Theory of Stochastic Processes
Vol. 28 (44), no. 2, 2024, pp. 30–38

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ON THE LIMITING BEHAVIOR OF REDUCED PROCESSES GENERATED FROM BRANCHING PROCESSES POSSIBLY HAVING INFINITELY VARIANCE

This paper considers a critical reduced process generated by a Galton-Watson branching process in which the number offspring of one particle possibly has infinite variance and conditional limit theorem proved. The rate of weak convergence of the critical reduced processes to the limit law is also obtained in the case when the number offspring of one particle has a finite variance.

1. Introduction and statement of results.

Let $\{Z(0)=1,\ Z(k), k\geq 1\}$ be a Galton-Watson branching process in which the number offspring of one particle has a generating function

$$f(s)=Es^{\xi}=\sum_{k=0}^{\infty}p_k s^k,\ \ 0\leq s\leq 1,\quad \text{where }\ p_k=P(\xi=k),\ k=0,1,2,...$$

Let $A=f'(1)<\infty$. The branching process $\{Z(k), k\geq 0\}$ is called subcritical, critical or supercritical if $A<1,\ A=1$ or $A>1$ respectively.

We denote by $Z(m,n)$ the number of particles at a moment m $(m\leq n)$ in the process $\{Z(k), k\geq 0\}$ whose offsprings exist at the moment n.

The random process $\{Z(m,n), 0\leq m\leq n\}$ is called a reduced process generated by the process $\{Z(k), k\geq 0\}$. The reduced process $\{Z(m,n), 0\leq m\leq n\}$ is called subcritical, critical or supercritical if the corresponding Galton-Watson branching process $\{Z(k), k\geq 0\}$ is such, respectively. Reduced processes generated by Galton–Watson branching processes were introduced by Fleischmann and Prehn [1]. Fleischmann and Siegmund-Schultze [2] in case when $\{Z(m,n), 0\leq m\leq n\}$ is critical reduced process and $f''(1)<\infty$ proved a conditional functional limit theorem, which established the weak convergence as $n\to\infty$ of stepwise random processes $\{Z([nt], n), 0\leq t\leq 1\}$ under condition $\{Z(0)=1,\ Z(n)>0\}$ to the Yule process (with a modified time scale) in the space $D_{[0,1]}$ with the Skorokhod J-topology, where the sign $[a]$, here and henceforth, denotes integer part number a. Liu and Vatutin [3] for critical reduced processes proved conditional limit theorems under conditions $Z(0)=1, 0<Z(n)\leq\tau(n)$, where $\tau(n)=O(n)$, or $\tau(n)=o(n)$ for $n\to\infty$. In [6], a similar problem was solved for reduced critical Bellman-Harris branching processes. In the mentioned works it is assumed that the variance of the number offspring of one particle is finite.

Estimating the rate of convergence to the limit law in limit theorems for branching processes is an important and urgent problem. In work [8] V. Nagaev and R.I. Mukhamedkhanova obtained an estimate of the rate of convergence in Yaglom's limit theorem for critical branching processes. In [9], using a modified Stein method, upper bounds were obtained for the Kolmogorov and Kantorovich–Wasserstein metrics in Yaglom's theorem. In this paper, we obtain an estimate of the rate of convergence to the

2000 *Mathematics Subject Classification.* 60J80; 60J85.
Key words and phrases. Critical branching process, reduced process, limit theorem.

limit law for reduced critical processes in the Kolmogorov metric in the case when the number offspring of one particle has a finite variance.

We denote by $f_n(s)$ the n-th iteration $f(s)$:

$$f_0(s) = s,\ f_1(s) = f(s), ..., f_n(s) = f_{n-1}(f(s))$$

and let $B = \{Z(0) = 1,\ Z(n) > 0\}$. Through P_B, E_B we denote respectively a conditional probability and conditional mathematical expectation for given B.
If

$$A = f'(1) = E\xi = 1,\ \ 0 < \sigma^2 = f''(1) < \infty,$$

it is known (see, for example, [4], Chapter I, paragraph 9) that

$$P(Z(n) > 0/Z(0) = 1) = 1 - f_n(0) \sim \frac{2}{\sigma^2 n} \quad \text{as } n \to \infty, \tag{1}$$

and Yaglom's conditional limit theorem also holds:

$$\lim_{n\to\infty} P_B\left(\frac{2Z(n)}{\sigma^2 n} < y\right) = 1 - e^{-y},\ y \geq 0.$$

