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DESCRIPTION OF ALL LIMIT DISTRIBUTIONS OF SOME MARKOV CHAINS WITH MEMORY 2

There are situations where the data sequence does not depend on past values. As can be expected, the additional history of memory often has the advantage of offering a more precise predictive value. By bringing more memory into the random process, one can build a higher order Markov model. In this paper we consider 2-state Markov chain with memory 2 generated by Hamiltonian with competing interactions and describe its all limit distributions.

1. Introduction

A Markov chain is a stochastic process $\{X_n\}_{n=0}^{\infty}$ over a finite state space S, where the conditional probability distribution of future states in the process depends upon the present or past states. The classical "Markov property" specifies that the probability of transition to the next state s_{n+1} depends only on the probability of the current state s_n. That is,

$$Pr(X_{n+1} = s_{n+1} | X_n = s_n, \cdots, X_1 = s_1, X_0 = s_0) =$$
$$Pr(X_{n+1} = s_{n+1} | X_n = s_n)$$

A Markov chain with memory 2 is a process satisfying

$$Pr(X_{n+1} = s_{n+1} | X_n = s_n, \cdots, X_1 = s_1, X_0 = s_0) =$$
$$Pr(X_{n+1} = s_{n+1} | X_n = s_n, X_{n-1} = s_{n-1})$$

For simplicity, identify the states as $S = \{1, 2, \cdots, r\}$ and assume that the chain is time homogeneous. Then a transition probability matrix $\Pi = [p_{ij}]$ defined by

$$p_{ij} := Pr(X_{n+1} = j | X_n = i)$$

is independent of n and is row stochastic.

Assuming again time homogeneity, a Markov chain with memory 2 can be conveniently represented via the order-3 tensor $\mathcal{P} = [p_{i_1 i_2 i_3}]$ [7],defined by

$$p_{i_1 i_2 i_3} := Pr(X_{n+1} = i_3 | X_n = i_2, X_{n-1} = i_1),$$

where $\mathcal{P}$ is called a transition probability tensor. Necessarily we have the properties $0 \le p_{i_1 i_2 i_3} \le 1$ and that

$$\sum_{i_3=1}^{r} p_{i_1 i_2 i_3} = 1$$

for every fixed 2-tuple (i_1, i_2).

As shown in [5], the theory of finite Markov chains with positive transition probabilities can be embedded into the theory of limit Gibbs distributions as a trivial particular case,

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and the Hamiltonians can be considered as a natural generalization of the transition probabilities or, more exactly, of their logarithms.

2. Hamiltonian for Finite Markov Chain

Let $\Omega = S^{Z_+}$ be the sample space of a finite Markov chain with r states. That is, the range space S consists of r elements and P is a measure on Ω corresponding to a homogeneous Markov chain with a stationary transition matrix $\Pi = (p_{ij})$ and with stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_r)$. We consider the case when all $p_{ij} > 0$, i.e. $\Pi = (p_{ij})$ is the regular matrix, and there exists unique the limiting distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_r)$.

For arbitrary $\Lambda \subset Z_+$ a configuration σ on Λ is defined as $\sigma(\Lambda) : \Lambda \to S$. For $\Lambda_{km} = \{k, k+1, \cdots, k+m\}$, the probability of an arbitrary configuration $\sigma(\Lambda_{km}) = (\sigma(k), \sigma(k+1), \cdots, \sigma(k+m))$ is equal to

$$\pi_{\sigma(k)} \cdot p_{\sigma(k)\sigma(k+1)} \cdot p_{\sigma(k+1)\sigma(k+2)} \cdots p_{\sigma(k+m-1)\sigma(k+m)}$$
$$= \exp\{\ln \pi_{\sigma(k)} + \sum_{i=k}^{k+m-1} \ln p_{\sigma(i)\sigma(i+1)}\}$$

Introducing the Hamiltonian

$$H(\sigma) = -\sum_{i=0}^{\infty} \ln p_{\sigma(i)\sigma(i+1)}.$$

we see that only the interactions between neighbouring spins $< i, i+1 >$ are taken into account.

