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CONSISTENT ESTIMATES OF THE PARAMETERS OF THE MULTIPARAMETER FRACTIONAL BROWNIAN MOTION

The consistent estimators of the multiplicative parameter c and Hurst parameter H of the covariance function of the multiparameter fractional Brownian motion are constructed.

1. Introduction

The multiparameter fractional Brownian Motion (MFBM) is a multiparameter Gaussian random process $\{X_H(t), t = (t_1, \dots, t_d)\}$ with zero mean and covariance function

$$r(t,s) = \frac{c}{2}\left(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}\right), t, s \in R^d, \tag{1}$$

where $\|t\| = \sqrt{t_1^2 + \dots + t_d^2}$ is the Euclidean norm of vector $t = (t_1, \dots, t_d)$, $H \in (0,1)$ is the Hurst parameter, $c > 0$ is a multiplicative parameter. In the case of $d = 1, c = 1$ this is the fractional Brownian Motion with Hurst parameter H [5].

MFBM was studied by many authors. Thus the Hausdorff measure and multiple points of the trajectories of MFBM are studied in the articles [6],[7]. The series expansion of the MFBM is obtained in [4]. The problem of estimating the Hurst parameter arises in applied models, is relevant and attracts the attention of many scientists in the field of statistics. Thus, in the article [3] the strong consistent estimate of the Hurst parameter H of the MFBM is constructed by using the Levy–Baxter theorems.

The problem of estimating the parameters c, H of fractional Brownian motion based on observations with errors was studied in article [1].

2. Problem Statement

Observing fractional Brownian field $\left\{X(t_1,t_2), (t_1,t_2) \in R^2\right\}$ at the points

$$\{(k,j) \in \{0,1,\dots,n\}\}, n \geq 1,$$

we need to estimate the unknown parameters of the covariance function (1), such as the multiplicative parameter $c > 0$ and Hurst parameter $H \in (0,1)$.

For $k, j \geq 1$ we put

$$\Delta_1 X(k,j) = X(k,j) - X(k-1,j), \tag{2}$$

$$\Delta X(k,j) = X(k-1,j-1) - X(k-1,j) - X(k,j-1) + X(k,j). \tag{3}$$

These increments of the stochastic field can be written as:

$$\Delta_1 X(k,j) = \sum_{\alpha=0}^{1} (-1)^\alpha X(k-\alpha, j), \tag{4}$$

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$$\Delta X(k,j) = \sum_{\alpha,\beta=0}^{1} (-1)^{\alpha+\beta} X(k-\alpha, j-\beta). \tag{5}$$

To obtain consistent estimates of the multiplicative parameter c and Hurst parameter H of the covariance function of the multiparameter fractional Brownian motion, we use the following statistics:

$$S_n^{(1)} = \frac{1}{n^2} \sum_{k,j=1}^{n} \left(\Delta_1 X(k,j)\right)^2, \tag{6}$$

$$S_n^{(2)} = \frac{1}{n^2} \sum_{k,j=1}^{n} \left(\Delta X(k,j)\right)^2. \tag{7}$$

3. Calculation of means and variances of statistics $S_n^{(1)}, S_n^{(2)}$

Lemma 3.1. *The expected values of statistics $S_n^{(1)}, S_n^{(2)}$ are equal to*

$$ES_n^{(1)} = c, \tag{8}$$

$$ES_n^{(2)} = 2c\left(2 - 2^H\right). \tag{9}$$

Proof. To calculate the means of the squares of the increments (4)–(5) we apply the formula (1) for the covariance function. For every $k, j \geq 1$ we get:

$$E\left(\Delta_1 X(k,j)\right)^2 = E\left(\sum_{\alpha_1,\alpha_2=0}^{1} (-1)^{\alpha_1+\alpha_2} X(k-\alpha_1, j) X(k-\alpha_2, j)\right) =$$

$$= -\frac{c}{2} \sum_{\alpha_1,\alpha_2=0}^{1} (-1)^{\alpha_1+\alpha_2} \left|\alpha_1 - \alpha_2\right|^{2H} = c;$$