Fleischmann and Sigmund-Schulze [2] considered critical reduced processes with $0 < \sigma^2 < \infty$ and proved that for $s \in [0,1]$ and for any $t \in [0,1)$

$$\lim_{n\to\infty} E_B s^{Z([nt],n)} = \frac{(1-t)s}{1-ts}. \tag{2}$$

Suppose that the generating function $f(s)$ of the form

$$f(s) = s + (1-s)^{1+\alpha} L(1-s),\ \ s \in [0,1], \tag{3}$$

where $\alpha \in (0,1]$ is some fixed number, a function $L(x)$ is slowly varying at zero, i.e. such that for any constant $c > 0$

$$\frac{L(cx)}{L(x)} \to 1,\ \ x \to 0.$$

It is easy to see that if $f''(1) < \infty$, then representation (1) holds with $\alpha = 1$ (by virtue of the Taylor formula). In the case $\alpha = 1$ a random variable with generating function (3) may have a finite variance (for example, in the case $f(s) = s + \frac{1}{2}(1-s)^2$ and may has a infinite variance (for example, ([5]), in the case

$$f(s) = s + \frac{1}{2}(1-s)^2\left(\frac{1}{2} - \frac{1}{4}\log(1-s)\right).$$

If (3) holds then, as is known from [5] for any $\lambda \geq 0$ the following limit relation holds:

$$\lim_{n\to\infty} E_B e^{-\lambda(1-f_n(0))Z(n)} = 1 - \lambda(1+\lambda^\alpha)^{-1/\alpha}. \tag{4}$$

The next the question arises: what the asymptotic behavior the critical reduced process in the case when the generating function of the number offspring of one particle has the form (3). The following theorems answer that question.

Theorem 1.1. *If (3) holds, then for $s \in [0,1]$ and for any $t \in [0,1)$*

$$E_B s^{Z([nt],n)} \to 1 - \left(\frac{(1-s)^\alpha}{1-t+t(1-s)^\alpha}\right)^{1/\alpha},\ \ n \to \infty \tag{5}$$

and for any $0 \leq t_1 \leq t_2 < 1$ and $j \in N, s \in [0,1]$

$$E_B\left[s^{Z([nt_2],n)}\Big/Z([nt_1],n) = j\right] \to \left[1 - \left(\frac{t_2-t_1}{1-t_1} + \frac{1-t_2}{(1-t_1)(1-s)^\alpha}\right)^{-1/\alpha}\right]^j \tag{6}$$

as $n \to \infty$.

Corollary 1.1. *Finite-dimensional distributions of the random process* $\{Z([nt], n), 0 \leq t \leq 1\}$*, under the condition* $\{Z(0) = 1, \ Z(n) > 0\}$*, weakly converge as* $n \to \infty$ *to the corresponding finite-dimensional distributions inhomogeneous Markov process* $\{X(t), 0 \leq t \leq 1\}$*, the transition probabilities of which are determined by a generating function of the form as in the limit* (6).

In the case when $\alpha = 1$ (5) implies (2).

Comment. Repeating verbatim the second part of the argument, the proof of the theorem from [2], one can be convinced that the processes under the condition are relatively compact. Therefore, taking into account the corollary 1.1, we come to the conclusion that these processes weakly converge to a process in space with the Skorokhod J-topology. At the same time, it must be taken into account that

$$P(X(t+h) = 1/X(t) = 1) = \psi'(t, s)_{s=0} = \frac{1-t-h}{1-t}$$

Theorem 1.2. *Let the condition* (3) *be satisfied and* $Z(0) = \psi(x, n) = [xnL(1 - f_n(0))]^{\frac{1}{\alpha}}$ *with probability 1, where* $x > 0$ *is a constant. Then for* $s \in [0, 1]$ *and for any* $t \in [0, 1)$

$$E\left[s^{Z([nt],n)} \Big/ Z(0) = \psi(x, n)\right] \to e^{-\left(\frac{x}{\alpha}\right)^{1/\alpha}\left(t + \frac{1-t}{(1-s)^{\alpha}}\right)^{-1/\alpha}} \quad \text{as } n \to \infty.$$

If we set $t = 0$, then in the limit we obtain the generating function of the Poisson distribution, which is intuitively clear due to the independence and identical distribution of the number of descendants of particles in the Galton-Watson branching process.