Below we add one more interaction, namely interactions of second neighbours $> i, i+2 <$. Let $\Pi^2 = (p_{ij}^{(2)})$ and model specified by the following Hamiltonian

$$H(\sigma) = -\sum_{i=0}^{\infty} \ln p_{\sigma(i)\sigma(i+1)} - \sum_{i=0}^{\infty} \ln p^{(2)}_{\sigma(i)\sigma(i+2)}, \tag{1}$$

Such model is called the model with competing interactions [6].

Consideration of models with competing interactions defined on Z^2 has proved to be fruitful in many fields of physics, ranging from the determination of phase diagrams in metallic alloys and exhibition of new types of phase transition, to site percolation. The axial next-nearest neighbour Ising (ANNNI) model, originally introduced by Elliot [1] to describe the sinusoidal magnetic structure of Erbirum, the chiral Potts model, introduced by Ostlund [4] and Huse [2] in connection with monolayers adsorbed on rectangular substrates, and others, attract much attention both from a purely theoretical point of view because of their applications.

They have a rich phase diagram demonstrating a countable set of different modulated structures. However, exact solutions for these models are unknown, and most of the results are only obtained numerically.

In our paper we are considering model (1) defined on the set Z_+ of non-negative integers. We suppose that the points of Z_+ are the vertices of the infinite graph $\Gamma^1_+ = (Z_+, L)$, where $L = \{l =< i, i+1 >: i \in Z_+\}$ is set of edges. Note that this graph is semi-infinite Cayley tree of first order, i.e. an infinite graph without cycles with 2 edges issuing from each vertex except for vertex 0 which has only 1 edge.

In recent years models on a Cayley tree has been studied extensively because it turns out that there are physically interesting solutions correspond to the attractors of the mapping. This simplifies the numerical work considerably and detailed study of the whole phase diagram becomes feasible. Apart from the intrinsic interest attached to the study of models on trees, it is possible to argue that the results obtained on trees provide

a useful guide to the more involved study of their counterparts on crystal lattices.
Note that statistical mechanics on trees involve nonlinear recursion equations and are naturally connected to the rich world of dynamical systems, a world presently under intense investigation.

3. Recursion Equations

For $\Gamma^1_+ = (Z_+, L)$ let $\Lambda_n = \{0, 1, \cdots, n\}$ is a finite subset of Z_+ and L_n is the set of edges on Λ_n.
Note that $\Lambda_{n+1} = \Lambda_n \cup \{n+1\}$ and $L_{n+1} = L'_n \cup \{< 0, 1 >\}$, where L'_n is the set of edges on $\Lambda'_n = \{1, 2, \cdots, n+1\}$.
We will apply this equality to produce recurrence equations.

Below we consider case with state space $S = \{1, 2\}$, i.e. $r = 2$.

We consider configurations $\sigma : Z_+ \to \{1, 2\}$. The restriction of σ on any subset $\Lambda_n \subset Z_+$ is denoted by $\sigma(\Lambda_n) \equiv \sigma_n$.

Any configuration σ one can split into two sub-configurations $\sigma(\Lambda_n)$ and $\sigma^n(\Lambda^c_n)$, where $\Lambda^c_n = Z_+ \setminus \Lambda_n$ is still infinite set.
Let us fix some configuration $\bar{\sigma}^n(\Lambda^c_n) : \Lambda^c_n \to \{1, 2\}$ and call it boundary configuration. Then we consider the set of all configurations that vary on Λ_n but fixed on Λ^c_n. Note that this set is finite. In the theory of random processes, every process is determined by the family of its finite-dimensional probability distributions. In problems of classical statistical mechanics we find ourselves faced with a different situation. Here the theory is based upon a formal expression, called the Hamiltonian. By its help, all possible conditional probability distributions of the random process or field inside any finite domain can be found under the condition that its values outside the domain are fixed.

