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REMARK ON RIGHT CONTINUOUS EXPONENTIAL MARTINGALES

Using $\langle M^c \rangle$, jump measure μ and its compensator ν we characterize the event where the stochastic exponential $\mathcal{E}(M)$ equals to zero.

1. Introduction. Let us introduce a basic probability space (Ω, \mathcal{F}, P) and a right continuous filtration $(\mathcal{F}_t)_{0 \leq t < \infty}$ satisfying usual conditions. Let \mathcal{F}_∞ be the smallest σ -Algebra containing all \mathcal{F}_t for $t < \infty$ and let $M = (M_t)_{t \geq 0}$ be a local martingale on the stochastic interval $[[0; T[$, where T is a stopping time. Denote by $\Delta M_t = M_t - M_{t-}$ jumps of M and by $\mathcal{E}(M)$ the stochastic exponential of the local martingale M :

$$\mathcal{E}(M) = \exp \left\{ M_t - \frac{1}{2} \langle M^c \rangle_t \right\} \prod_{0 < s \leq t} (1 + \Delta M_s) e^{-\Delta M_s},$$

where M^c denotes a continuous local martingale part of M . Notice, that $M = M^c + M^d$ where M^d is a purely discontinuous local martingale part of M , which means that M^d is orthogonal to any continuous local martingale. With this we know that $M_t^d = \int_0^t \int_{-1}^\infty x d(\mu - \nu)$, where $\mu(\omega, t, x)$ is the jump measure of M and $\nu(\omega, t, x)$ is its compensator.

Through this paper we will integrate with respect to μ over the set $(-1; 1) \setminus \{0\}$ and we will write it as $\int_0^T \int_{-1}^1 \cdot d\mu$.

It is well known that $\mathcal{E}_t(M) = 1 + \int_0^t \mathcal{E}_{s-}(M) dM_s$, so it is clear that for local martingale M the associated stochastic exponential $\mathcal{E}(M)$ is a local martingale. Throughout of this paper we assume that $\Delta M_t \geq -1$ which implies that $\mathcal{E}(M)$ is a non-negative local martingale and therefore a supermartingale. In case when $\mathcal{E}(M)$ is a uniformly integrable martingale on $[[0; T]$, we can define using $\mathcal{E}(M)$ and the Radon-Nikodym derivative a new probability measure: $dQ = \mathcal{E}_T(M) dP$. It is clear that $Q \ll P$ and if $P\{\mathcal{E}_T(M) > 0\} = 1$, then P and Q will be equivalent probability measures ($P \sim Q$). To know whether $P \sim Q$ or not, we must study the set $\{\mathcal{E}_T(M) = 0\}$. In case when $M = M^c$ it was shown by Kazamaki [2] in 1994 that $\{\mathcal{E}_T(M^c) = 0\} = \{\langle M^c \rangle_T = \infty\}$. For general M , in 1978 it was proved by J. Jacod [1] that

$$\{\mathcal{E}_\infty(M) > 0\} = \left\{ \langle M^c \rangle_\infty + \int_0^\infty \int_{-1}^\infty \frac{x^2}{1 + |x|} d\nu + \int_0^\infty \frac{1}{\mathcal{E}_{s-}(M)} dB_s < \infty \right\}$$

where B_s is the predictable, non-decreasing process from the Doob-Meyer decomposition of $\mathcal{E}(M)$. In 2019 M. Larsson and J. Ruf [3] proved the set inclusion

$$\left\{ \lim_{t \uparrow \tau} \mathcal{E}_t(M) = 0 \right\} \subset \left\{ \lim_{t \uparrow \tau} M_t = -\infty \right\} \cup \{[M]_\tau = \infty\} \cup \{\Delta M_t = -1, t \in [0; \tau)\}$$

holds true for any predictable stopping time τ . With this they proved, that if in addition $\Delta M \geq -1$ and $\lim_{t \uparrow \tau} M_t < \infty$, then the reverse set inclusion also holds.

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The aim of this paper is to characterize the set $\{\mathcal{E}_T(M) = 0\}$ using $\langle M^c \rangle$, $\mu(\omega, t, x)$ and $\nu(\omega, t, x)$, for any stopping time T .