Let F and G be two distribution functions. Let us $d_K(F, G)$ denote the Kolmogorov distance between the distribution functions F and G:

$$d_K(F, G) = \sup_{x \in R} |F(x) - G(x)|.$$

Theorem 1.3. *Let* $\{Z(m, n), 0 \leq m \leq n\}$ *be a reduced process generated by a critical branching process* $Z = \{Z(k), k \geq 0\}$ *for which* $0 < \sigma^2$ *and* $K = f'''(1) < \infty$*. Then*

$$d_K(W_{t,n}, W) = C\left|\frac{2K}{3\sigma^4} - 1\right| \frac{\log[n(1-t)]}{n(1-t)^2}(1 + o(1)),$$

where
$W_{t,n}(x) = P(Z([nt], n) < x/Z_0 = 1, Z_n > 0)$, $W(x) = (1-t)\sum_{k<x} t^k$ *and here in after* C *is a constant, not always the same.*

Comment. In [8], S.V. Nagaev and R.I. Mukhamedkhanova proved that if $0 < \sigma^2$ and $K = f'''(1) < \infty$, then the following estimate for the rate of convergence in Yaglom's theorem is valid:

$$d_K(F_n, E) = C\frac{\log^2 n}{n}(1 + o(1)),$$

where $F_n(y) = P_B(\frac{2Z(n)}{\sigma^2 n} < y)$, $E(y) = 1 - e^{-y}$, $y \geq 0$. In [9], using the Stein method, an upper estimate for the Wasserstein distance in Yaglom's theorem was obtained.

2. Proof of theorems

For proof of aur main results we will need the following lemmas.

Lemma 2.1. *[5]. If* (3) *holds and* a_n, b_n, $n \in \mathbb{N}$ *both sequences of positive numbers tend to zero as* $n \to \infty$ *and there exist constant numbers* $0 < K_1 < K_2 < \infty$ *such that for sufficiently large* n

$$0 < K_1 < \frac{a_n}{b_n} < K_2 < \infty,$$

then

$$\frac{L(a_n)}{L(b_n)} \to 1 \quad as\ n \to \infty.$$

Lemma 2.2. *[5]. If (3) holds, then*

$$\alpha n(1 - f_n(0))^\alpha L(1 - f_n(0)) \to 1 \quad as\ n \to \infty.$$

Lemma 2.3. *[8] Let $0 < \sigma^2$ and $K = f'''(1) < \infty$. Then*

$$P(Z_n > 0) = 1 - f_n(0) = \frac{2}{\sigma^2 n} + \frac{2}{\sigma^2}\left(\frac{2K}{3\sigma^4} - 1\right)\frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right).$$

Proof of Theorem 1.1.

We have

$$E_B s^{Z([nt],n)} = \frac{E\left[s^{Z([nt],n)} I(Z(n) > 0)/Z(0) = 1\right]}{P(Z(n) > 0/Z(0) = 1)} =$$

$$= \frac{E\left[s^{Z([nt],n)}/Z(0) = 1\right] - E\left[s^{Z([nt],n)} I(Z(n) = 0)/Z(0) = 1\right]}{1 - f_n(0)} =$$

$$= \frac{f_{[nt]}\left(f_{n-[nt]}(0) + \left(1 - f_{n-[nt]}(0)\right)s\right) - f_n(0)}{1 - f_n(0)}. \tag{7}$$

By Lemma 1.2

$$1 - f_k(0) \sim \frac{1}{[\alpha k L(1 - f_k(0))]^{1/\alpha}} \to 0,\ k \to \infty. \tag{8}$$

Therefore, for large enough k we have

$$f_k(0) + (1 - f_k(0))s \sim 1 - \frac{1 - s}{[\alpha k L(1 - f_k(0))]^{1/\alpha}}. \tag{9}$$

Considering (3) and the property of a slowly varying function we obtain

$$\frac{1 - f_{k+1}(0)}{1 - f_k(0)} = \frac{1 - f(f_k(0))}{1 - f_k(0)} = \frac{1 - f_k(0) - (1 - f_k(0))^{1+\alpha} L(1 - f_k(0))}{1 - f_k(0)} =$$

$$= 1 - (1 - f_k(0))^\alpha L(1 - f_k(0)) \to 1 \tag{10}$$

as $n \to \infty$. It follows that for any $\varepsilon > 0$ there is a number N_0 that is for everyone $n \geq N_0$

$$1 - \varepsilon \leq \frac{1 - f_{n-[nt]+1}(0)}{1 - f_{[n(1-t)]}(0)} \leq 1 + \varepsilon. \tag{11}$$

