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ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS IN MULTIDIMENSIONAL SPACE

Consider the multidimensional SDE

$$\mathrm{d}X(t) = a(X(t))\mathrm{d}t + b(X(t))\mathrm{d}W(t).$$

We study the asymptotic behavior of its solution $X(t)$ as $t \to \infty$, namely, we study sufficient conditions of transience of its solution $X(t)$, stabilization of its multidimensional angle $X(t)/|X(t)|$, and asymptotic equivalence of solutions of the given SDE and the following ODE without noise:

$$\mathrm{d}x(t) = a(x(t))\mathrm{d}t.$$

1. Introduction

Usually, there are two modes of behavior of SDE solutions as $t \to \infty$: transience and recurrence. In this article, we study the transience of solutions.

Consider a one-dimensional SDE of the form

$$\mathrm{d}X(t) = a(X(t))\mathrm{d}t + b(X(t))\mathrm{d}W(t). \tag{1}$$

Gikhman and Skorokhod (see § 16, 17 of Part I in [4]) were the first who started studying its non-random asymptotics, i.e., a function $x(t)$ such that $X(t) \sim x(t),\ t \to \infty$, a.s. Later this problem was studied by Keller, Kersting, and Rösler [7]. Buldygin, Indlekofer, Klesov, Stainebach, and Tymoshenko (see [2], [1]) considered some types of non-autonomous SDEs and studied asymptotic behavior of their solutions; in particular, they considered the problem of asymptotic equivalence of SDE and ODE solutions. Pilipenko, Proske, and Pavlyukevich (see [12], [10]) considered SDEs with non-Gaussian noise.

Unlike the one-dimensional case, the asymptotic behavior of the multidimensional SDE solution differs even provided its transience. Friedman [3] and Khasminskii [8] studied conditions of transience and recurrence for systems of linear SDEs. Friedman also studied the behavior of the polar angle of the two-dimensional SDE solution (see § 12.7 in [3]). Spitzer [14] studied the limit distribution (as $t \to \infty$) of the polar angle of the Wiener process on the plane. Samoilenko, Stanzhitskii, Novak [13] and Pilipenko, Proske [11] studied the transience of solutions to multidimensional SDEs.

Consider an n-dimensional ($n \geq 2$) autonomous SDE of the form

$$\mathrm{d}X(t) = a(X(t))\mathrm{d}t + b(X(t))\mathrm{d}W(t),\ X(0) = x_0 \in \mathbb{R}^n, \tag{2}$$

where $a\colon \mathbb{R}^n \to \mathbb{R}^n$, $b\colon \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally bounded and W is an m-dimensional Wiener process

In this article, we study the asymptotic behavior of solutions $X(t)$ of the SDE (2) as $t \to \infty$. Namely, we search for sufficient conditions such that:

- the solution $X(t)$ is transient, i.e., almost surely

$$|X(t)| \to \infty,\ t \to \infty;$$

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- the angle of the solution's growth stabilizes, i.e., there exists a random variable Φ_∞ (*the limit angle*) such that the limit
$$\lim_{t\to\infty}\frac{X(t)}{|X(t)|} =: \Phi_\infty$$
exists almost surely;
- there exists a non-random function of two variables $r_\varphi(t),\ \varphi\in\mathbb{R}^n,\ t\ge 0$, that describes the asymptotics of $|X(t)|$, i.e., almost surely
$$|X(t)| \sim r_{\Phi_\infty}(t),\ t\to\infty,$$
where Φ_∞ is the limit angle.

For convenience, denote $|X(t)| =: R(t)$ and $X(t)/|X(t)| =: \Phi(t)$. We will call $R(t)$ the *radius* process and $\Phi(t)$ the *angle* process.

It is known that if the diffusion is non-degenerate ($\inf_x \det\left(b^{\mathrm{T}}b\right)(x) > 0$) then the solution of a multidimensional ($n\ge 2$) SDE starting at $x_0\neq 0$ never hits zero with probability 1. Without loss of generality, we will assume that $X(0)=x_0\neq 0$. Applying the Itô formula to the radius and angle processes, we get

$$\mathrm{d}R(t) = \mu(R(t),\Phi(t))\mathrm{d}t + \sigma(R(t),\Phi(t))\mathrm{d}W(t),\ R(0)=r_0=|x_0|>0, \tag{3}$$
$$\mathrm{d}\Phi(t) = \nu(R(t),\Phi(t))\mathrm{d}t + \chi(R(t),\Phi(t))\mathrm{d}W(t),\ \Phi(0)=\varphi_0=\frac{x_0}{|x_0|}, \tag{4}$$

where $\mu\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $\sigma\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$, $\nu\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $\chi\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ are some functions.

Let's write down formulae for coefficients μ, σ, ν, χ of equations (3)–(4) in terms of coefficients a, b of the initial SDE (2). For this, define the *radial* and *tangential* components of the vector field a at the point $x\neq 0$ by
$$a_{\mathrm{rad}}(x) := \frac{xx^{\mathrm{T}}}{|x|^2}a(x), \qquad a_{\tan}(x) := a(x) - a_{\mathrm{rad}}(x),$$
respectively. Similarly, define the radial and tangential components of the matrix field b at the point $x\neq 0$ by
$$b_{\mathrm{rad}}(x) := \frac{xx^{\mathrm{T}}}{|x|^2}b(x), \qquad b_{\tan}(x) := b(x) - b_{\mathrm{rad}}(x),$$
respectively. Then
$$\mu(r,\varphi) = \varphi^{\mathrm{T}}a_{\mathrm{rad}}(r\varphi) + \frac{|b_{\tan}(r\varphi)|^2}{2r}, \qquad \sigma(r,\varphi) = \varphi^{\mathrm{T}}b_{\mathrm{rad}}(r\varphi),$$
$$\nu(r,\varphi) = \frac{a_{\tan}(r\varphi)}{r} - \frac{\left(2(b(r\varphi)b^{\mathrm{T}}(r\varphi))_{\tan}\right) + |b_{\tan}(r\varphi)|^2)\varphi}{2r^2},$$
$$\chi(r,\varphi) = \frac{b_{\tan}(r\varphi)}{r}.$$

Further, we focus on studying the system of SDEs

$$\mathrm{d}R(t) = \mu(R(t),\Phi(t))\mathrm{d}t + \sigma(R(t),\Phi(t))\mathrm{d}W(t),\ R(0)=r_0, \tag{5}$$
$$\mathrm{d}\Phi(t) = \nu(R(t),\Phi(t))\mathrm{d}t + \chi(R(t),\Phi(t))\mathrm{d}W(t),\ \Phi(0)=\varphi_0, \tag{6}$$

considering coefficients μ,σ,ν,χ to be arbitrary (not related to the coefficients a,b of the initial SDE (2)). Nevertheless, we will keep calling the processes R and Φ the radius and the angle, respectively. Results about $R(t)$ and $\Phi(t)$ obtained below will describe the asymptotic behavior of the solution $X(t)$ to the SDE (2).