Theorem 1. Let M be a local martingale with $\Delta M \geq -1$. Then the following set equalities hold true P a. s.:

$$(i) \quad \{\mathcal{E}_T(M) = 0\} = \left\{ \langle M^c \rangle_T + \int_0^T \int_{-1}^1 \frac{x^2}{1+x} d\mu + \int_0^T \int_1^{+\infty} \frac{x^2}{1+x} d\nu = \infty \right\};$$

(ii) If $E \frac{1}{1+\Delta M_\sigma} 1_{\{|\Delta M_\sigma| \leq 1\}} < \infty$, for any $\sigma < \infty$, then

$$\{\mathcal{E}_T(M) = 0\} = \left\{ \langle M^c \rangle_T + \int_0^T \int_{-1}^{+\infty} \frac{x^2}{1+x} d\nu = \infty \right\};$$

(iii) If $E \Delta M_\sigma < \infty$, for any $\sigma < \infty$, then

$$\{\mathcal{E}_T(M) = 0\} = \left\{ \langle M^c \rangle_T + \int_0^T \int_{-1}^{+\infty} \frac{x^2}{1+x} d\mu = \infty \right\}.$$

Remark 1. In the contrary to the result from Jacod [1], we are not using the additional increasing process B_t , which is not in terms of M . In their result Larsson and Ruf [3] used the predictable stopping time τ and they have additional restriction on M to obtain the set equality. In part (i) of Theorem 1 we have the set equality without any restriction on M and in part (ii) we have the set equality with predictable characteristics of M , but with integrability restriction on jumps of M . With this let us mention that we use any kind of stopping times T , while Larsson and Ruf [3] used only predictable stopping times.

Proof of the Theorem 1: If $\Delta M_s = -1$ for some $s \leq T$, then it is obvious that $\mathcal{E}_T(M^d) = 0$ and $\int_0^T \int_{-1}^1 \frac{x^2}{1+x} d\mu = \sum_{s \leq T} \frac{(\Delta M_s)^2}{1+\Delta M_s} = \infty$, so we can prove Theorem 1 when $\Delta M_s > -1$.

Define local martingales

$$M_t^1 = \int_0^t \int_{-1}^1 x d(\mu - \nu); \quad M_t^2 = \int_0^t \int_1^{+\infty} x d(\mu - \nu).$$

It is clear that $|\Delta M_t^1| \leq 1$, $\Delta M_t^2 \geq 1$ and $M_t^d = M_t^1 + M_t^2$, so we have $M = M^c + M^1 + M^2$. It is easy to check that $\mathcal{E}_T(M) = \mathcal{E}_T(M^c) \mathcal{E}_T(M^1) \mathcal{E}_T(M^2)$, so

$$\{\mathcal{E}_T(M) = 0\} = \{\mathcal{E}_T(M^c) = 0\} \cup \{\mathcal{E}_T(M^1) = 0\} \cup \{\mathcal{E}_T(M^2) = 0\}.$$

It is well known from Kazamaki [2] that $\{\mathcal{E}_T(M^c) = 0\} = \{\langle M^c \rangle_T = \infty\}$, so to prove part (i) of Theorem 1 it is sufficient to show the set equalities

$$(1) \quad \{\mathcal{E}_T(M^1) = 0\} = \left\{ \int_0^T \int_{-1}^1 \frac{x^2}{1+x} d\mu = \infty \right\},$$

$$(2) \quad \{\mathcal{E}_T(M^2) = 0\} = \left\{ \int_0^T \int_1^{+\infty} \frac{x^2}{1+x} d\nu = \infty \right\}.$$

First let us show that $\left\{ \int_0^T \int_{-1}^{+\infty} \frac{x^2}{1+x} d\mu = \infty \right\} \subset \{\mathcal{E}_T(M^d) = 0\}$ for any local martingale M . An easy calculations give us:

$$\mathcal{E}_T^2\left(\frac{1}{2}M^d\right) = \exp \left\{ M_T^d + \int_0^T \int_{-1}^{+\infty} \left[2 \ln\left(1 + \frac{x}{2}\right) - x \right] d\mu = \right.$$

$$\mathcal{E}_T(M^d) \exp \left\{ \int_0^T \int_{-1}^{+\infty} \ln \frac{(1 + \frac{x}{2})^2}{1+x} d\mu \right\}$$

and from this we obtain:

$$\mathcal{E}_T(M^d) = \mathcal{E}_T^2\left(\frac{1}{2}M^d\right) \exp \left\{ - \int_0^T \int_{-1}^{+\infty} \ln \left(1 + \frac{1}{4} \cdot \frac{x^2}{1+x}\right) d\mu \right\}.$$