$$E\left(\Delta X(k,j)\right)^2 = E\Big(\sum_{\alpha_1,\beta_1=0}^{1} (-1)^{\alpha_1+\beta_1} X(k-\alpha_1, j-\beta_1)\times$$

$$\times \sum_{\alpha_2,\beta_2=0}^{1} (-1)^{\alpha_2+\beta_2} X(k-\alpha_2, j-\beta_2)\Big) =$$

$$= -\frac{c}{2} \sum_{\alpha_1,\beta_1=0}^{1} (-1)^{\alpha_1+\beta_1} \sum_{\alpha_2,\beta_2=0}^{1} (-1)^{\alpha_2+\beta_2} \left((\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2\right)^H = 2c\left(2 - 2^H\right).$$

Therefore, we obtain equalities (8)–(9). The Lemma is proved. □

Lemma 3.2. *The variances of statistics $S_n^{(1)}, S_n^{(2)}$ are equal to:*

$$VarS_n^{(1)} = \frac{2}{n^4} \sum_{k_1,j_1=1}^{n} \sum_{k_2,j_2=1}^{n} \left(E\left(\Delta_1 X(k_1,j_1)\Delta_1 X(k_2,j_2)\right)\right)^2, \tag{10}$$

$$VarS_n^{(2)} = \frac{2}{n^4} \sum_{k_1,j_1=1}^{n} \sum_{k_2,j_2=1}^{n} \left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2. \tag{11}$$

Proof. For the expected value of the product of random variables $\eta_1, \eta_2, \eta_3, \eta_4$, which have a compatible Gaussian distribution with zero mean the next formula holds true

$$E\eta_1\eta_2\eta_3\eta_4 = E(\eta_1\eta_2)E(\eta_3\eta_4) + E(\eta_1\eta_3)E(\eta_2\eta_4) + E(\eta_1\eta_4)E(\eta_2\eta_3).$$

This formula is called the Isserlis formula. It is a partial case of the formula for the expected value of the product of an even number of random variables with zero means and jointly Gaussian distribution [2].

For the statistics $S_n^{(1)}$ the variance is calculated as follows

$$VarS_n^{(1)} = E\left(S_n^{(1)}\right)^2 - \left(ES_n^{(1)}\right)^2 =$$

$$= \frac{1}{n^4}\sum_{k_1,j_1=1}^{n}\sum_{k_2,j_2=1}^{n}\left(\left(E\left((\Delta_1 X(k_1,j_1))^2\right)(\Delta_1 X(k_2,j_2))^2\right) - \right.$$

$$\left. -E\left(\Delta_1 X(k_1,j_1)\right)^2 E\left(\Delta_1 X(k_2,j_2)\right)^2\right). \tag{12}$$

Since the multiparameter fractional Brownian motion is a Gaussian random field with zero mean, then the random vector

$$(\Delta_1 X(k_1,j_1), \Delta_1 X(k_1,j_1), \Delta_1 X(k_2,j_2), \Delta_1 X(k_2,j_2))$$

has a jointly Gaussian distribution with zero mean. In the Isserlis formula, we put $\eta_1 = \eta_2 = \Delta_1 X(k_1,j_1)$, $\eta_3 = \eta_4 = \Delta_1 X(k_2,j_2)$ and get an equality

$$\left(E\left((\Delta_1 X(k_1,j_1))^2\right)(\Delta_1 X(k_2,j_2))^2\right) - E\left(\Delta_1 X(k_1,j_1)\right)^2 E\left(\Delta_1 X(k_2,j_2)\right)^2 =$$

$$= 2\left(E\left(\Delta_1 X(k_1,j_1)\Delta_1 X(k_2,j_2)\right)\right)^2.$$

From this equality and equality (12) the equality (10) follows. Similarly, the equality (11) is proved. The Lemma is proved.