Notice that the problems stated previously now can be reformulated in terms of the radius and the angle processes; namely, we search for sufficient conditions such that the following hold almost surely:

- $R(t) \to \infty,\ t \to \infty$;
- $\exists \lim_{t\to\infty} \Phi(t) =: \Phi_\infty$;
- $\exists r_\varphi(t)\colon R(t) \sim r_{\Phi_\infty}(t),\ t \to \infty.$

This article has the following structure. In Section 2, we prove a general theorem about the asymptotic equivalence of SDE and ODE solutions in the one-dimensional non-autonomous case. In Section 3, we state sufficient conditions that guarantee the transience of the radius. In Section 4, we prove a theorem about angle stabilization. In Section 5, we prove the main result about radius asymptotics. The Appendix contains some auxiliary results.

2. Asymptotic Behavior of One-Dimensional SDEs

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a filtered probability space, B be a one-dimensional $(\mathcal{F}_t)_t$-adapted Wiener process.

Let's prove the next lemma about the asymptotics of Itô integrals.

Lemma 2.1. *Let $b = b(t,\omega)$ be a progressively measurable stochastic process and $C > 0$, $\beta > 0$ be constants such that*

$$\mathbb{E}b^2(t) \le C(1 + t^{2\beta}),\ t \ge 0.$$

Then for any $\gamma > \beta + \frac{1}{2}$, almost surely

$$\frac{1}{t^\gamma}\int_0^t b(s)\mathrm{d}B(s) \to 0,\ t \to \infty.$$

Proof. Let $\varepsilon > 0$, $k \in \mathbb{N}_0$. Then using Doob's martingale inequality, the Itô isometry, and Fubini's theorem, we have

$$\mathbb{P}\left\{\sup_{2^k \le t \le 2^{k+1}} \frac{1}{t^\gamma}\left|\int_0^t b(s)\mathrm{d}B(s)\right| \ge \varepsilon\right\} \le$$

$$\le \mathbb{P}\left\{\sup_{2^k \le t \le 2^{k+1}} \frac{1}{(2^k)^\gamma}\left|\int_0^t b(s)\mathrm{d}B(s)\right| \ge \varepsilon\right\} \le$$

$$\le \mathbb{P}\left\{\sup_{0 \le t \le 2^{k+1}} \left|\int_0^t b(s)\mathrm{d}B(s)\right| \ge \varepsilon 2^{\gamma k}\right\} \le$$

$$\le \frac{1}{(\varepsilon 2^{\gamma k})^2}\mathbb{E}\left(\int_0^{2^{k+1}} b(s)\mathrm{d}B(s)\right)^2 = \frac{1}{\varepsilon^2 2^{2\gamma k}}\mathbb{E}\int_0^{2^{k+1}} b^2(s)\mathrm{d}s =$$

$$= \frac{1}{\varepsilon^2 2^{2\gamma k}}\int_0^{2^{k+1}} \mathbb{E}b^2(s)\mathrm{d}s \le \frac{1}{\varepsilon^2 2^{2\gamma k}}\int_0^{2^{k+1}} C(1 + s^{2\beta})\mathrm{d}s =$$

$$= \frac{1}{\varepsilon^2 2^{2\gamma k}}\left(2^{k+1} + \frac{(2^{k+1})^{2\beta+1}}{2\beta+1}\right) =$$

$$= \frac{2C}{\varepsilon^2}2^{(1-2\gamma)k} + \frac{2^{2\beta+1}C}{(2\beta+1)\varepsilon^2}2^{(2\beta-2\gamma+1)k}.$$

Hence, for any $n \in \mathbb{N}$,

$$\mathbb{P}\left\{\limsup_{t\to\infty} \frac{1}{t^\gamma}\left|\int_0^t b(s)\mathrm{d}B(s)\right| \ge \varepsilon\right\} \le \mathbb{P}\left\{\sup_{t \ge 2^n} \frac{1}{t^\gamma}\left|\int_0^t b(s)\mathrm{d}B(s)\right| \ge \varepsilon\right\} \le$$

$$\le \sum_{k=n}^\infty \mathbb{P}\left\{\sup_{2^k \le t \le 2^{k+1}} \frac{1}{t^\gamma}\left|\int_0^t b(s)\mathrm{d}B(s)\right| \ge \varepsilon\right\} \le$$

$$\le \frac{2C}{\varepsilon^2} \sum_{k=n}^{\infty} 2^{(1-2\gamma)k} + \frac{2^{2\beta+1}C}{(2\beta+1)\varepsilon^2} \sum_{k=n}^{\infty} 2^{(2\beta-2\gamma+1)k}.$$

The last two series converge to 0 as $n \to \infty$ (since $2\gamma - 2\beta - 1 > 0$ by condition) so the right-hand side converges to 0 as $n \to \infty$.

Since $\varepsilon > 0$ is arbitrary,

$$\mathbb{P}\left\{ \limsup_{t\to\infty} \frac{1}{t^\gamma} \left| \int_0^t b(s)\mathrm{d}B(s) \right| > 0 \right\} = 0.$$

Therefore,

$$\mathbb{P}\left\{ \limsup_{t\to\infty} \frac{1}{t^\gamma} \left| \int_0^t b(s)\mathrm{d}B(s) \right| = 0 \right\} = 1,$$

that is

$$\mathbb{P}\left\{ \lim_{t\to\infty} \frac{1}{t^\gamma} \int_0^t b(s)\mathrm{d}B(s) = 0 \right\} = 1,$$

and the lemma is proved. □

Let X be a solution of the one-dimensional non-autonomous SDE

$$\mathrm{d}X(t) = a(X(t), t, \omega)\mathrm{d}t + b(X(t), t, \omega)\mathrm{d}B(t), \qquad X(0) = x_0 \in (x_1, x_2), \tag{7}$$

where $a = a(x, s, \omega), b = b(x, s, \omega)$ are such that for any $t \ge 0$, their restrictions to $\mathbb{R} \times [0, t] \times \Omega$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable.

The next theorem generalizes the results of Gikhman and Skorokhod (see § 17 of Part I in [4]).