Then, by Lemma 1.1 and (11)

$$\frac{L\left(1 - f_{n-[nt]+1}(0)\right)}{L\left(1 - f_{[n(1-t)]}(0)\right)} \to 1 \quad \text{as } n \to \infty. \tag{12}$$

Now from (9), (11)-(12) we obtain

$$f_{n-[nt]}(0) + \left(1 - f_{n-[nt]}(0)\right)s \sim 1 - \frac{1 - s}{\left[\alpha(1 - t)nL\left(1 - f_{[n(1-t)]}(0)\right)\right]^{1/\alpha}} \tag{13}$$

as $n \to \infty$.

From (13) and from $f_k(0) \to 1,\ k \to \infty$, we conclude that there is a natural number $r = r(t, s, n) \to \infty$ such that

$$f_r(0) \leq f_{n-[nt]}(0) + \left(1 - f_{n-[nt]}(0)\right)s \leq f_{r+1}(0). \tag{14}$$

From Lemma 1.2 and from (13)-(14) we obtain

$$\frac{1}{\left[\alpha\left(r+1\right)L\left(1-f_{r+1}\left(0\right)\right)\right]^{1/\alpha}}\leq\frac{1-s}{\left[\alpha\left(1-t\right)nL\left(1-f_{[n(1-t)]}\left(0\right)\right)\right]^{1/\alpha}}\leq$$

$$\leq\frac{1}{\left[\alpha rL\left(1-f_{r}\left(0\right)\right)\right]^{1/\alpha}}. \tag{15}$$

Now we proof that

$$\frac{L\left(1-f_{[n(1-t)]}\left(0\right)\right)}{L\left(1-f_{[n(1-t)]+m}\left(0\right)\right)}\rightarrow 1 \quad\text{as } n\rightarrow\infty, \tag{16}$$

where $m\sim n\left(1-t\right)\left(\left(1-s\right)^{-\alpha}-1\right)$. To prove the relation (16), by Lemma 1.1, it is enough to show that the sequence $\frac{1-f_{[n(1-t)]+m}(0)}{1-f_{[n(1-t)]}(0)}$, $n\geq 1$ limited from below and above. Indeed, is obvious

$$\frac{1-f_{[n(1-t)]+m}\left(0\right)}{1-f_{[n(1-t)]}\left(0\right)}\leq\frac{1-f_{[n(1-t)]}\left(f_{m}\left(0\right)\right)}{1-f_{[n(1-t)]}\left(0\right)}\leq\frac{1-f_{[n(1-t)]}\left(0\right)}{1-f_{[n(1-t)]}\left(0\right)}=1.$$

Further, it is clear that

$$\frac{1-f_{[nt]+m}\left(0\right)}{1-f_{[nt]}\left(0\right)}=\prod_{i=0}^{m-1}\frac{1-f_{[nt]+i+1}\left(0\right)}{1-f_{[nt]+i}\left(0\right)}. \tag{17}$$

By virtue of (8) and Lemma 1.2 for sufficiently large n we have

$$\frac{1-f_{[nt]+i+1}\left(0\right)}{1-f_{[nt]+i}\left(0\right)}=1-\left(1-f_{[nt]+i}\left(0\right)\right)^{\alpha}L\left(1-f_{[nt]+i}\left(0\right)\right)>1-\frac{2}{\alpha\left([nt]+i\right)}.$$

From here and from (17) we get

$$\frac{1-f_{[nt]+m}\left(0\right)}{1-f_{[nt]}\left(0\right)}>\prod_{i=0}^{m-1}\left(1-\frac{2}{\alpha\left([nt]+i\right)}\right)>\left(1-\frac{2}{\alpha\left[nt\right]}\right)^{m}\sim e^{-\frac{2m}{\alpha nt}}\sim e^{-\frac{2(1-t)}{\alpha(1-s)^{\alpha}t}}$$

as $n\rightarrow\infty$.