□

Lemma 3.3. *For the variance of the statistic $S_n^{(1)}$, the following upper estimate holds:*

- *for $H \in \left(0, \frac{1}{2}\right)$:*

$$VarS_n^{(1)} \le \frac{8c^2}{n} + \frac{16c^2H^2}{n^2}\sum_{p,q=1}^{\infty}\frac{1}{(p^2+q^2)^{2-2H}};$$

- *for $H = \frac{1}{2}$:*

$$VarS_n^{(1)} \le \frac{8c^2}{n} + \frac{16c^2H^2}{n^2}\left(1 + \frac{\pi}{2}\ln\left(n\sqrt{2}\right)\right);$$

- *for $H \in \left(\frac{1}{2}, 1\right)$:*

$$VarS_n^{(1)} \le \frac{8c^2}{n} + \frac{16c^2H^2}{n^2}\left(1 + \frac{2^{2H-2}\pi}{(4H-2)n^{2-4H}}\right).$$

Remark 3.1. A double series $\sum_{p,q=1}^{\infty}\frac{1}{(p^2+q^2)^{\gamma}}$ converges for $\gamma > 1$.

Proof. Let us prove Lemma 3.3. For a natural number n, we put $C_n = \{1, 2, \ldots, n\}^4$,

$$A_n = \left\{(k_1,j_1,k_2,j_2) \in C_n \big| |k_2 - k_1| \le 1 \; or \, |j_2 - j_1| = 0\right\},$$

$$B_n = \left\{(k_1,j_1,k_2,j_2) \in C_n \big| |k_2 - k_1| \ge 2 \; or \, |j_2 - j_1| \ge 1\right\}.$$

Note that $A_n \cup B_n = C_n$, $A_n \cap B_n = \emptyset$. Denote by *card*(M) the number of elements of the finite set M. Let us count the number of elements of the set B_n. We have:

$$card\left\{(k_1,k_2) \in \{1,2,\ldots,n\}^2 \big| \, |k_2 - k_1| \ge 2\right\} = n^2 - 3n + 2,$$

$$card\left\{(j_1,j_2) \in \{1,2,\ldots,n\}^2 \big| \, |j_2 - j_1| \ge 1\right\} = n^2 - n,$$

hence

$$card\,(B_n) = (n^2 - 3n + 2)(n^2 - n),$$
$$card\,(A_n) = n^4 - card\,(B_n) = 4n^3 - 5n^2 + 2n \le 4n^3, n \ge 1.$$

Divide the amount on the right–hand side of equation (10) into two amounts. The first sum will include terms with summation indexes $(k_1, j_1, k_2, j_2) \in A_n$, and the second sum with summation indexes $(k_1, j_1, k_2, j_2) \in B_n$:

$$VarS_n^{(1)} = \frac{2}{n^4} \sum_{(k_1,j_1,k_2,j_2)\in A_n} \left(E\left(\Delta_1 X(k_1, j_1)\Delta_1 X(k_2, j_2)\right)\right)^2 +$$

$$+\frac{2}{n^4} \sum_{(k_1,j_1,k_2,j_2)\in B_n} \left(E\left(\Delta_1 X(k_1, j_1)\Delta_1 X(k_2, j_2)\right)\right)^2. \tag{13}$$

Let us apply the Cauchy–Buniakovsky inequality to estimate the first sum for the mathematical expectation $E\left(\Delta_1 X(k_1, j_1)\Delta_1 X(k_2, j_2)\right)$:

$$\left(E\left(\Delta_1 X(k_1, j_1)\Delta_1 X(k_2, j_2)\right)\right)^2 \le E\left(\Delta_1 X(k_1, j_1)\right)^2 E\left(\Delta_1 X(k_2, j_2)\right)^2.$$

In the proof of Lemma 3.1, it was established for every $k, j \ge 1$ that $E\left(\Delta_1 X(k, j)\right)^2 = c$. Then

$$\left(E\left(\Delta_1 X(k_1, j_1)\Delta_1 X(k_2, j_2)\right)\right)^2 \le c^2$$

and

$$\frac{2}{n^4} \sum_{(k_1,j_1,k_2,j_2)\in A_n} \left(E\left(\Delta_1 X(k_1, j_1)\Delta_1 X(k_2, j_2)\right)\right)^2 \le \frac{2c^2}{n^4} card\,(A_n) \le \frac{8c^2}{n}. \tag{14}$$