We define the conditional Hamiltonian with fixed boundary configuration $\bar{\sigma}^n(\Lambda^c_n)$ as follows

$$
\begin{aligned}
(2) \qquad H(\sigma_n|\bar{\sigma}^n) &= -\sum_{i=0}^{n-1} \ln p_{\sigma_n(i)\sigma_n(i+1)} - \ln p_{\sigma_n(n)\bar{\sigma}^n(n+1)} \\
&- \sum_{i=0}^{n-2} \ln p^{(2)}_{\sigma_n(i)\sigma_n(i+2)} - \ln p^{(2)}_{\sigma_n(n-1)\bar{\sigma}^n(n+1)} - \ln p^{(2)}_{\sigma_n(n)\bar{\sigma}^n(n+2)}
\end{aligned}
$$

Then the conditional Gibbs state on finite subset Λ_n with boundary configuration $\bar{\sigma}^n(\Lambda^c_n)$ to be the measure μ_n given by

$$
(3) \qquad \mu_n(\sigma(\Lambda_n)|\bar{\sigma}^n(\Lambda^c_n)) = \frac{e^{-\beta H(\sigma_n|\bar{\sigma}^n)}}{Z_n(\bar{\sigma}^n)}
$$

for any configuration $\sigma(\Lambda_n) \in \Sigma(\Lambda_n)$, where

$$
Z_n(\bar{\sigma}^n) = \sum_{\sigma(\Lambda_n) \in \Sigma(\Lambda_n)} e^{-\beta H(\sigma(\Lambda_n)|\bar{\sigma}^n(\Lambda^c_n))}.
$$

Now we will define a limit Gibbs state on Ω by the following way.
We will say that μ is a limit Gibbs state on Ω, if for any finite subset Λ_n and for arbitrary boundary configuration $\bar{\sigma}^n(\Lambda \setminus \Lambda_n)$ the conditional probability with respect to μ given that the configuration $\sigma \in \Omega$ is $\bar{\sigma}^n(\Lambda \setminus \Lambda_n)$ on $\Lambda \setminus \Lambda_n$ is the same as the conditional Gibbs state on $\Omega(\Lambda)$ given above:

$$
\mu_n(\sigma(\Lambda_n)|\bar{\sigma}^n(\Lambda \setminus \Lambda_n)) = \frac{e^{-\beta H(\sigma_n|\bar{\sigma}^n)}}{Z_n(\bar{\sigma}^n)}
$$

The main problem of equilibrium statistical physics is to describe all limit Gibbs states of given Hamiltonian, i.e. to investigate the existence and uniqueness of such measures. Below we investigate the limit Gibbs measures corresponding to the Hamiltonian (3).

Let us split a set of configurations $\Sigma(\Lambda_n)$ into 4 subsets $\Sigma^{i_0,i_1}(\Lambda_n) = \{\sigma_n \in \Sigma(\Lambda_n) : \sigma(0) = i_0; \sigma(1) = i_1\}$ where $i_0, i_1 \in \{1, 2\}$.
Assume

$$Z_n^{i_0,i_1}(\bar{\sigma}^n) = \sum_{\sigma(\Lambda_n)\in\Sigma^{i_0,i_1}(\Lambda_n)} e^{-\beta H(\sigma(\Lambda_n)|\bar{\sigma}^n(\Lambda_n^c))},$$

and

$$z_1 = Z_n^{1,1}(\bar{\sigma}^n), z_2 = Z_n^{1,2}(\bar{\sigma}^n), z_3 = Z_n^{2,1}(\bar{\sigma}^n), z_4 = Z_n^{2,2}(\bar{\sigma}^n)$$

with

$$Z_n(\bar{\sigma}^n) = z_1 + z_2 + z_3 + z_4.$$

Let

$$z_1' = Z_{n+1}^{1,1}(\bar{\sigma}^{n+1}), z_2' = Z_{n+1}^{1,2}(\bar{\sigma}^{n+1}), z_3' = Z_{n+1}^{2,1}(\bar{\sigma}^{n+1}), z_4' = Z_{n+1}^{2,2}(\bar{\sigma}^{n+1})$$