The supermartingale property of $\mathcal{E}(\frac{1}{2}M^d)$ implies $P\{\mathcal{E}_T(\frac{1}{2}M^d) < \infty\} = 1$, so we obtain that $\{\int_0^T \int_{-1}^{+\infty} \ln(1 + \frac{1}{4} \cdot \frac{x^2}{1+x}) d\mu = \infty\} \subset \{\mathcal{E}_T(M^d) = 0\}$. Now the set equalities below are obvious and the first set inclusion follows from the inequality $\ln(1 + \sum_n x_n) \leq \sum_n \ln(1 + x_n)$, where $x_n \geq 0$:

$$\begin{aligned} \left\{ \int_0^T \int_{-1}^{+\infty} \frac{x^2}{1+x} d\mu = \infty \right\} &= \left\{ 1 + \frac{1}{4} \int_0^T \int_{-1}^{+\infty} \frac{x^2}{1+x} d\mu = \infty \right\} = \\ &\left\{ \ln \left(1 + \frac{1}{4} \int_0^T \int_{-1}^{+\infty} \frac{x^2}{1+x} d\mu \right) = \infty \right\} \subset \\ &\left\{ \int_0^T \int_{-1}^{+\infty} \ln \left(1 + \frac{1}{4} \cdot \frac{x^2}{1+x} \right) d\mu = \infty \right\} \subset \{\mathcal{E}_T(M^d) = 0\}. \end{aligned}$$

So we proved that $\{\int_0^T \int_{-1}^{+\infty} \frac{x^2}{1+x} d\mu = \infty\} \subset \{\mathcal{E}_T(M^d) = 0\}$, for any local martingale M . It is clear that from this we can deduce as a particular case

$$(3) \quad \left\{ \int_0^T \int_{-1}^1 \frac{x^2}{1+x} d\mu = \infty \right\} \subset \{\mathcal{E}_T(M^1) = 0\}.$$

Now it is time to prove the reverse set inclusion:

$$\{\mathcal{E}_T(M^1) = 0\} \subset \left\{ \int_0^T \int_{-1}^1 \frac{x^2}{1+x} d\mu = \infty \right\}.$$

$$\begin{aligned} \mathcal{E}_T(M^1) \mathcal{E}_T^2\left(-\frac{1}{2}M^1\right) &= \exp \left\{ M_T^1 + \int_0^T \int_{-1}^1 [\ln(1+x) - x] d\mu - M_T^1 + \right. \\ &\left. \int_0^T \int_{-1}^1 [2 \ln(1 - \frac{x}{2}) + x] d\mu \right\} = \exp \left\{ \int_0^T \int_{-1}^1 \ln [(1+x)(1 - \frac{x}{2})^2] d\mu \right\}. \end{aligned}$$

From the last equality and the supermartingale property of $\mathcal{E}(-\frac{1}{2}M^1)$ we deduce that

$$\{\mathcal{E}_T(M^1) = 0\} \subset \left\{ - \int_0^T \int_{-1}^1 \ln [(1+x)(1 - \frac{x}{2})^2] d\mu = \infty \right\}.$$

Using Lemma 1 from Appendix we obtain $-\ln(1+x)(1 - \frac{x}{2})^2 \leq \frac{2x^2}{1+x}$ and this gives us an inclusion:

$$\{\mathcal{E}_T(M^1) = 0\} \subset \left\{ \int_0^T \int_{-1}^1 \frac{2x^2}{1+x} d\mu = \infty \right\} = \left\{ \int_0^T \int_{-1}^1 \frac{x^2}{1+x} d\mu = \infty \right\}$$

which with (3) implies the equality (1).

Now we prove the set equality $\{\mathcal{E}_T(M^2) = 0\} = \{\int_0^T \int_{(1;+\infty)} \frac{x^2}{1+x} d\nu = \infty\}$.

