) = u - \widetilde{m}_0.$$

From the inequality

$$\gamma_0\left(\mu_t, \mu_0 - m_0\right) \le \int_{\mathbb{R}^d} \frac{\|x(u,t) - u - \widetilde{m}_0\|}{1 + \|x(u,t) - u - \widetilde{m}_0\|}\mu_0(du).$$

we get

$$\lim_{t\to\infty} \gamma_0\left(\mu_t, \mu_0 - \widetilde{m}_0\right) = 0.$$

Theorem is proved. □

Theorem 2.3. *Let $x_0 \in \mathbb{R}^d$ be a critical point of the type "A" of the system (3), all eigenvalues of the matrix $J_{f(x,x_0)}(x_0)$ have negative real parts, R_2 satisfies Lipschitz condition by $x_2 \in \mathbb{R}^d$ with Lipschitz constant L, such that $L \le \frac{M\left(J_{f(x,x_0)}(x_0)\right)}{2\|m_0\|K(J_{f(x,x_0)}(x_0))}$ and*

$$\sup_{x_1,x_2\in\mathbb{R}^d} \|R_1(x_1,x_2)\| = M_1 < \frac{1}{2}M\left(J_{f(x,x_0)}(x_0)\right) \text{ uniformly as } x_1,x_2 \in \mathbb{R}^d, \tag{21}$$

(22)

$$\sup_{x_1,x_2\in\mathbb{R}^d} \|R_2(x_1,x_2) - R_2(x_0,x_2)\| = M_2 < \frac{1}{2}M\left(J_{f(x,x_0)}(x_0)\right) \text{ uniformly as } x_1,x_2 \in \mathbb{R}^d,$$

Then in Wasserstain distance

$$\lim_{t\to\infty} \mu_t = \delta_{\{x_0\}}.$$

Proof. Because x_0 is a critical point of the type "A" of the system (3) we have from (9)

(23)
$$f(x_1, x_2) = J_{f(x,x_0)}(x_0)(x_1 - x_0) + (R_2(x_1, x_2) - R_2(x_0, x_2))(x_2 - x_0) + R_1(x_1, x_2)(x_1 - x_0).$$

Then it follows from (23) that the function $\widetilde{m}_t$ satisfies the equation

$$d\widetilde{m}_t = J_{f(x,x_0)}(x_0)\widetilde{m}_t dt +$$
(24)
$$+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left((R_2(v,u) - R_2(x_0,u))(u - x_0) + R_1(v,u)(v - x_0) \right) \mu_t(dv)\mu_t(du)dt.$$

Because all eigenvalues of the matrix $J_{f(x_0,x)}(x_0)$ have negative real parts it follows using conditions on R_1 and R_2 from theorem 2.1 that m_t satisfies the estimation (15).

Let
$$A = J_{f(x,x_0)}(x_0).$$

Using (3) we have

$$\frac{d\widetilde{x}(u,t)}{dt} = A\widetilde{x}(u,t) + \widetilde{x}(u,t)\int_{\mathbb{R}^d} R_1(\widetilde{x}(u,t) + x_0, v)\mu_t(dv) +$$
(25)
$$+ \int_{\mathbb{R}^d} (R_2(\widetilde{x}(u,t) + x_0, v) - R_2(x_0, v))(v - x_0)\mu_t(dv).$$

Because all eigenvalues $\lambda_1, ..., \lambda_d$ of the matrix A have negative real part, we can take $\alpha > 0$ such that

$$Re(\lambda_i) < -\alpha < 0 \quad \forall i = 1, ..., d.$$

and $K > 0$ such that

$$\left\| e^{At} \right\| < Ke^{-\alpha t} \quad \text{for all } t \geq 0,$$

where $M(A) = \frac{\alpha}{K}$, $K = K(A)$.

We have from (25), that

$$\widetilde{x}(u,t) = (u - x_0)e^{At} + \int_0^t e^{A(t-s)} \left(\widetilde{x}(u,s)\int_{\mathbb{R}^d} R_1(\widetilde{x}(u,s) + x_0, v)\mu_s(dv) + \right.$$
(26)
$$\left. + \int_{\mathbb{R}^d} (R_2(\widetilde{x}(u,s) + x_0, v) - R_2(x_0, v))(v - x_0)\mu_s(dv) \right) ds.$$

Using (21) and Lipshitz condition on R_2 it follows from (26), that

$$\|\widetilde{x}(u,t)\| \leq Ke^{-\alpha t}\|u - x_0\| + \int_0^t Ke^{-\alpha(t-s)}(M + LK\|m_0\|)\|\widetilde{x}(u,s)\| ds,$$

where $M = M\left(J_{f(x,x_0)}(x_0)\right)$.