Then we can produce the following recurrent equations

$$z_1' = p_{11}p_{11}^{(2)}z_1 + p_{11}p_{12}^{(2)}z_2 \tag{4}$$

$$z_2' = p_{12}p_{11}^{(2)}z_3 + p_{12}p_{12}^{(2)}z_4 \tag{5}$$

$$z_3' = p_{21}p_{21}^{(2)}z_1 + p_{21}p_{22}^{(2)}z_2 \tag{6}$$

$$z_4' = p_{22}p_{21}^{(2)}z_3 + p_{22}p_{22}^{(2)}z_4 \tag{7}$$

Renormalizing as follows [6]

$$x = \frac{z_2 + z_3}{z_1 + z_4}, \quad y_1 = \frac{z_1 - z_4}{z_1 + z_4}, \quad y_2 = \frac{z_2 - z_3}{z_1 + z_4},$$

one can produce the following recurrent equations

$$x' = \frac{p_{12}p_{11}^{(2)}z_3 + p_{12}p_{12}^{(2)}z_4 + p_{21}p_{21}^{(2)}z_1 + p_{21}p_{22}^{(2)}z_2}{p_{11}p_{11}^{(2)}z_1 + p_{11}p_{12}^{(2)}z_2 + p_{22}p_{21}^{(2)}z_3 + p_{22}p_{22}^{(2)}z_4} =$$
$$\frac{p_{12}p_{11}^{(2)}(x - y_2) + p_{12}p_{12}^{(2)}(1 - y_1) + p_{21}p_{21}^{(2)}(1 + y_1) + p_{21}p_{22}^{(2)}(x + y_2)}{p_{11}p_{11}^{(2)}(1 + y_1) + p_{11}p_{12}^{(2)}(x + y_2) + p_{22}p_{21}^{(2)}(x - y_2) + p_{22}p_{22}^{(2)}(1 - y_1)} =$$
$$\frac{(p_{12}p_{11}^{(2)} + p_{21}p_{22}^{(2)})x + (p_{21}p_{21}^{(2)} - p_{12}p_{12}^{(2)})y_1 + (p_{21}p_{22}^{(2)} - p_{12}p_{11}^{(2)})y_2 + D_1}{(p_{11}p_{12}^{(2)} + p_{22}p_{21}^{(2)})x + (p_{11}p_{11}^{(2)} - p_{22}p_{22}^{(2)})y_1 + (p_{11}p_{12}^{(2)} - p_{22}p_{21}^{(2)})y_2 + D},$$

where $D_1 = p_{12}p_{12}^{(2)} + p_{21}p_{21}^{(2)}, D = p_{11}p_{11}^{(2)} + p_{22}p_{22}^{(2)}$,

$$y_1' = \frac{p_{11}p_{11}^{(2)}z_1 + p_{11}p_{12}^{(2)}z_2 - p_{22}p_{21}^{(2)}z_3 - p_{22}p_{22}^{(2)}z_4}{p_{11}p_{11}^{(2)}z_1 + p_{11}p_{12}^{(2)}z_2 + p_{22}p_{21}^{(2)}z_3 + p_{22}p_{22}^{(2)}z_4} =$$
$$\frac{p_{11}p_{11}^{(2)}(1 + y_1) + p_{11}p_{12}^{(2)}(x + y_2) - p_{22}p_{21}^{(2)}(x - y_2) - p_{22}p_{22}^{(2)}(1 - y_1)}{p_{11}p_{11}^{(2)}(1 + y_1) + p_{11}p_{12}^{(2)}(x + y_2) + p_{22}p_{21}^{(2)}(x - y_2) + p_{22}p_{22}^{(2)}(1 - y_1)} =$$
$$\frac{(p_{12}p_{12}^{(2)} - p_{21}p_{21}^{(2)})x + (p_{12}p_{11}^{(2)} + p_{21}p_{22}^{(2)})y_1 + (p_{12}p_{12}^{(2)} + p_{21}p_{21}^{(2)})y_2 + D_2}{(p_{11}p_{12}^{(2)} + p_{22}p_{21}^{(2)})x + (p_{11}p_{11}^{(2)} - p_{22}p_{22}^{(2)})y_1 + (p_{11}p_{12}^{(2)} - p_{22}p_{21}^{(2)})y_2 + D},$$