Theorem 2.1. *Suppose that:*

- *the coefficient a is bounded and*
$$\lim_{\substack{x\to+\infty \\ t\to\infty}} a(x, t) = A \ \textit{a.s.},$$
where $A > 0$ is a positive random variable;
- *there exist constants $\beta \in \left(0, \frac{1}{2}\right)$ and $C > 0$ such that*
$$\mathbb{P}\{\forall x \in \mathbb{R}\ \forall t \ge 0\ |b(x, t)| \le C(1 + |x|^\beta)\} = 1;$$
- *$X(t) \to +\infty$ a.s.*

Then almost surely

$$X(t) \sim At, \ t \to \infty.$$

Remark 2.1. *Here, $X(t)$ is a weak solution, not necessarily unique.*

Proof. Consider SDE (7) in the integral form:

$$X(t) = x_0 + \int_0^t a(X(s), s)\mathrm{d}s + \int_0^t b(X(s), s)\mathrm{d}B(s).$$

Estimate the expectation of $X^2(t)$:

$$\mathbb{E}X^2(t) = \mathbb{E}\left(\left(x_0 + \int_0^t a(X(s), s)\mathrm{d}s \right) + \int_0^t b(X(s), s)\mathrm{d}B(s) \right)^2 \le$$

$$\le 2\mathbb{E}\left(x_0 + \int_0^t a(X(s), s)\mathrm{d}s \right)^2 + 2\mathbb{E}\left(\int_0^t b(X(s), s)\mathrm{d}B(s) \right)^2.$$

Since a is bounded, for the first expectation we have the estimate

$$\mathbb{E}\left(x_0 + \int_0^t a(X(s), s)\mathrm{d}s \right)^2 \le (C_1 t + C_2)^2$$

for some $C_1 > 0,\ C_2 > 0$. Estimate the second expectation using the Itô isometry and Jensen's inequality:

$$\mathbb{E}\left(\int_0^t b(X(s),s)\mathrm{d}B(s)\right)^2 \le C_3 t + C_3\int_0^t (\mathbb{E}X^2(s))^\beta \mathrm{d}s$$

for some $C_3 > 0$. Hence, we obtain the estimate

$$\mathbb{E}X^2(t) \le (C_4 t + C_5)^2 + C_6\int_0^t (\mathbb{E}X^2(s))^\beta \mathrm{d}s$$

for some $C_4, C_5, C_6 > 0$, or denoting $u(t) := \mathbb{E}X^2(t)$,

$$u(t) \le (C_4 t + C_5)^2 + C_6\int_0^t u^\beta(s)\mathrm{d}s.$$

Using a generalization of Grönwall's inequality (Lemma 6.1 in Appendix), we get

$$u(t) \le \tilde{C}\left((1-\beta)t + (C_4 t + C_5)^{2-2\beta}\right)^{\frac{1}{1-\beta}}$$

for some $\tilde{C} > 0$. So

$$\limsup_{t\to\infty}\frac{u(t)}{t^2} \le C_7$$

for some $C_7 > 0$. From the last inequality and local boundedness of the function u, it follows that

$$u(t) = \mathbb{E}X^2(t) \le C_8(t^2+1)$$

for some $C_8 > 0$. Then using Jensen's inequality,

$$\mathbb{E}b^2(X(t),t) \le \left(C\left(1+\mathbb{E}|X(t)|^\beta\right)\right)^2 \le 2C^2\left(1+\left(\mathbb{E}|X(t)|^\beta\right)^2\right) \le$$
$$\le 2C^2\left(1+(\mathbb{E}X^2(t))^\beta\right) \le 2C^2\left(1+(C_8(t^2+1))^\beta\right) \le C_9(1+t^{2\beta})$$

for some $C_9 > 0$.

Hence, by Lemma 2.1, almost surely

$$\frac{1}{t}\int_0^t b(X(s),s)\mathrm{d}B(s) \to 0,\ t\to\infty.$$

Therefore, almost surely for large t,

$$\frac{X(t)}{t} = \frac{x_0}{t} + \frac{1}{t}\int_0^t a(X(s),s)\mathrm{d}s + \frac{1}{t}\int_0^t b(X(s),s)\mathrm{d}B(s).$$

Going to the limit as $t\to\infty$, we get almost surely

$$\lim_{t\to\infty}\frac{X(t)}{t} = A$$

and the theorem is proved. □

The next theorem states that the previous one holds if the coefficients have those properties for large x.

Theorem 2.2. *Suppose that:*

- *the coefficient $a = a(x,t)$ is bounded for $x \ge x_* > 0$, $t \ge 0$ and there exists the limit*
$$\lim_{x,t\to\infty} a(x,t) = A \text{ a.s.},$$
where $A > 0$ is a positive random variable;
- *there exist $\beta \in \left(0,\frac{1}{2}\right)$, $C > 0$, and $x_* > 0$ such that almost surely for any $x \ge x_* > 0$ and $t \ge 0$,*
$$|b(x,t)| \le Cx^\beta;$$
- *$X(t) \to +\infty$ a.s.*

Then almost surely

$$X(t) \sim At, \; t \to \infty.$$

Proof. Construct a twice continuously differentiable function f such that

$$f(x) = \begin{cases} 0, \; x < x_*, \\ x, \; x > 2x_*. \end{cases}$$

Apply Itô's lemma to the process $\tilde{X}(t) := f(X(t))$:

$$\mathrm{d}\tilde{X}(t) = \tilde{a}(t)\mathrm{d}t + \tilde{b}(t)\mathrm{d}B(t),$$

where

$$\tilde{a}(t) := a(X(t),t)f'(X(t)) + \frac{1}{2}b^2(X(t),t)f''(X(t)),$$
$$\tilde{b}(t) := b(X(t),t)f'(X(t)).$$

Considering each of the cases $X(t) < x_*$, $x_* \le X(t) \le 2x_*$, and $X(t) > 2x_*$, it is easy to see that $\tilde{a}$ is bounded, almost surely

$$\lim_{t\to\infty} \tilde{a}(t) = \lim_{t\to\infty} a(X(t),t) = A$$

and almost surely

$$|\tilde{b}(t)| \le \mathrm{const}\,(1 + \tilde{X}^\beta(t)).$$

It is clear that $X(t) \to \infty, \; t \to \infty$, a.s. iff $\tilde{X}(t) \to \infty, \; t \to \infty$, a.s.

Applying the previous theorem to $\tilde{X}$, we obtain $\tilde{X}(t) \sim At, \; t \to \infty$, a.s., which implies that $X(t) \sim At, \; t \to \infty$, a.s. □

The following theorem lets to find the asymptotics of the solution in the case when the drift coefficient has power growth.