Now from (15) and (16) we have

$$r\sim\frac{n\left(1-t\right)}{\left(1-s\right)^{\alpha}} \quad\text{as } n\rightarrow\infty. \tag{18}$$

Relations (14)-(18) and (8) allow us to conclude

$$f_{[nt]}\left(f_{n-[nt]}\left(0\right)+\left(1-f_{n-[nt]}\left(0\right)\right)s\right)\sim f_{[nt]}\left(f_{r}\left(0\right)\right)=f_{[nt]+r}\left(0\right)\sim$$

$$\sim 1-\frac{1-s}{\left[\alpha\left([nt]+r\right)nL\left(1-f_{[nt]+r}\left(0\right)\right)\right]^{1/\alpha}}\sim$$

$$\sim 1-\left[\alpha\left(t+\frac{1-t}{\left(1-s\right)^{\alpha}}\right)nL\left(1-f_{[nt]+r}\left(0\right)\right)\right]^{-1/\alpha}. \tag{19}$$

Similar to (16), we can proof the ratio

$$\frac{L\left(1-f_{n}\left(0\right)\right)}{L\left(1-f_{[nt]+r}\left(0\right)\right)}\rightarrow 1 \quad\text{as } n\rightarrow\infty.$$

From here and from (19) we obtain

$$f_{[nt]}\left(f_{n-[nt]}\left(0\right)+\left(1-f_{n-[nt]}\left(0\right)\right)s\right)\sim 1-\frac{1-f_{n}\left(0\right)}{\left(t+\frac{1-t}{(1-s)^{\alpha}}\right)^{1/\alpha}}. \tag{20}$$

Now from (8), (20) and (7) it follows

$$E_B s^{Z([nt],n)} = \frac{f_{[nt]}\left(f_{n-[nt]}(0) + \left(1 - f_{n-[nt]}(0)\right)s\right) - f_n(0)}{1 - f_n(0)} =$$

$$\to 1 - \left(t + \frac{1-t}{(1-s)^{\alpha}}\right)^{-1/\alpha}, \ n \to \infty.$$

Now we prove (4). For ease of recording, instead of $[nt]$ will write nt. We have

$$E_B\left(s^{Z(nt_2,n)} \Big/ Z(nt_1,n) = j\right) =$$

$$= \left[E\left(s^{Z(n(t_2-t_1),n(1-t_1))} \Big/ Z(0) = 1, Z(n(1-t_1)) > 0\right)\right]^j.$$

Obviously that from last relation and from (3) immediately follows (4).

The proof of Theorem 1.1 is complete.

Proof of corollary 1.1

Due the Markov property of the process we have for any

$$m_i, k_i \in N, \ \ i = \overline{1,r}, \ \ m_1 < ... < m_r < m, \ \ k,j \in N$$

$$P_B\left(Z(m_1,n) = k_i, \ \ i = \overline{1,m_{r-1}}, \ \ Z(m_r,n) = j, \, Z(m,n) = k\right) =$$

$$P_B\left(Z(m_1,n) = k_i, i = \overline{1,m_{r-1}}, \ \ Z(m_r,n) = j\right) P_B\left(Z(m,n) = k / Z(m_r,n) = j\right).$$

Now the weak convergence of finite-dimensional distributions of the process $\{Z([nt],n), \ 0 \le t \le 1\}$ (for given $\{Z(0) = 1, Z(n) > 0\}$) follows from the last relation, from (3)-(4) and from the method of mathematical induction.

Proof of Theorem 1.2.

Let us denote by $\{Z_i(k), k \ge 0\}$ the Galton-Watson branching process generated by i-th the particle from the initial particles $i = 1, ..., \psi(x,n)$. Let $\{Z_i(m,n), \ 0 \le m \le n\}$ the reduced process generated by the process $\{Z_i(k), k \ge 0\}$. Taking into account the additivity properties of Galton-Watson processes, it is easy to see that

$$E\left[s^{Z_i(m,n)} \Big/ Z_i(0) = 1\right] = E\left[E\left[s^{Z_i(m,n)} \Big/ Z_i(0) = 1, Z_i(m)\right]\right] =$$

$$= E\left[\left[f_{n-m}(0) + (1 - f_{n-m}(0))s\right]^{Z_i(m)} \Big/ Z_i(0) = 1\right] =$$

$$= f_m\left(f_{n-m}(0) + (1 - f_{n-m}(0))s\right). \tag{21}$$

It's clear that

$$Z(m,n) = \sum_{i=1}^{\psi(x,n)} Z_i(m,n), \tag{22}$$

moreover, $Z_i(m,n)$ are independent and identical distributed. From (21)-(22) it follows that

$$E\left[s^{Z([nt],n)} \Big/ Z(0) = \psi(x,n)\right] = \left[f_{[nt]}\left(f_{n-[nt]}(0) + \left(1 - f_{n-[nt]}(0)\right)s\right)\right]^{\psi(x,n)}.$$