where $D_2 = p_{12}p_{11}^{(2)} - p_{21}p_{22}^{(2)}, D = p_{11}p_{11}^{(2)} + p_{22}p_{22}^{(2)}$,

$$y_2' = \frac{p_{12}p_{11}^{(2)}z_3 + p_{12}p_{12}^{(2)}z_4 - p_{21}p_{21}^{(2)}z_1 - p_{21}p_{22}^{(2)}z_2}{p_{11}p_{11}^{(2)}z_1 + p_{11}p_{12}^{(2)}z_2 + p_{22}p_{21}^{(2)}z_3 + p_{22}p_{22}^{(2)}z_4} =$$

$$\frac{p_{12}p_{11}^{(2)}(x-y_2) + p_{12}p_{12}^{(2)}(1-y_1) - p_{21}p_{21}^{(2)}(1+y_1) - p_{21}p_{22}^{(2)}(x+y_2)}{p_{11}p_{11}^{(2)}(1+y_1) + p_{11}p_{12}^{(2)}(x+y_2) + p_{22}p_{21}^{(2)}(x-y_2) + p_{22}p_{22}^{(2)}(1-y_1)} =$$

$$\frac{(p_{12}p_{11}^{(2)} - p_{21}p_{22}^{(2)})x + (p_{21}p_{21}^{(2)} + p_{12}p_{12}^{(2)})y_1 + (p_{21}p_{22}^{(2)} + p_{12}p_{11}^{(2)})y_2 + D_3}{(p_{11}p_{12}^{(2)} + p_{22}p_{21}^{(2)})x + (p_{11}p_{11}^{(2)} - p_{22}p_{22}^{(2)})y_1 + (p_{11}p_{12}^{(2)} - p_{22}p_{21}^{(2)})y_2 + D},$$

where $D_3 = p_{12}p_{12}^{(2)} - p_{21}p_{21}^{(2)}, D = p_{11}p_{11}^{(2)} + p_{22}p_{22}^{(2)}$,

4. Markov Chain with double-stochastic transition matrix

Assume $r = 2$ and a matrix $\Pi = [p_{ij}]$ is a double-stochastic transition matrix, that is

$$\Pi = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$$

and

$$\Pi^2 = \begin{bmatrix} p^2+q^2 & 2pq \\ 2pq & p^2+q^2 \end{bmatrix},$$

where $p + q = 1$ with $0 < p < 1$
and the limiting distribution $\pi = (\pi_1, \pi_2)$ is a uniform distribution $\pi_1 = \pi_2 = 1/2$.

Then

$$x' = \frac{(p^2+q^2)qx + 2pq^2}{2p^2qx + p(p^2+q^2)}$$

$$y_1' = \frac{q(p^2+q^2)y_1 + 2pq^2y_2}{2p^2qx + p(p^2+q^2)}$$

$$y_2' = \frac{2pq^2y_1 + q(p^2+q^2)y_2}{2p^2qx + p(p^2+q^2)}$$

Since $q = 1 - p$, we have

$$x' = \frac{(1-p)(2p^2-2p+1)x + 2p(1-p)^2}{2(1-p)p^2x + p(2p^2-2p+1)}$$

$$y_1' = \frac{(1-p)(2p^2-2p+1)y_1 + 2p(1-p)^2y_2}{2(1-p)p^2x + p(2p^2-2p+1)}$$

$$y_2' = \frac{2p(1-p)^2y_1 + (1-p)(2p^2-2p+1)y_2}{2(1-p)p^2x + p(2p^2-2p+1)}$$

Starting from random initial conditions (with $y_1, y_2 \neq 0$), one iterates the recurrence relations and observes their behaviour after a large number of iterations. In the simplest situation a fixed point (x^*, y_1^*, y_2^*) is reached. In this case according the definition of conditional Gibbs measure (3) and recurrent equations (4)-(7)one can see that the conditional Gibbs measure and corresponding limit Gibbs measure will be the translation-invariant measure. Then respectively the limit Gibbs measure is a measure on $\Omega = S^{Z_+}$ corresponding to a homogeneous Markov chain with a stationary transition matrix $\Pi = [p_{ij}]$ and with stationary distribution $\pi = (1/2, 1/2)$. Using terminology of statistical physics, the set of stationary Markov measures is divided into two classes: paramagnetic phase (P) if $y_1^* = y_2^* = 0$ or to a ferromagnetic phase (F) if $y_1^*, y_2^* \neq 0$.
A limit cycle with a period a multiple of the distance between sites corresponds to periodic Markov measure or modulated limit Gibbs measure (M). Finally, the system may