Theorem 2.3. *Let $\alpha \in (-1,1)$. Suppose that:*

(A) there exists a positive random variable $A > 0$ such that almost surely

$$a(x,t) \sim Ax^\alpha, \; x, t \to \infty;$$

(B) there exist constants $K \ge 0$ and $x_* > 0$ such that

$$a(x,t) \le Kx^\alpha, \; x > x_*, \; t \ge 0;$$

(C) there exist constants $C > 0$, $\beta \in (0, \frac{\alpha+1}{2})$, $x_* > 0$ such that almost surely for any $x \ge x_*$, $t \ge 0$,

$$|b(x,t)| \le Cx^\beta;$$

(D) $X(t) \to +\infty, \; t \to \infty$, a.s.

Then almost surely

$$X(t) \sim ((1-\alpha)At)^{\frac{1}{1-\alpha}}.$$

Remark 2.2. *Notice that the asymptotics of X is nothing but the asymptotics of the following ODE solution x:*

$$\mathrm{d}x(t) = Ax^\alpha(t)\mathrm{d}t.$$

Proof. Construct a twice continuously differentiable function f such that

$$f(x) = \frac{x^{1-\alpha}}{1-\alpha}, \; x \ge x_*.$$

Apply Itô lemma to the process $\tilde{X}(t) := f(X(t))$:

$$\mathrm{d}\tilde{X}(t) = \tilde{a}(t)\mathrm{d}t + \tilde{b}(t)\mathrm{d}B(t),$$

where

$$\tilde{a}(t) := a(X(t),t)f'(X(t)) + \frac{1}{2}b^2(X(t),t)f''(X(t)),$$

$$\tilde{b}(t) := b(X(t), t) f'(X(t)).$$

For large t,

$$\tilde{a}(t) = \frac{a(X(t), t)}{X^{\alpha}(t)} - \frac{\alpha}{2} \frac{b^2(X(t), t)}{X^{\alpha+1}(t)},$$

$$\tilde{b}(t) = \frac{b(X(t), t)}{X^{\alpha}(t)}.$$

We have almost surely

$$\lim_{t \to \infty} \frac{a(X(t), t)}{X^{\alpha}(t)} = \lim_{t \to \infty} \frac{A X^{\alpha}(t)}{X^{\alpha}(t)} = A$$

and

$$\lim_{t \to \infty} \frac{b^2(X(t), t)}{X^{\alpha+1}(t)} \le \lim_{t \to \infty} \frac{C^2 X^{2\beta}(t)}{X^{\alpha+1}(t)} = C^2 \lim_{t \to \infty} \frac{1}{X^{\alpha - 2\beta + 1}(t)} = 0$$

since $\alpha - 2\beta + 1 > 0$ and $X(t) \to \infty,\ t \to \infty$, a.s. by condition. Hence, almost surely

$$\lim_{t \to \infty} \tilde{a}(t) = A.$$

For sufficiently large t,

$$|\tilde{b}(t)| = \frac{|b(X(t), t)|}{X^{\alpha}(t)} \le C \frac{X^{\beta}(t)}{X^{\alpha}(t)} = C X^{\beta - \alpha}(t) = C \left((A(1-\alpha) \tilde{X}(t))^{\frac{1}{1-\alpha}} \right)^{\beta - \alpha} =$$

$$= \text{const} \cdot (\tilde{X}(t))^{\frac{\beta - \alpha}{1-\alpha}} =: \text{const} \cdot (\tilde{X}(t))^{\tilde{\beta}}.$$

Hence, we have the following equivalences:

$$\tilde{\beta} < \frac{1}{2} \Leftrightarrow \frac{\beta - \alpha}{1 - \alpha} < \frac{1}{2} \Leftrightarrow 2\beta - 2\alpha < 1 - \alpha \Leftrightarrow \beta < \frac{\alpha + 1}{2}.$$

It is clear that $X(t) \to \infty,\ t \to \infty$, a.s. iff $\tilde{X}(t) \to \infty,\ t \to \infty$, a.s. Applying the previous theorem to the process $\tilde{X}$, we obtain $\tilde{X}(t) \sim At,\ t \to \infty$, a.s., i.e., $f(X(t)) \sim At,\ t \to \infty$, a.s.

For sufficiently large t,

$$f^{-1}(t) = ((1-\alpha)t)^{\frac{1}{1-\alpha}},$$

which is a power function. Applying f^{-1} to both parts of the last equivalence, we obtain almost surely

$$f^{-1}(f(X(t))) \sim f^{-1}(At),\ t \to \infty,$$

i.e., almost surely

$$X(t) \sim ((1-\alpha)At)^{\frac{1}{1-\alpha}},\ t \to \infty.$$

□

Similarly to the proof of the previous theorem, one can prove the following result.

Theorem 2.4. *If conditions (B), (C), and (D) hold then a.s.*

$$((1-\alpha)A_-)^{\frac{1}{1-\alpha}} \le \liminf_{t \to \infty} \frac{X(t)}{t^{\frac{1}{1-\alpha}}} \le \limsup_{t \to \infty} \frac{X(t)}{t^{\frac{1}{1-\alpha}}} \le ((1-\alpha)A_+)^{\frac{1}{1-\alpha}},$$

where the random variables A_- and A_+ are defined as follows:

$$A_- := \liminf_{x, t \to \infty} \frac{a(x,t)}{x^{\alpha}}, \qquad A_+ := \limsup_{x, t \to \infty} \frac{a(x,t)}{x^{\alpha}}.$$

3. Transience of Solutions

Consider the system (5)-(6) again:

$$\mathrm{d}R(t) = \mu(R(t),\Phi(t))\mathrm{d}t + \sigma(R(t),\Phi(t))\mathrm{d}W(t),\ R(0) = r_0,$$
$$\mathrm{d}\Phi(t) = \nu(R(t),\Phi(t))\mathrm{d}t + \chi(R(t),\Phi(t))\mathrm{d}W(t),\ \Phi(0) = \varphi_0,$$

where coefficients μ,σ,ν,χ are arbitrary (not related to the coefficients a,b of the initial SDE (2)).

Define the next operators for the radius SDE (3):

$$L_\varphi[V](r) := \mu(r,\varphi)V'(r) + \frac{1}{2}|\sigma(r,\varphi)|^2 V''(r),\ r>0,\ \varphi\in\mathbb{R}^n,\ V\in \mathrm{C}^2(0,\infty),$$

where $|\sigma| := \sqrt{\sigma\sigma^{\mathrm{T}}}$ is a norm of the vector σ, $\mathrm{C}^2(0,\infty)$ is the set of all twice continuously differentiable function on $(0,\infty)$.