Estimation of the second sum of equality (13) is more complicated. First, we calculate the expectation of the product of random variables $\Delta_1 X(k_1, j_1), \Delta_1 X(k_2, j_2),\ k, j \ge 1$. Using formula (1) for the covariance function, we obtain:

$$E\left(\Delta_1 X(k_1, j_1)\Delta_1 X(k_2, j_2)\right) = \sum_{\alpha,\beta=0}^{1} (-1)^{\alpha+\beta} E\left(\Delta_1 X(k_1 - \alpha, j_1)\Delta_1 X(k_2 - \beta, j_2)\right) =$$

$$= \frac{c}{2} \sum_{\alpha,\beta=0}^{1} (-1)^{\alpha+\beta} \Big(\left((k_1 - \alpha)^2 + j_1^2\right)^H + \left((k_2 - \beta)^2 + j_2^2\right)^H -$$

$$- \left((k_1 - \alpha - k_2 + \beta)^2 + (j_1 - j_2)^2\right)^H \Big) =$$

$$= \frac{c}{2}\Big(\left((k_1 - k_2 - 1)^2 + (j_1 - j_2)^2\right)^H - 2\left((k_1 - k_2)^2 + (j_1 - j_2)^2\right)^H +$$

$$+ \left((k_1 - k_2 + 1)^2 + (j_1 - j_2)^2\right)^H \Big).$$

Further,

$$\sum_{(k_1,j_1,k_2,j_2)\in B_n} \left(E\left(\Delta_1 X(k_1, j_1)\Delta_1 X(k_2, j_2)\right)\right)^2 =$$

$$= \frac{c^2}{4} \sum_{(k_1,j_1,k_2,j_2)\in B_n} \Big(\left((k_1 - k_2 - 1)^2 + (j_1 - j_2)^2\right)^H - 2\left((k_1 - k_2)^2 + (j_1 - j_2)^2\right)^H +$$

$$+ \left((k_1 - k_2 + 1)^2 + (j_1 - j_2)^2\right)^H \Big)^2. \tag{15}$$

For $(k_1, j_1, k_2, j_2) \in B_n$, let us put $p = k_1 - k_2, q = j_1 - j_2;\ 2 \le |p| \le n - 1$, $1 \le |q| \le n - 1$. The system of equations

$$\begin{cases} k_1 - k_2 = p, \\ j_1 - j_2 = q \end{cases}$$

with respect to $(k_1, j_1, k_2, j_2) \in B_n$ has $(n-|p|)\cdot(n-|q|)$ solutions. Therefore

$$\Sigma_{B_n} := \sum_{(k_1,j_1,k_2,j_2)\in B_n} \left(E\left(\Delta_1 X(k_1,j_1)\Delta_1 X(k_2,j_2)\right)\right)^2 =$$

$$= \frac{c^2}{2}\sum_{|p|=2}^{n-1}\sum_{|q|=1}^{n-1}(n-|p|)(n-|q|)\Big(\left((p-1)^2+q^2\right)^H - 2(p^2+q^2)^H +$$

$$+\left((p+1)^2+q^2\right)^H\Big)^2 = 2c^2\sum_{p=2}^{n-1}\sum_{q=1}^{n-1}(n-p)(n-q)\Big(\left((p-1)^2+q^2\right)^H -$$

$$-2(p^2+q^2)^H + \left((p+1)^2+q^2\right)^H\Big)^2. \tag{16}$$

The expression

$$\left((p-1)^2+q^2\right)^H - 2(p^2+q^2)^H + \left((p+1)^2+q^2\right)^H$$

is an increment of the second order of the function $f(x) = (x^2+q^2)^{2H}, x \ge 1$ on the interval $[p-1, p+1]$. Thus, there is an intermediate point $\theta_p \in (p-1,p+1)$ such that

$$f(p-1) - 2f(p) + f(p+1) = f''(\theta_p).$$

A second derivative of the function f

$$f''(x) = 2H\left((2H-1)x^2+q^2\right)\left(x^2+q^2\right)^{H-2}, x \ge 1. \tag{17}$$