remain aperiodic which corresponds aperiodic Markov measure or incommensurate phase (M).

The resultant phase diagram is shown in the following figure.

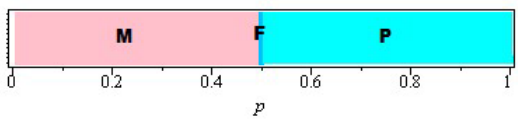

Figure 1. Phase Diagram for Double Stochastic Matrix

In this phase diagram P and F mean stationary and M -non stationary Markov measure.
The transition lines may be obtained by linearising the system around the fixed point $(x^*, 0, 0)$, where x^* is given by

$$x^* = \frac{(1-p)(2p^2-2p+1)x^* + 2p(1-p)^2}{2(1-p)p^2x^* + p(2p^2-2p+1)}.$$

Solving quadratic equation one can find

$$x^* = \frac{(q-p)(p^2+q^2) + \sqrt{(p-q)^2(p^2+q^2)^2 + 16p^3q^3}}{4p^2(1-p)}.$$

The variable x^* is unaffected in y_1 and y_2, and the linearized equations are, in matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \frac{(1-p)(2p^2-2p+1)}{A} & \frac{2p(1-p)^2}{A} \\ \frac{2p(1-p)^2}{A} & \frac{(1-p)(2p^2-2p+1)}{A} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

where $A = 2(1-p)p^2x^* + p(2p^2-2p+1)$.

The fixed point is linearly stable if the corresponding eigenvalues have moduli smaller than one.

Computing eigenvalues we have

$$\lambda_1 = \frac{(1-p)}{2(1-p)p^2x^* + p(2p^2-2p+1)}.$$

and

$$\lambda_2 = \frac{(1-p)(2p-1)^2}{2(1-p)p^2x^* + p(2p^2-2p+1)}.$$

Substituting x^* we have

$$\lambda_1 = \frac{2(1-p)}{2p^2 - 2p + 1 + \sqrt{(2p-1)^2(2p^2-2p+1)^2 + 16p^3(1-p)^3}}.$$

and

$$\lambda_2 = \frac{2(1-p)(2p-1)^2}{2p^2 - 2p + 1 - \sqrt{(2p-1)^2(2p^2-2p+1)^2 + 16p^3(1-p)^3}}.$$

Plotting graphs of the functions $\lambda_1(p)$ and $\lambda_2(p)$ one can see that $\lambda_1 > 1$ and $\lambda_2 > 1$ for $p \in (0, 0.5)$ and $\lambda_1 < 1$ and $\lambda_2 < 1$ for $p \in (0.5, 1)$.

Therefore the point $p = 0.5$ on segment $[0, 1]$ is the stationary-non-stationary transition point. Thus we have proved the following statement.

Theorem 4.1. *For Markov chain with memory 2 generated by Hamiltonian (2) with double stochastic matrix there exists stationary limit distribution for $p \in [0.5, 1)$ and non-stationary limit distribution for $p \in (0, 0.5)$.*

5. Dynamical System for general case

Now consider arbitrary stochastic matrix

$$\Pi = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

and

$$\Pi^2 = \begin{bmatrix} p^2 + (1-p)(1-q) & (1-p)(p+q) \\ (1-q)(p+q) & (1-p)(1-q) + q^2 \end{bmatrix},$$

where $0 < p, q < 1$.