Theorem 3.1. *Suppose that:*

(1) for any starting point (r_0,φ_0), $r_0\neq 0$, there exists a unique solution (R,Φ) of the system (5)-(6), which is a strong Markov process;
(2) μ,σ are continuous, $|\sigma(r,\varphi)|\geq\sigma_*>0$, $r>0$, $\varphi\in\mathbb{R}^n$;
(3) there exist a non-decreasing function V_0 and $\delta>0$ such that $V_0(0)=-\infty$ and

$$\forall r\in(0,\delta)\ \forall\varphi\in\mathbb{R}^n\ L_\varphi[V_0](r)\leq 0;$$

(4) there exist a decreasing function V_∞ and a constant $\Delta>\delta$ such that $|V_\infty(\infty)|<\infty$ and

$$\forall r>\Delta\ \forall\varphi\in\mathbb{R}^n\ L_\varphi[V_\infty](r)\leq 0.$$

Then almost surely:

(1) $R(t)>0,\ t\geq 0$;
(2) $R(t)\to\infty,\ t\to\infty$.

Remark 3.1. *If coefficients μ,σ,ν,χ are locally Lipschitz and have linear growth at infinity then it is known that the SDE solution exists, is unique, and is a strong Markov process (see § 10 of Part I in [4]).*

Remark 3.2. *Notice that there are no requirements for the behavior of the generator in the interval $r\in[\delta,\Delta]$. It may be possible to find a common function V in the interval $(0,\infty)$ instead of two functions V_0 and V_∞ such that $V(0)=-\infty$, $V(\infty)\in\mathbb{R}$, and*

$$\forall r>0\ \forall\varphi\in\mathbb{R}^n\ L_\varphi[V](r)\leq 0.$$

Proof. Notice that since coefficients μ,σ are continuous, they are bounded on compact sets, hence, by Corollary 6.1, the process R exits any interval $[a,b]\subset(0,\infty)$ almost surely. Define $\tau_r := \inf\{t\geq 0\colon R(t)=r\}$, $r\geq 0$.

Step 1. Suppose first that $R(0)=r_0\in(0,\delta)$. Let $\varepsilon\in(0,r_0)$. Since the solution exits any interval a.s. (by Corollary 6.1), $\tau_\varepsilon\wedge\tau_\delta<\infty$ a.s. Then by Lemma 6.3, we have

$$\mathbb{P}\{\tau_\delta<\tau_\varepsilon\}\geq\frac{V_0(r_0)-V_0(\varepsilon)}{V_0(\delta)-V_0(\varepsilon)}\to 1,\ \varepsilon\to 0.$$

Therefore, $\tau_\delta<\infty$ a.s. and $\mathbb{P}\{\tau_0<\tau_\delta\}=0$. This implies (by virtue of continuity of R) that $\mathbb{P}\{R(t)>0,\ t\geq 0\}=1$.

Step 2. By the strong Markov property, the distribution of the process[1] $(R(\tau_\delta+t),\Phi(\tau_\delta+t))_{r\geq 0}$ is the same as the distribution of the process $(R_\delta(t),\Phi_\xi(t))$, where $\xi\sim\Phi(\tau_\delta)$, ξ is independent of $\Phi(\tau_\delta)$. Therefore, without loss of generality, we suppose now that $R(0)=\delta$. Let $\tilde\Delta>\Delta$. Since μ and σ are continuos and $|\sigma|\geq\sigma_*>0$, μ and σ are

[1]By R_δ we denote the solution of the corresponding SDE with the starting point $R(0)=\delta$. Similarly, we define Φ_ξ.

bounded for $r \in [\delta/2, \tilde{\Delta}]$ ($|\mu| \le M$, $|\sigma| \le \Sigma$ for some $M,\ \Sigma > 0$). Choose some decreasing function V such that $L_\varphi[V](r) \le 0$ for $r \in [\delta/2, \tilde{\Delta}]$ and $\varphi \in \mathbb{R}^n$ (e.g., $V(r) = e^{-\frac{2M}{\Sigma^2} r}$). Since the solution exits any interval a.s. (by Corollary 6.1), $\tau_{\tilde{\Delta}} \wedge \tau_{\delta/2} < \infty$ a.s. Then by Lemma 6.3,

$$\mathbb{P}_\delta \left\{ \tau_{\tilde{\Delta}} < \tau_{\delta/2} \right\} \ge \frac{V(\delta) - V(\delta/2)}{V(\tilde{\Delta}) - V(\delta/2)} =: p > 0,$$

$$\mathbb{P}_\delta \left\{ \tau_{\delta/2} < \tau_{\tilde{\Delta}} \right\} \ge \frac{V(\tilde{\Delta}) - V(\delta)}{V(\tilde{\Delta}) - V(\delta/2)} = 1 - p < 1.$$

Using the strong Markov property k times, one can show that the probability of exiting the interval $\left[\frac{\delta}{2}, \tilde{\Delta}\right]$ k times from the left end and returning into the interval $[\delta, \tilde{\Delta}]$ is not greater than $(1-p)^k \to 0,\ k \to \infty$. Thus, $\tau_{\tilde{\Delta}} < \infty$ a.s.

Step 3. By the strong Markov property, the distribution of the process $((R(\tau_{\tilde{\Delta}} + t), \Phi(\tau_{\tilde{\Delta}} + t)))_t$ is the same as the distribution of the process $((R_{\tilde{\Delta}}(t), \Phi_\xi(t)))_{t \ge 0}$, where $\xi \sim \Phi(\tau)$ and ξ is independent of $\Phi(\tau)$. Therefore, without loss of generality, we suppose now that $R(0) = \tilde{\Delta}$. Let $L > \tilde{\Delta}$ and $\Delta^* \in (\Delta, \tilde{\Delta})$. Since the solution exits any interval (by Corollary 6.1), $\tau_L \wedge \tau_{\Delta^*} < \infty$ a.s. By Lemma 6.3,

$$\mathbb{P}_{\tilde{\Delta}} \left\{ \tau_{\Delta^*} < \tau_L \right\} \le \frac{V_\infty(L) - V_\infty(\tilde{\Delta})}{V_\infty(L) - V_\infty(\Delta^*)}.$$

As $L \to \infty$, we obtain

$$\mathbb{P}_{\tilde{\Delta}} \left\{ \inf_{t \ge 0} R(t) \le \Delta^* \right\} \le \frac{V_\infty(\infty) - V_\infty(\tilde{\Delta})}{V_\infty(\infty) - V_\infty(\Delta^*)}.$$

Then we have the following estimates:

$$\mathbb{P}_{\tilde{\Delta}} \left\{ \liminf_{t \to \infty} R(t) \le \Delta^* \right\} \le \mathbb{P}_{\tilde{\Delta}} \left\{ \inf_{t \ge 0} R(t) \le \Delta^* \right\} \le \frac{V_\infty(\infty) - V_\infty(\tilde{\Delta})}{V_\infty(\infty) - V_\infty(\Delta^*)} =$$

$$= 1 - \frac{V_\infty(\tilde{\Delta}) - V_\infty(\Delta^*)}{V_\infty(\infty) - V_\infty(\Delta^*)} =: p_{\tilde{\Delta}}.$$