It follows from equalities (15), (16) that

$$\Sigma_{B_n} = 8c^2H^2\sum_{p=2}^{n-1}\sum_{q=1}^{n-1}(n-p)(n-q)\frac{\left((2H-1)\theta_p^2+q^2\right)^2}{\left(\theta_p^2+q^2\right)^{4-2H}}. \tag{18}$$

Since $n-p<n, n-q<n, \left|(2H-1)\theta_p^2+q^2\right| \le \theta_p^2+q^2$, then

$$\Sigma_{B_n} \le 8c^2H^2n^2\sum_{p=2}^{n-1}\sum_{q=1}^{n-1}\frac{1}{\left(\theta_p^2+q^2\right)^{2-2H}}$$

$$\le 8c^2H^2n^2\sum_{p=2}^{n-1}\sum_{q=1}^{n-1}\frac{1}{\left((p-1)^2+q^2\right)^{2-2H}} \le$$

$$\le 8c^2H^2n^2\sum_{p=1}^{n-2}\sum_{q=1}^{n-1}\frac{1}{\left(p^2+q^2\right)^{2-2H}}. \tag{19}$$

For $H \in \left(0,\frac{1}{2}\right)$, the double series $\sum_{p,q=1}^{\infty}\frac{1}{(p^2+q^2)^{2-2H}}$ converges, since $2-2H>1$. So, in this case

$$\Sigma_{B_n} \le 8c^2H^2n^2\sum_{p,q=1}^{\infty}\frac{1}{\left(p^2+q^2\right)^{2-2H}}. \tag{20}$$

Now let $H \in \left[\frac{1}{2},1\right)$. Then we have:

$$\sum_{p,q=1}^{n-1}\frac{1}{\left(p^2+q^2\right)^{2-2H}} = \frac{1}{2^{2-2H}} + \sum_{\substack{p,q=1,\\ p+q>2}}^{n-1}\frac{1}{\left(p^2+q^2\right)^{2-2H}}.$$

The first term on the right–hand side of the last equality does not exceed 1. We estimate the second term using a double integral. We have:

$$\sum_{\substack{p,q=1,\\ p+q>2}}^{n-1} \frac{1}{\left(p^2+q^2\right)^{2-2H}} \le \iint_{\substack{1\le x^2+y^2\le 2n^2,\\ x\ge 0, y\ge 0}} \frac{dxdy}{\left(x^2+y^2\right)^{2-2H}}. \tag{21}$$

We calculate the double integral in the right–hand side of inequality (21) using the transition to the polar coordinate system. For $H=\frac{1}{2}$:

$$\iint_{\substack{1\le x^2+y^2\le 2n^2,\\ x\ge 0, y\ge 0}} \frac{dxdy}{x^2+y^2} = \frac{\pi}{2}\int_1^{n\sqrt{2}} \frac{dr}{r} = \frac{\pi}{2}\ln(n\sqrt{2}). \tag{22}$$

If $H\in\left(\frac{1}{2},1\right)$, then

$$\iint_{\substack{1\le x^2+y^2\le 2n^2,\\ x\ge 0, y\ge 0}} \frac{dxdy}{\left(x^2+y^2\right)^{2-2H}} = \frac{\pi}{2}\int_1^{n\sqrt{2}} \frac{dr}{r^{3-4H}} < \frac{\pi}{2}\cdot\frac{2^{2H-1}}{4H-2}\cdot\frac{1}{n^{2-4H}}. \tag{23}$$

From the relations (13), (14), (20)–(23) it follows the statement of Lemma 3.3. The Lemma is proved. □

Corollary 3.1. *The following upper bounds for the rate of convergence of the sequence* $VarS_n^{(1)}$ *hold:*

- *for* $H\in\left(0,\frac{3}{4}\right]$*:*
$$VarS_n^{(1)} = O\left(\frac{1}{n}\right),\ n\to\infty;$$
- *for* $H\in\left(\frac{3}{4},1\right)$*:*
$$VarS_n^{(1)} = O\left(\frac{1}{n^{4-4H}}\right),\ n\to\infty.$$