Then produce the following recurrent equations with parameters $p, q \in (0, 1)$.

$$x' = \frac{A_1 x + B_1 y_1 + C_1 y_2 + D_1}{Ax + By_1 + Cy_2 + D}$$

$$y_1' = \frac{A_2 x + B_2 y_1 + C_2 y_2 + D_2}{Ax + By_1 + Cy_2 + D}$$

$$y_2' = \frac{A_3 x + B_3 y_1 + C_3 y_2 + D_3}{Ax + By_1 + Cy_2 + D}$$

where

$$A_1 = (1-p)[p^2 + (1-p)(1-q)] + (1-q)[(1-p)(1-q) + q^2]$$

$$B_1 = (q-p)(q+p-2)(p+q)$$

$$C_1 = (p+q)(p-q)[1-(p+q)]$$

$$D_1 = [(1-p)^2 + (1-q)^2](p+q)$$

and

$$A_2 = [(1-p)^2 - (1-q)^2](p+q)$$

$$B_2 = (1-p)[p^2 + (1-p)(1-q)] + (1-q)[(1-p)(1-q) + q^2]$$

$$C_2 = [(1-p)^2 + (1-q)^2](p+q)$$

$$D_2 = (1-p)[p^2 + (1-p)(1-q)] - (1-q)[(1-p)(1-q) + q^2]$$

and

$$A_3 = (1-p)[p^2 + (1-p)(1-q)] - (1-q)[(1-p)(1-q) + q^2]$$

$$B_3 = (q-p)(q+p-2)(p+q)$$

$$C_3 = p(1-p)(p+q) + q(1-q)(p+q)$$
$$D_3 = [(1-p)^2 - (1-q)^2](p+q)$$

and

$$A = [p(1-p) + q(1-q)](p+q)$$
$$B = p[p^2 + (1-p)(1-q)] - q[(1-p)(1-q) + q^2]$$
$$C = [p(1-p) + q(1-q)]](p+q)$$
$$D = [(1-p)^2 + (1-q)^2](p+q)[(1-p)^2 - (1-q)^2](p+q)$$

The resultant phase diagram is shown in the Fig.2. As one can see the phase diagram consist of the same three phases, namely, ferromagnetic, paramagnetic and modulated phases.

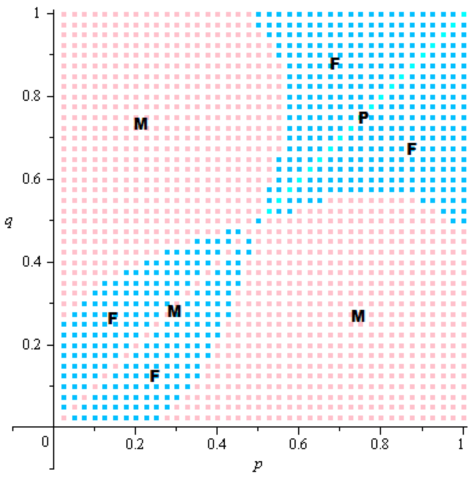

FIGURE 2. Phase Diagram for arbitrary transition matrix

According this phase diagram for $(p,q) \in I^2$ corresponding Gibbs state is translation-invariant if $(p,q) \in F \cup P$. One can see that we have symmetry for plotted phase diagram. On segment $\{(p,p) : p \in (0.5,1)\}$ we reach paramagnetic phase. The problem of describing $F - M$ transition line is rather difficult problem, since the problem of finding fixed points also is difficult. Thus we have proved the following statement.

Theorem 5.1. *For Markov chain with memory 2 generated by Hamiltonian (2) with arbitrary stochastic matrix there exists stationary limit distribution for* $(p,q) \in F \cup P$ *and non-stationary limit distribution for* $(p,q) \in I^2 \setminus (F \cup P)$*, where* $I^2 = (0,1) \times (0,1)$.