It follows from Jacod [1], that $\{\int_0^T \int_1^{+\infty} \frac{x^2}{1+|x|} d\nu = \infty\} \subset \{\mathcal{E}_T(M^2) = 0\}$. But it is clear that $\int_0^T \int_1^{+\infty} \frac{x^2}{1+|x|} d\nu = \int_0^T \int_1^{+\infty} \frac{x^2}{1+x} d\nu$, because $x \geq 1$. So we have

$$\left\{ \int_0^T \int_1^{+\infty} \frac{x^2}{1+x} d\nu = \infty \right\} \subset \{\mathcal{E}_T(M^2) = 0\}.$$

For the reverse inclusion it is clear that $\mathcal{E}_T(M^2) = \exp\left\{-\int_0^T \int_1^{+\infty} x d\nu + \int_0^T \int_1^{+\infty} \ln(1+x) d\mu\right\}$ and from this we deduce $\{\mathcal{E}_T(M^2) = 0\} \subset \{\int_0^T \int_1^{+\infty} x d\nu = \infty\}$. For $x \geq 1$ the inequality $x \leq \frac{2x^2}{1+x}$ holds true, which implies that

$$\left\{ \int_0^T \int_1^{+\infty} x d\nu = \infty \right\} \subset \left\{ \int_0^T \int_1^{+\infty} \frac{x^2}{1+x} d\nu = \infty \right\}.$$

So we will have inclusion $\{\mathcal{E}_T(M^2) = 0\} \subset \left\{ \int_0^T \int_1^{+\infty} \frac{x^2}{1+x} d\nu = \infty \right\}$ and finally we get the set equality (2). So the proof of part (i) is completed.

Now we shall prove part (ii) and part (iii) of Theorem 1. To prove part (ii) we need the set equality

$$(4) \quad \left\{ \int_0^T \int_{-1}^1 \frac{x^2}{1+x} d\mu = \infty \right\} = \left\{ \int_0^T \int_{-1}^1 \frac{x^2}{1+x} d\nu = \infty \right\}$$

and for part (iii)

$$(5) \quad \left\{ \int_0^T \int_1^{+\infty} \frac{x^2}{1+x} d\mu = \infty \right\} = \left\{ \int_0^T \int_1^{+\infty} \frac{x^2}{1+x} d\nu = \infty \right\}.$$

Inequality $\frac{(\Delta M_\sigma)^2}{1+\Delta M_\sigma} 1_{\{|\Delta M_\sigma| \leq 1\}} \leq \frac{1}{1+\Delta M_\sigma} 1_{\{|\Delta M_\sigma| \leq 1\}}$ and the integrability condition

$$E \frac{1}{1+\Delta M_\sigma} 1_{\{|\Delta M_\sigma| \leq 1\}} < \infty$$

from part (ii) gives us possibility to use part 2) of Theorem 2.6.1 from [4] to obtain (4). Inequality $\frac{x^2}{1+x} \leq x$ for $x \geq 1$ and condition $E(\Delta M_\sigma) < \infty$ from (iii) will imply the integrability of $\frac{(\Delta M_\sigma^2)^2}{1+\Delta M_\sigma^2}$. Indeed

$$E \frac{(\Delta M_\sigma^2)^2}{1+\Delta M_\sigma^2} = E \frac{(\Delta M_\sigma)^2}{1+\Delta M_\sigma} 1_{\{\Delta M_\sigma \geq 1\}} \leq E(\Delta M_\sigma 1_{\{\Delta M_\sigma \geq 1\}}) \leq 1 + E(\Delta M_\sigma) < \infty.$$

So we can apply part 2) of Theorem 2.6.1 from [4] and we obtain (5). \square

4. Appendix.

Lemma 1. $k(x) = \frac{2x^2}{1+x} + \ln(1+x)(1 - \frac{1}{2}x)^2 \geq 0$ for any $x \in (-1; 1)$.

Proof.

$$\begin{aligned} k'(x) &= \frac{4x(1+x) - 2x^2}{(1+x)^2} + \frac{(1 - \frac{1}{2}x)^2 - (1+x)(1 - \frac{1}{2}x)}{(1+x)(1 - \frac{1}{2}x)^2} = \\ &= \frac{2x^2 + 4x}{(1+x)^2} - \frac{3x}{(1+x)(2-x)} = \frac{-2x^3 - 3x^2 + 5x}{(1+x)^2(2-x)} = \frac{x(2x+5)(1-x)}{(1+x)^2(2-x)}. \end{aligned}$$

It is obvious that $k'(0) = 0$, $k'(x) < 0$ when $x \in (-1; 0)$ and $k'(x) > 0$ when $x \in (0; 1)$. So $x = 0$ is a minimum point and because $k(0) = 0$, we can deduce that $k(x) \geq 0$ for $x \in (-1; 1)$. \square

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