Notice that $p_{\tilde{\Delta}} \to 0$ as $\tilde{\Delta} \to \infty$. As $\tilde{\Delta} \to \infty$, we obtain

$$\mathbb{P} \left\{ \liminf_{t \to \infty} R(t) \le \Delta^* \right\} = 0.$$

As $\Delta^* \to \infty$, we obtain

$$\mathbb{P} \left\{ \liminf_{t \to \infty} R(t) < +\infty \right\} = 0 \Rightarrow \mathbb{P} \left\{ \liminf_{t \to \infty} R(t) = +\infty \right\} = 1 \Rightarrow$$

$$\Rightarrow \mathbb{P} \left\{ \lim_{t \to \infty} R(t) = +\infty \right\} = 1.$$

□

Example 3.1. *Consider the following n-dimensional ($n \ge 2$) SDE:*

$$\mathrm{d}X(t) = |X(t)|^{\alpha - 1} X(t) \mathrm{d}t + \mathrm{d}W(t),\ \ X(0) = x_0 \ne 0,$$

where $-1 < \alpha < 1$, W is an n-dimensional Wiener process.

If for $x \in \mathbb{R}^n \backslash \{0\}$ and $-1 < \alpha < 1$ we denote $x^\alpha := |x|^{\alpha-1} x$, then the previous SDE can be written in the following form:

$$\mathrm{d}X(t) = X^\alpha(t) \mathrm{d}t + \mathrm{d}W(t).$$

Let's prove that almost surely:

- $\forall t \ge 0\ X(t) \ne 0$;
- $|X(t)| \to \infty,\ t \to \infty$.

Proof. Since the coefficients of the SDE are locally Lipschitz and have linear growth at infinity, there exists a unique solution until it hits 0. One can obtain the SDE for the radius $R = |X|$ of the process X using the general formula given in Section 1:

$$\mathrm{d}R(t) = \left(R^{\alpha}(t) + \frac{n-1}{2R(t)}\right)\mathrm{d}t + \mathrm{d}W^{(1)}(t),$$

where $W^{(1)}$ is some one-dimensional Wiener process.

The scale function for this SDE is

$$s(r) = \int_1^r \frac{1}{u^{n-1}} \exp\frac{2(1-u^{\alpha+1})}{\alpha+1}\mathrm{d}u.$$

Since $|s(0)| = \infty$ and $|s(+\infty)| < \infty$, classic results imply that the process R never hits zero and $R(t) \to \infty,\ t \to \infty$, almost surely. This means that the process X never hits the origin and $|X(t)| \to \infty,\ t \to \infty$, almost surely. □

Example 3.2. *Let's perturb the drift coefficient of the SDE from the previous example:*

$$\mathrm{d}X(t) = (X^{\alpha}(t) + f(X(t)))\mathrm{d}t + \mathrm{d}W(t),$$

where the function f is such that:

- $|f_{\mathrm{rad}}(x)| = o\left(\frac{1}{|x|}\right),\ |x| \to 0, \qquad |f_{\mathrm{rad}}(x)| = o(|x|^{\alpha}),\ |x| \to \infty;$
- $|f_{\tan}(x)| \le C_2|x|^{\alpha-\varepsilon}$ *for large* $|x|$, *where* $C_2 > 0$, $\varepsilon \in (0, 1+\alpha)$.

Check that the solution of this SDE has the same properties as the one from the previous example.

Proof. Notice that

$$|f_{\mathrm{rad}}(x)| = o(|x|^{\alpha}),\ |x| \to \infty \qquad \Rightarrow \qquad |\langle f(x), x\rangle| = o(|x|^{1+\alpha}),\ |x| \to \infty.$$

The SDE for the radius $|X|$ has the form

$$\mathrm{d}|X(t)| = \left(|X(t)|^{\alpha} + \frac{\langle f(X(t)), X(t)\rangle}{|X(t)|} + \frac{n-1}{2|X(t)|}\right)\mathrm{d}t + \mathrm{d}W^1(t).$$

Use Theorem 3.1 to prove that the solution does not hit zero and goes to infinity. Consider a Lyapunov function $V_0(r) := -\frac{1}{r^{n-1}}$, which is increasing, $V_0(0) = 0$, and a Lyapunov function $V_\infty(r) := \frac{1}{r}$, which is decreasing, $|V_\infty(+\infty)| < \infty$. We have:

$$LV_0(r) \le \frac{n-1}{r^{n-\alpha}} - \frac{n-1}{2r^{n+1}} + o\left(\frac{1}{r^{n+1}}\right) \le 0$$

for $r \to 0$,

$$LV_\infty(r) \le -\frac{1}{r^{2-\alpha}} - \frac{n-3}{2r^3} + o(r^{\alpha-2}) \le 0$$

for $r \to \infty$. Hence,

$$\mathbb{P}\{X(t) \ne 0,\ t \ge 0\} = 1, \qquad \mathbb{P}\{|X(t)| \to \infty,\ t \to \infty\} = 1.$$

□

4. Stabilization of the Angle

Consider SDE (6) for Φ.

Theorem 4.1. *Suppose that:*

(1) $\mathbb{P}\{\forall t \ge 0\ R(t) > 0\} = 1;$

(2) $\liminf_{t\to\infty} \frac{R(t)}{t^{\gamma}} \ge C^*$ *for some random variable* $C^* > 0$ *and some non-random constant* $\gamma > 0$;

(3) $|\nu(r,\varphi)| \le \frac{\mu^*}{r^{\delta_1}}, |\chi(r,\varphi)| \le \frac{\chi^*}{r^{\delta_2}}$ *for all* $\varphi \in \mathbb{R}^n$ *and large* r, *where* $\nu^*, \chi^* > 0, \delta_1 > \frac{1}{\gamma}, \delta_2 > \frac{1}{2\gamma}$.

Then there exists the limit $\lim_{t\to\infty}\Phi(t)$ *a.s.*

Proof. Rewrite SDE (6) in the integral form:

$$\Phi(t) = \varphi_0 + \int_0^t \nu(R(s), \Phi(s))\mathrm{d}s + \int_0^t \chi(R(s), \Phi(s))\mathrm{d}W(s).$$

Then the limit

$$\Phi_\infty = \varphi_0 + \int_0^\infty \nu(R(t), \Phi(t))\mathrm{d}t + \int_0^\infty \chi(R(t), \Phi(t))\mathrm{d}W(t)$$

exist a.s. if the integrals in the right-hand side are convergent.