From here and from (20) we obtain

$$E\left[s^{Z([nt],n)} \Big/ Z(0) = \psi(x,n)\right] = \left(f_{[nt]}\left(f_{n-[nt]}(0) + \left(1 - f_{n-[nt]}(0)\right)s\right)\right)^{\psi(x,n)} =$$

$$= \left(1 - \left[\alpha\left(t + \frac{1-t}{(1-s)^{\alpha}}\right)nL(1 - f_n(0))\right]^{-1/\alpha}\right)^{\psi(x,n)} \to e^{-\left(\frac{x}{\alpha}\right)^{1/\alpha}\left(t + \frac{1-t}{(1-s)^{\alpha}}\right)^{-1/\alpha}}$$

as $n \to \infty$, which completes the proof of Theorem 1.2.

Proof of Theorem 1.3.

For convenience of notation, we will write everywhere instead $[a]$ of a. It is easy to see

$$\varphi_n(t,s) = E\left[s^{Z(nt,n)} \big/ Z_0 = 1, Z_n > 0\right] =$$

$$\frac{f_{nt}\left(f_{n(1-t)}(0) + \left(1 - f_{n(1-t)}(0)\right)s\right) - f_n(0)}{1 - f_n(0)} = 1 - \frac{1 - f_{nt}(\psi_n(t,s))}{1 - f_n(0)}, \tag{23}$$

where

$$\psi_n(t,s) = f_{n(1-t)}(0) + \left(1 - f_{n(1-t)}(0)\right)s.$$

It's obvious that

$$d_K(W_n, W) = \sup_{k \in N}\left|P\left(Z([nt]\,n) = k/Z_0 = 1, Z_n > 0\right) - (1-t)\,t^k\right|.$$

According to Theorem 3 [5] (section 3 of Chapter 1)

$$d_K(W_n, W) \le \sup_{0 \le s \le 1}\left|\varphi_n(t,s) - \frac{(1-t)\,s}{1-ts}\right|.$$

Considering (23) we have

$$d_K(W_n, W) \le \sup_{0 \le s \le 1}\left|\varphi_n(t,s) - \frac{(1-t)\,s}{1-ts}\right| \le \sup_{0 \le s \le 1}\left|\frac{1 - f_{nt}(\psi_n(t,s))}{1 - f_n(0)} - \frac{1-s}{1-ts}\right| \le$$

$$\le \sup_{0 \le s \le 1}\left|\frac{1 - f_{nt}(\psi_n(t,s))}{1 - f_n(0)} - \frac{1 - f_{nt}\left(f_{n\frac{1-t}{1-s}}(0)\right)}{1 - f_n(0)}\right| + \sup_{0 \le s \le 1}\left|\frac{1 - f_{n\frac{1-ts}{1-s}}(0)}{1 - f_n(0)} - \frac{1-s}{1-ts}\right| =$$

$$\sup_{0 \le s \le 1}\left|\frac{f_{nt}(\psi_n(t,s)) - f_{nt}\left(f_{n\frac{1-t}{1-s}}(0)\right)}{1 - f_n(0)}\right| + \sup_{0 \le s \le 1}\left|\frac{1 - f_{n\frac{1-ts}{1-s}}(0)}{1 - f_n(0)} - \frac{1-s}{1-ts}\right|. \tag{24}$$

Applying the formula about the average value and considering that $f_n'(1) = 1$ and also monotonicity $f_n^{'}(s)$, we have

$$\left|f_{nt}(\psi_n(t,s)) - f_{nt}\left(f_{\frac{n(1-t)}{1-s}}(0)\right)\right| \le f_{nt}'(\theta)\left|\psi_n(t,s) - f_{\frac{n(1-t)}{1-s}}(0)\right| \le$$

$$\le \left|\psi_n(t,s) - f_{\frac{(1-t)n}{1-s}}(0)\right|. \tag{25}$$

By Lemma 1.3

$$\psi_n(t,s) - f_{\frac{(1-t)n}{1-s}}(0) = -\left(\frac{4K}{3\sigma^6} - \frac{2}{\sigma^2}\right)(1-s)\cdot$$

$$\cdot\left\{\frac{s\ln[n(1-t)]}{[n(1-t)]^2} + (1-s)\frac{\ln(1-s)}{[n(1-t)]^2}\right\} + o\left(\frac{\ln[n(1-t)]}{[n(1-t)]^2}\right).$$