Lemma 3.4. *The variance* $VarS_n^{(2)}$ *satisfies the following inequality:*

$$VarS_n^{(2)} \le \frac{48c^2}{n} + \frac{2c^2K^2}{n^2}\sum_{p,q=1}^{\infty}\frac{1}{\left(p^2+q^2\right)^2},\ n\ge 1, \tag{24}$$

where

$$K = 4\max_{H\in[0,1]}\left(H(1-H)(H^2-7H+11)\right).$$

Proof. At the beginning, let us put $C_n=\{1,2,\dots,n\}^4$, $n\ge 1$;

$$A_n = \left\{(k_1,j_1,k_2,j_2)\in C_n \big|\, |k_2-k_1|\le 1\ or\, |j_2-j_1|\le 1\right\},$$
$$B_n = \left\{(k_1,j_1,k_2,j_2)\in C_n \big|\, |k_2-k_1|\ge 2\ or\, |j_2-j_1|\ge 2\right\}.$$

Note that $A_n\cup B_n=C_n, A_n\cap B_n=\emptyset$. Let $card\,(M)$ be the number of elements of the finite set M. Let us count the number of elements of the set B_n. Since

$$card\left\{(k_1,k_2)\in\{1,2,\dots,n\}^2 \big|\, |k_2-k_1|\ge 2\right\} = n^2-3n+2,$$

then

$$card\,(B_n) = (n^2-3n+2)^2,$$
$$card\,(A_n) = n^4 - card\,(B_n) = (3n-2)(2n^2-3n+2)\le 6n^3, n\ge 1.$$

Now, divide the amount on the right–hand side of equation (11) into two amounts. The first sum will include terms with summation indexes $(k_1,j_1,k_2,j_2)\in A_n$, and the second sum with summation indexes $(k_1,j_1,k_2,j_2)\in B_n$:

$$VarS_n^{(2)} = \frac{2}{n^4}\sum_{(k_1,j_1,k_2,j_2)\in A_n}\left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2 +$$

$$+\frac{2}{n^4} \sum_{(k_1,j_1,k_2,j_2)\in B_n} \left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2. \tag{25}$$

The first sum is estimated using the Cauchy–Buniakovsky inequality for mathematical expectation $E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)$:

$$\left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2 \leq E\left(\Delta X(k_1,j_1)\right)^2 E\left(\Delta X(k_2,j_2)\right)^2.$$

In the proof of Lemma 3.1, it was established that for every $k,j\geq 1$:

$$E\left(\Delta X(k,j)\right)^2 = 2c(2-2^H).$$

So

$$\left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2 \leq 4c^2(2-2^H)^2$$

and

$$\frac{2}{n^4} \sum_{(k_1,j_1,k_2,j_2)\in A_n} \left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2 \leq \frac{8c^2(2-2^H)^2}{n^4} card\,(A_n) \leq \frac{48c^2}{n}. \tag{26}$$

Let us proceed to the evaluation of the second sum of the right–hand side of equality (25). First, let us calculate the mathematical expectation of the product of increments $\Delta X(k_1,j_1), \Delta X(k_2,j_2)$, $k,j\geq 1$. We get:

$$E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right) =$$

$$E\left(\sum_{\alpha_1,\beta_1=0}^{1}(-1)^{\alpha_1+\beta_1}\Delta X(k_1-\alpha_1,j_1-\beta_1)\sum_{\alpha_2,\beta_2=0}^{1}(-1)^{\alpha_2+\beta_2}\Delta X(k_2-\alpha_2,j_2-\beta_2)\right) =$$

$$= \frac{c}{2}\sum_{\alpha_1,\beta_1=0}^{1}(-1)^{\alpha_1+\beta_1}\sum_{\alpha_2,\beta_2=0}^{1}(-1)^{\alpha_2+\beta_2}\Big(\left((k_1-\alpha_1)^2+(j_1-\beta_1)^2\right)^H +$$

$$+\left((k_2-\alpha_2)^2+(j_2-\beta_2)^2\right)^H - \left((k_2-k_1+\alpha_2-\alpha_1)^2+(j_2-j_1+\beta_2-\beta_1)^2\right)^H\Big) =$$