Then

$$e^{\alpha t}\|\widetilde{x}(u,t)\| \leq K\|u - x_0\| + \int_0^t 2MKe^{\alpha s}\|\widetilde{m}_s\| ds.$$

From this inequality by Gronwall-Bellman lemma we get, that $\widetilde{x}(u,t)$ satisfies the estimation

$$\|\widetilde{x}(u,t)\| \leq K\|u - x_0\|e^{(-\alpha + 2MK)t} \quad \forall t \geq 0.$$

Then because $-\alpha + 2MK < 0$ we get

$$\lim_{t\to\infty} \widetilde{x}(u,t) = 0.$$

Using the inequality

$$\gamma_0\left(\mu_t, \delta_{\{x_0\}}\right) \leq \int\limits_{\mathbb{R}^d} \frac{\|x(u,t)-x_0\|}{1+\|x(u,t)-x_0\|}\mu_0\left(du\right).$$

we get, that

$$\lim_{t\to\infty} \gamma_0\left(\mu_t, \delta_{\{x_0\}}\right) = 0.$$

Theorem is proved. □

Let us notice, that in case $f(x_1, x_2) \equiv f(x_1)$ we have, that $R_2 \equiv 0$ Theorem 2.3 is exactly the result of classical theory of ODE as $x_0 = 0$ (look, for example, [10], p. 88) that claims

$$\lim_{t\to\infty} x(t,u) = 0.$$

Let us consider examples for Theorem 2.3 and Theorem 2.4.

Example 2.2. Consider the equation

$$\begin{cases} dx(u,t) = \int\limits_{\mathbb{R}^2} \left(\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} + \frac{1}{80} \begin{pmatrix} \sin\|v\| & \cos\|v\| \sin\|2v\| \\ \sin\|3v\| & \sin\|4v\| \end{pmatrix} \right) v\mu_t(dv)dt, \\ x(u,0) = u, \\ \mu_t = \mu_0 \circ x(\cdot,t)^{-1}, \end{cases}$$

where initial measure μ_0 is such, that $\int\limits_{\mathbb{R}^d} ||v||\mu_0(dv) < +\infty$.

Here we have

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}. \tag{27}$$

Then we can take $M(A) = \frac{1}{20}$. So, conditions of Theorem 1 and Theorem 2 are fulfilled and by Theorem 2.2

$$\lim_{t\to\infty} \mu_t = \mu_0 - \int\limits_{\mathbb{R}^d} v\mu_0(dv)$$

in Wasserstain distance.

This equation has a unique critical point zero and it is of the type "B".

Let us note, that if we consider the system

$$dy = \left(\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} + \frac{1}{80} \begin{pmatrix} \sin\|y\| & \cos\|y\| \sin\|2y\| \\ \sin\|3y\| & \sin\|4y\| \end{pmatrix} \right) ydt,$$

then zero solution of this system is globally asymptotically stable by Corollary 6.27 in [10], that give us, that all solutions to this system vanish on infinity.

Example 2.3. Consider the equation

$$\begin{cases} dx(u,t) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(u,t) + \right. \\ \left. + \begin{pmatrix} \sin\|x(u,t)\| & 0 \\ 0 & \sin\|4x(u,t)\| \end{pmatrix} \int\limits_{\mathbb{R}^2} \frac{v}{3200\|m_0\|(1+\|v\|)} \mu_t(dv) \right) dt, \\ x(u,0) = u, \\ \mu_t = \mu_0 \circ x(\cdot,t)^{-1}, \end{cases}$$

where initial measure μ_0 is such, that $\int\limits_{\mathbb{R}^d} ||v||\mu_0(dv) =< +\infty$, $m_0 = \int\limits_{\mathbb{R}^d} ||v||\mu_0(dv)$.

Here we also have (27) and take $M(A) = \frac{1}{20}$. So, conditions of Theorem 2.1 and Theorem 2.3 are fulfilled and by Theorem 2.3

$$\lim_{t\to\infty} \mu_t = \delta_0$$

in Wasserstain distance.

This equation has a unique critical point zero and it is of the type "A".

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