Hence, considering the elementary inequality $s(1-s) \le \frac{1}{4}$ we get

$$\left|\psi_n(t,s) - f_{\frac{(1-t)n}{1-s}}(0)\right| \le \left|\frac{4K}{3\sigma^6} - \frac{2}{\sigma^2}\right|(1-s)\cdot$$

$$\cdot\left|\frac{s\ln[n(1-t)]}{[n(1-t)]^2} + (1-s)\frac{\ln(1-s)}{[n(1-t)]^2}\right| + o\left(\left|\frac{\ln[n(1-t)]}{[n(1-t)]^2}\right|\right) \le$$

$$\le C\left|\frac{4K}{3\sigma^6} - \frac{2}{\sigma^2}\right|(1-s)\frac{s\ln[n(1-t)]}{[n(1-t)]^2} \le \frac{C}{\sigma^2}\left|\frac{2K}{3\sigma^4} - 1\right|\frac{\ln[n(1-t)]}{[n(1-t)]^2}.$$

Therefore, considering (1) we have

$$\sup_{0 \le s \le 1}\left|\frac{\psi_n(t,s) - f_{\frac{(1-t)n}{1-s}}(0)}{1 - f_n(0)}\right| \le C\left|\frac{2K}{3\sigma^4} - 1\right|\frac{\ln[n(1-t)]}{n(1-t)^2} \tag{26}$$

According to Lemma 1.3 and considering the property of generating functions, we have

$$f_{nt}\left(f_{\frac{n(1-t)}{1-s}}(0)\right) = f_{n\left(t+\frac{1-t}{1-s}\right)}(0) = f_{n\frac{1-ts}{1-s}}(0) =$$
$$= 1 - \tfrac{2(1-s)}{\sigma^2(1-ts)n} - \left(\tfrac{4K}{3\sigma^6} - \tfrac{2}{\sigma^2}\right)\tfrac{(1-s)^2\{\ln[n(1-ts)]-\ln(1-s)\}}{[n(1-ts)]^2} +$$
$$+ o\left(\tfrac{\ln[n(1-t)]}{[n(1-t)]^2}\right).$$

That's why

$$\sup_{0\le s\le 1}\left|\frac{1-f_{nt}\left(f_{\frac{n(1-t)}{1-s}}(0)\right)}{1-f_n(0)} - \frac{1-s}{1-ts}\right| = \sup_{0\le s\le 1}\left|\frac{1-f_{n\left(t+\frac{1-t}{1-s}\right)}(0)}{1-f_n(0)} - \frac{1-s}{1-ts}\right| =$$

$$= \sup_{0\le s\le 1}\left|\frac{1-f_{n\frac{1-ts}{1-s}}(0)}{1-f_n(0)} - \frac{1-s}{1-ts}\right| =$$

$$(27)\qquad = \left(\frac{2K}{3\sigma^4}-1\right)\frac{\ln[n(1-t)]}{n(1-t)^2} + o\left(\frac{\ln[n(1-t)]}{n(1-t)^2}\right) \le C\left(\frac{2K}{3\sigma^4}-1\right)\frac{\ln[n(1-t)]}{n(1-t)^2}.$$

Now the relations from (24)-(27) lead to the statement of Theorem 1.3. The proof of the theorem is completed.

Let us consider an important special case when the generating function $f(s)$ is fractionally linear. In this case

$$f(s) = c + \frac{(1-c)^2 s}{1-cs},\ 0 \le s \le 1,$$

where $A\,(0 < c < 1)$ is constant. Obviously, that $f'(1) = 1$. As is well known (see, for example, [4], p. 6), that in the case under consideration

$$f_n(s) = \frac{nc - (nc+c-1)s}{1-c+nc-ncs}.$$

It is easy to verify that in the particular case under consideration we have the estimate

$$d_K(W_{t,n}, W) \le \frac{12(1-c)}{nc(1-t)^2},$$

which is much better than the general case.

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