$$= -\frac{c}{2}\sum_{\alpha_1,\beta_1=0}^{1}(-1)^{\alpha_1+\beta_1}\sum_{\alpha_2,\beta_2=0}^{1}(-1)^{\alpha_2+\beta_2}\Big(\Big((k_2-k_1+\alpha_2-\alpha_1)^2+$$

$$+(j_2-j_1+\beta_2-\beta_1)^2\Big)^H\Big).$$

Let us put $p=k_2-k_1, q=j_2-j_1$, $2\leq |p|,|q|\leq n-1$. After summing similar terms in the last expression for $E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)$, we get:

$$E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right) = -\frac{c}{2}\Big(\left((p-1)^2+(q-1)^2\right)^H + \left((p-1)^2+(q+1)^2\right)^H +$$

$$+\left((p+1)^2+(q-1)^2\right)^H + \left((p+1)^2+(q+1)^2\right)^H - 2\left(p^2+(q-1)^2\right)^H -$$

$$-2\left(p^2+(q+1)^2\right)^H - 2\left((p-1)^2+q^2\right)^H - 2\left((p+1)^2+q^2\right)^H + 4\left(p^2+q^2\right)^H\Big).$$

Consider the function

$$f(x,y) = \left(x^2+y^2\right)^H, (x,y)\in[p-1,p+1]\times[q-1,q+1].$$

It is not difficult to verify by direct integration that

$$\sigma(p,q) := \int_{q-1}^{q}dy\int_{y}^{y+1}dt\int_{p-1}^{p}dx\int_{x}^{x+1}\frac{\partial^4 f(s,t)}{\partial s^2\partial t^2}ds =$$

$$= \Big(\left((p-1)^2+(q-1)^2\right)^H + \left((p-1)^2+(q+1)^2\right)^H + \left((p+1)^2+(q-1)^2\right)^H +$$

$$+\left((p+1)^2+(q+1)^2\right)^H - 2\left(p^2+(q-1)^2\right)^H - 2\left(p^2+(q+1)^2\right)^H -$$

$$-2\left((p-1)^2+q^2\right)^H - 2\left((p+1)^2+q^2\right)^H + 4\left(p^2+q^2\right)^H\Big).$$

So,

$$E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)=-\frac{c}{2}\int_{q-1}^{q}dy\int_{y}^{y+1}dt\int_{p-1}^{p}dx\int_{x}^{x+1}\frac{\partial^4 f(s,t)}{\partial s^2\partial t^2}ds.$$

The partial derivative of the fourth order is equal to

$$\frac{\partial^4 f(x,y)}{\partial x^2\partial y^2}=4H(H-1)\left(x^2+y^2\right)^{H-2}+8H(H-1)(H-2)\left(x^2+y^2\right)^{H-2}+$$
$$+16x^2y^2H(H-1)(H-2)(H-3)\left(x^2+y^2\right)^{H-4}.$$

To evaluate this partial derivative, we apply the inequality

$$x^2y^2\le\frac{1}{4}\left(x^2+y^2\right)^2,\ x,y\in R.$$

Then for $p,q\ge 2$

$$\max_{(x,y)\in[p-1,p+1]\times[q-1,q+1]}\left|\frac{\partial^4 f(x,y)}{\partial x^2\partial y^2}\right|\le$$
$$\le 4H(1-H)(H^2-7H+11)\left((p-1)^2+(q-1)^2\right)^{H-2}\le$$
$$\le K\left((p-1)^2+(q-1)^2\right)^{H-2},$$

where $K=4\max_{H\in[0,1]}\left(H(1-H)(H^2-7H+11)\right)$.