To prove convergence of the first and the second integrals, use conditions 2 and 3 of the theorem:

$$\delta_1 > \frac{1}{\gamma} \Rightarrow \delta_1\gamma > 1 \Rightarrow \int_1^\infty \frac{\mathrm{d}t}{t^{\delta_1\gamma}} < \infty \Rightarrow \int_0^\infty |\nu(R(t), \Phi(t))|\,\mathrm{d}t < \infty \text{ a.s.};$$

$$\delta_2 > \frac{1}{2\gamma} \Rightarrow 2\delta_2\gamma > 1 \Rightarrow \int_1^\infty \frac{\mathrm{d}t}{t^{2\delta_2\gamma}}\mathrm{d}t < \infty \Rightarrow \int_0^\infty \chi^2(R(t), \Phi(t))\mathrm{d}t < \infty \text{ a.s.} \Rightarrow$$

$$\Rightarrow \int_0^\infty \chi(R(t), \Phi(t))\mathrm{d}W(t) \text{ is well-defined.}$$

□

Example 4.1. *Consider the SDE from Example 3.1. Let's prove that almost surely:*

- $|X(t)| \sim ((1-\alpha)t)^{\frac{1}{1-\alpha}},\ t\to\infty$;
- $\exists \lim_{t\to\infty}\frac{X(t)}{|X(t)|}$.

Proof. Let's find the asymptotics of the process $|X|$ using Theorem 2.3 being applied to the SDE for $R = |X|$. The drift coefficient $\mu(r) = r^\alpha + \frac{n-1}{2r} \sim r^\alpha,\ r\to\infty$ (here $A = 1,\ \alpha = \alpha$), the diffusion coefficient $\sigma^{(1)}(r) = 1 \le r^0$ (here $C = 1,\ \beta = 0$) and a.s. $|X(t)| \to \infty,\ t\to\infty$. Then by Theorem 2.3 almost surely

$$|X(t)| \sim ((1-\alpha)t)^{\frac{1}{1-\alpha}},\ t\to\infty.$$

Let's prove that the angle $\frac{X}{|X|}$ stabilizes using Theorem 4.1. From the general SDE for the angle (see Section 1) one can obtain the SDE for the angle in our case:

$$\mathrm{d}\frac{X(t)}{|X(t)|} = -\frac{2I_{\tan}(X(t)) + n - 1}{2|X(t)|^3}X(t)\mathrm{d}t + \frac{I_{\tan}(X(t))}{|X(t)|}\mathrm{d}W(t),$$

where I is a unit $n\times n$ matrix and $I_{\tan}(x) = I - \frac{xx^{\mathrm{T}}}{|x|^2}$. The first condition of the theorem is satisfied. The second condition (existence of the lower asymptotics) follows from existence of the exact asymptotics (here $\gamma = \frac{1}{1-\alpha}$). Check the third condition (find estimates for the coefficients):

$$\left|-\frac{2I_{\tan}(x) + n - 1}{2|x|^3}x\right| \le \frac{2\sqrt{n-1} + n - 1}{2|x|^2},$$

$$\left|\frac{I_{\tan}(x)}{|x|}\right| = \frac{\sqrt{n-1}}{|x|},$$

i.e., $\delta_1 = 2,\ \delta_2 = 1$. The third condition of the theorem is satisfied since for such δ_1, δ_2, and $\gamma = \frac{1}{1-\alpha}$,

$$\begin{cases}\delta_1 > \frac{1}{\gamma},\\ \delta_2 > \frac{1}{2\gamma}\end{cases} \Leftrightarrow \alpha > -1.$$

Hence, by Theorem 4.1, there exists a limit $\lim_{t\to\infty}\frac{X(t)}{|X(t)|}$ almost surely. □

Example 4.2. *Consider the SDE from Example 3.2. Check that the solution of this SDE has the same properties as one from the previous example.*

Proof. Let's find the asymptotics of $|X|$. Since

$$\frac{\langle f(r\varphi), r\varphi\rangle}{r} \le \frac{C_1|r\varphi|^{1+\alpha-\varepsilon}}{r} = Cr^{\alpha-\varepsilon},$$

the drift coefficient

$$\mu(r,\varphi) = r^\alpha + \frac{\langle f(r\varphi), r\varphi\rangle}{r} + \frac{n-1}{2r} \sim r^\alpha,\ r\to\infty$$

(here $A=1,\ \alpha=\alpha$), similarly to the previous example, by Theorem 2.3 we have almost surely

$$|X(t)| \sim ((1-\alpha)t)^{\frac{1}{1-\alpha}},\ t\to\infty.$$

Let's prove that the angle $\frac{X}{|X|}$ stabilizes. From the general SDE for the angle one can obtain the SDE for the angle in our case:

$$\mathrm{d}\frac{X(t)}{|X(t)|} = \left(\frac{f_{\tan}(X(t))}{|X(t)|} - \frac{2I_{\tan}(X(t)) + n - 1}{2|X(t)|^3}X(t)\right)\mathrm{d}t + \frac{I_{\tan}(X(t))}{|X(t)|}\mathrm{d}W(t),$$

Like in the previous example, the first and the second conditions of Theorem 4.1 are satisfied. Check the third condition (find estimates for the coefficients):

$$\left|\frac{f_{\tan}(x)}{|x|} - \frac{2I_{\tan}(x)+n-1}{2|x|^3}x\right| \le \frac{C_2}{|x|^{1-\alpha+\varepsilon}} + \frac{2\sqrt{n-1}+n-1}{2|x|^2} \le \frac{C_3}{|x|^{1-\alpha+\varepsilon}}$$

for large $|x|$, where $C_3>0$ (since $1-\alpha+\varepsilon<1$),

$$\left|\frac{I_{\tan}(x)}{|x|}\right| = \frac{\sqrt{n-1}}{|x|},$$

i.e., $\delta_1 = 1-\alpha+\varepsilon,\ \delta_2 = 1$. The third condition is satisfied, because for such δ_1, δ_2, and $\gamma = \frac{1}{1-\alpha}$,

$$\begin{cases}\delta_1 > \frac{1}{\gamma},\\ \delta_2 > \frac{1}{2\gamma}\end{cases} \Leftrightarrow \begin{cases}\varepsilon > 0,\\ \alpha > -1.\end{cases}$$

Hence, by Theorem 4.1, there exists the limit $\lim_{t\to\infty}\frac{X(t)}{|X(t)|}$ almost surely.

□

5. Asymptotics of the Radius

Let (R,Φ) be a solution of (5)-(6).