Therefore, for $p,q\ge 2$

$$\left(\sigma(p,q)\right)^2\le K^2\left((p-1)^2+(q-1)^2\right)^{2H-4}.$$

Similarly as in the proof of Lemma 3.3, the next equality is justified

$$\sum_{(k_1,j_1,k_2,j_2)\in B_n}\left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2=$$
$$=\frac{c^2}{4}\sum_{|p|,|q|=2}^{n-1}\left(n-|p|\right)\left(n-|q|\right)\left(\sigma(p,q)\right)^2.$$

Since the value $\sigma(p,q)$ is invariant with respect to changing the signs of the variables p,q, then

$$\sum_{(k_1,j_1,k_2,j_2)\in B_n}\left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2=$$
$$=c^2\sum_{p,q=2}^{n-1}\left(n-p\right)\left(n-q\right)\left(\sigma(p,q)\right)^2.$$

So,

$$\frac{2}{n^4}\sum_{(k_1,j_1,k_2,j_2)\in B_n}\left(E\left(\Delta X(k_1,j_1)\Delta X(k_2,j_2)\right)\right)^2\le$$

$$\le\frac{2c^2K^2}{n^2}\sum_{p,q=2}^{n-1}\frac{1}{\left((p-1)^2+(q-1)^2\right)^{4-2H}}<\frac{2c^2K^2}{n^2}\sum_{p,q=1}^{\infty}\frac{1}{\left(p^2+q^2\right)^2}. \tag{27}$$

The inequality (24) follows from relations (25), (26), (27). The Lemma is proved. □

Corollary 3.2. *For all values of the Hurst parameter $H\in(0,1)$, the following upper bound for the rate of convergence of the sequence of variances $VarS_n^{(2)}$ holds:*

$$VarS_n^{(2)}=O\left(\frac{1}{n}\right),\ n\to\infty.$$

4. Consistent estimates of the multiplicative parameter c and the Hurst parameter H

Theorem 4.1. *The statistics*

$$\widehat{C}_n = S_n^{(1)} \quad \text{and} \quad \widehat{H}_n = \log_2\left(2 - \frac{S_n^{(2)}}{2S_n^{(1)}}\right), n \geq 1$$

are consistent estimators of the multiplicative parameter c and the Hurst parameter H, respectively.

Proof. From Consequences 3.1 and 3.2 follow the following convergences

$$S_n^{(1)} \to c,\ S_n^{(2)} \to 2c(2-2^H)$$

with probability one as $n \to \infty$. Then

$$\widehat{C}_n \to c \quad \text{and} \quad \widehat{H}_n \to \log_2\left(2 - \frac{2c(2-2^H)}{2c}\right) = H$$

with probability one as $n \to \infty$. The Theorem is proved. □

Theorem 4.2. *The statistics*

$$\widetilde{C}_n = S_{2^n}^{(1)} \quad \text{and} \quad \widetilde{H}_n = \log_2\left(2 - \frac{S_{2^n}^{(2)}}{2S_{2^n}^{(1)}}\right), n \geq 1$$

are strongly consistent estimators of the multiplicative parameter c and the Hurst parameter H, respectively.

Proof. The following upper bounds for the rate of convergence of the sequence $S_{2^n}^{(1)}, S_{2^n}^{(2)}$ follow from Consequences 3.1 and 3.2:

$$VarS_{2^n}^{(1)} = O\left(\frac{1}{2^n}\right), n \to \infty \quad for \quad H \in \left(0, \frac{3}{4}\right];$$

$$VarS_{2^n}^{(1)} = O\left(\frac{1}{2^{n(4-4H)}}\right), n \to \infty \quad for \quad H \in \left(\frac{3}{4}, 1\right);$$

$$VarS_{2^n}^{(2)} = O\left(\frac{1}{2^n}\right), n \to \infty \quad for\ all \quad H \in (0,1).$$

From these relations follows the convergence of the series

$$\sum_{n=1}^{\infty} VarS_{2^n}^{(1)}, \sum_{n=1}^{\infty} VarS_{2^n}^{(2)}$$

for all $H \in (0,1)$. As a result, $S_{2^n}^{(1)} \to c$ and $S_{2^n}^{(2)} \to 2c(2-2^H)$ with probability one as $n \to \infty$. Therefore, $\widetilde{C}_n \to c$ and $\widetilde{H}_n = \log_2\left(2 - \frac{S_{2^n}^{(2)}}{2S_{2^n}^{(1)}}\right) \to H$ with probability one as $n \to \infty$. The Theorem is proved. □

5. Conclusion

In this article we obtained the consistent estimators of the multiplicative parameter c and the Hurst parameter H of the covariance function of the multiparameter fractional Brownian motion.

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