Using the Lévy theorem (see § 7 of Chapter II in [6]), one can find a one-dimensional Wiener process $W^{(1)}$ and a function $\sigma^{(1)}\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ such that

$$\sigma(R(t),\Phi(t))\mathrm{d}W(t) = \sigma^{(1)}(R(t),\Phi(t))\mathrm{d}W^{(1)}(t).$$

Hence, the SDE for the radius process can be written as follows:

$$\mathrm{d}R(t) = \mu(R(t),\Phi(t))\mathrm{d}t + \sigma^{(1)}(R(t),\Phi(t))\mathrm{d}W^{(1)}(t). \tag{8}$$

Theorem 5.1. *Consider SDE (8) for the radius:*

$$\mathrm{d}R(t) = \mu(R(t),\Phi(t))\mathrm{d}t + \sigma^{(1)}(R(t),\Phi(t))\mathrm{d}W^{(1)}(t).$$

Suppose that the following conditions hold:

(1) there exist a continuous bounded function $M\colon \mathbb{R}^n \to (0,\infty)$ and a constant $\alpha \in (-1,1)$ such that for any $\varphi_0 \in \mathbb{R}^n$,

$$\lim_{\substack{r\to\infty\\ \varphi\to\varphi_0}} \frac{\mu(r,\varphi)}{r^\alpha} = M(\varphi_0);$$

(2) *there exist constants* $C > 0$, $\beta \in (0, \frac{\alpha+1}{2})$, $r_* > 0$ *such that for any* $r \geq r_*$, $\varphi \in \mathbb{R}^n$,

$$|\sigma(r, \varphi)| \leq C r^{\beta};$$

(3) $R(t) \to \infty,\ t \to \infty$, *a.s.;*
(4) $\exists \Phi_\infty := \lim_{t\to\infty} \Phi(t)$ *a.s.*

Then

$$R(t) \sim ((1-\alpha)M(\Phi_\infty)t)^{\frac{1}{1-\alpha}},\ t \to \infty.$$

Proof. Notice that $\mu(r, \Phi(t)) \sim M(\Phi_\infty)r^{\alpha},\ r, t \to \infty$. Applying Theorem 2.3 to the process R, we obtain the statement of the theorem. □

Remark 5.1. *One can check that Theorem 5.1 holds for Example 4.2 with* $M(\varphi) = 1$.

6. Appendix. Auxiliary Results

Consider the following generalization of Grönwall's inequality (see § 1.7 in [9]).

Lemma 6.1. *Let* $u\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous function satisfying the inequality*

$$u(t) \leq a(t) + C \int_0^t u^{\beta}(s)\mathrm{d}s,$$

where $C > 0,\ 0 < \beta < 1$, *the function* $a\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is non-decreasing and continuous. Then*

$$u(t) \leq \tilde{C}\left((1-\beta)t + a^{1-\beta}(t)\right)^{\frac{1}{1-\beta}}, \qquad \text{where } \tilde{C} := C^{\frac{1}{1-\beta}}.$$

Let X be a solution of the following one-dimensional non-autonomous SDE:

$$\mathrm{d}X(t) = a(X(t), t, \omega)\mathrm{d}t + b(X(t), t, \omega)\mathrm{d}W(t), \qquad X(0) = x_0 \in (x_1, x_2), \tag{9}$$

where $a = a(x, s, \omega), b = b(x, s, \omega)$ are such that for any $t \geq 0$, their restrictions to $\mathbb{R} \times [0, t] \times \Omega$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable. For this SDE, define a family of operators

$$L_t[u](x) := a(x, t)u'(x) + \tfrac{1}{2}b^2(x, t)u''(x), \qquad x \in [x_1, x_2],\ t \geq 0, \tag{10}$$

and the exit time

$$\tau := \inf\{t \geq 0\colon X(t) \notin (x_1, x_2)\}.$$

The following lemma allows to prove that under some conditions, an SDE (9) solution exits any interval (x_1, x_2) after a finite time.

Lemma 6.2. *Let the functions* a *and* b *be bounded on* $[x_1, x_2]$. *Suppose that there exists a non-random function* u *such that*

$$L_t[u](x) \leq -1,\ x \in [x_1, x_2],\ t \geq 0.$$

Then[2] $\mathbb{E}_{x_0}\tau \leq 2\max_{x\in[x_1,x_2]} |u(x)|$. *As a consequence, almost surely* $\tau < \infty$.

The proof of the lemma is standard (e.g., see § 3.7 in [8]).

Corollary 6.1. *Let the functions* a *and* b *be bounded on* $[x_1, x_2]$ *and* $b > \delta > 0$ *on* $[x_1, x_2]$ *for some* $\delta > 0$. *Then* $\tau < \infty$ *almost surely.*

Proof. Let $\tilde{u}(x) = -e^{px},\ x \in [x_1, x_2]$, where $p > 0$. By condition, $|a| \leq C$ on $[x_1, x_2]$ for some $C > 0$. Write

$$L_t[\tilde{u}](x) = a(x, t)\tilde{u}'(x) + \frac{1}{2}b^2(x, t)\tilde{u}''(x) = -a(x, t)pe^{px} - \frac{1}{2}b^2(x, t)p^2e^{px}$$
$$= pe^{px}\left(-a(x, t) - \frac{1}{2}b^2(x, t)p\right) \leq pe^{px}\left(C - \frac{1}{2}\delta^2 p\right).$$

[2]Notation $\mathbb{E}_{x_0}$ and $\mathbb{P}_{x_0}$ emphasize that $X(0) = x_0$.

Choosing $p > \frac{2C}{\delta^2}$, we obtain $L_t[\tilde{u}](x) =: -\varepsilon(x) < 0$ for some $\varepsilon(x) > 0$, therefore $L_t[\tilde{u}](x) < -\varepsilon$ on $[x_1, x_2]$, where $\varepsilon := \min_{x\in[x_1,x_2]} \varepsilon(x)$. Then for $u(x) := \frac{1}{\varepsilon}\tilde{u}(x)$, $L_t[u](x) \le -1$ on $[x_1, x_2]$. Thus, by Lemma 6.2, $\tau < \infty$ almost surely. □

The next lemma allows to estimate probabilities of exiting through the left ot the right end of the interval $[x_1, x_2]$.

Lemma 6.3. *Let the conditions of Lemma 6.2 hold. Besides this, suppose that there exists a decreasing function V such that*

$$L_t[V](x) \le 0,\ x \in [x_1, x_2].$$

Then

$$\mathbb{P}_{x_0}\{X(\tau) = x_1\} \le \frac{V(x_0) - V(x_2)}{V(x_1) - V(x_2)}, \qquad \mathbb{P}_{x_0}\{X(\tau) = x_2\} \ge \frac{V(x_1) - V(x_0)}{V(x_1) - V(x_2)}.$$

Proofs of these lemmas are standard and exploit Itô's lemma on the interval $[0, \tau]$ (for the proof ideas, see § 16 of Part I in [4]).

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