6. Markov Chains with Memory 2

Recall that assuming time homogeneity, a Markov chain $\{\sigma(n)\}$ with memory 2 can be conveniently represented via the order-3 tensor $\mathcal{P} = [p_{i_1 i_2 i_3}]$ [7], defined by

$$p_{i_1 i_2 i_3} := Pr(\sigma(n+1) = i_3 | \sigma(n) = i_2, \sigma(n-1) = i_1),$$

where $\mathcal{P}$ is called a transition probability tensor. Necessarily we have the properties $0 \leq p_{i_1 i_2 i_3} \leq 1$ and that

$$\sum_{i_3=1}^{r} p_{i_1 i_2 i_3} = 1$$

for every fixed 2-tuple (i_1, i_2).

Note that

$$Pr(\sigma(2) = i_2|\sigma(1) = i_1, \sigma(0) = i_0) = \frac{Pr(\sigma(0) = i_0, \sigma(1) = i_1, \sigma(2) = i_2)}{Pr(\sigma(0) = i_0, \sigma(1) = i_1)},$$

and

$$Pr(\sigma(0) = i_0|\sigma(1) = i_1, \sigma(2) = i_2) = \frac{Pr(\sigma(0) = i_0, \sigma(1) = i_1, \sigma(2) = i_2)}{Pr(\sigma(1) = i_1, \sigma(2) = i_2)}.$$

From these we have

$$\begin{aligned} & Pr(\sigma(2) = i_2|\sigma(1) = i_1, \sigma(0) = i_0) \\ = & \frac{Pr(\sigma(0) = i_0|\sigma(1) = i_1, \sigma(2) = i_2)Pr(\sigma(1) = i_1, \sigma(2) = i_2)}{Pr(\sigma(0) = i_0, \sigma(1) = i_1)}. \end{aligned}$$

By given Hamiltonian (2) we can find the conditional Gibbs measure $Pr(X_0 = i_0|X_1 = i_1, X_2 = i_2)$ [5]. Thus to specify the order-3 tensor $\mathcal{P} = [p_{i_1i_2i_3}]$ we have to specify the two dimensional distributions $Pr(\sigma(n) = i_1, \sigma(n+1) = i_2)$.

Since we are considering the case of time homogeneity it is enough to compute $Pr(\sigma(0) = i_1, \sigma(1) = i_2)$.

We consider the case $r = 2$. In the case of double stochastic matrix for fixed point $(x^*, 0, 0)$ we have

$$\begin{aligned} Pr(\sigma(0) = 1, \sigma(1) = 1) &= \frac{1}{2(1+x^*)} \\ Pr(\sigma(0) = 1, \sigma(1) = 2) &= \frac{x^*}{2(1+x^*)} \\ Pr(\sigma(0) = 2, \sigma(1) = 1) &= \frac{x^*}{2(1+x^*)} \\ Pr(\sigma(0) = 2, \sigma(1) = 2) &= \frac{1}{2(1+x^*)} \end{aligned}$$

In the general case for fixed point (x^*, y_1^*, y_2^*) we have

$$\begin{aligned} Pr(\sigma(0) = 1, \sigma(1) = 1) &= \frac{1+y_1^*}{2(1+x^*)} \\ Pr(\sigma(0) = 1, \sigma(1) = 2) &= \frac{x^*+y_2^*}{2(1+x^*)} \\ Pr(\sigma(0) = 2, \sigma(1) = 1) &= \frac{x^*-y_2^*}{2(1+x^*)} \\ Pr(\sigma(0) = 2, \sigma(1) = 2) &= \frac{1-y_1^*}{2(1+x^*)} \end{aligned}$$

In the case of double stochastic matrix we can find the conditional Gibbs measure $Pr(\sigma(0) = i_0|\sigma(1) = i_1, \sigma(2) = i_2)$ as follows

$$\begin{aligned} P_{1,11} = Pr(\sigma(0) = 1|\sigma(1) = 1, \sigma(2) = 1) &= \frac{p(p^2+q^2)}{p(p^2+q^2) + 2pq^2} \\ P_{2,11} = Pr(\sigma(0) = 2|\sigma(1) = 1, \sigma(2) = 1) &= \frac{2pq^2}{p(p^2+q^2) + 2pq^2} \\ P_{1,12} = Pr(\sigma(0) = 1|\sigma(1) = 1, \sigma(2) = 2) &= \frac{2pq^2}{2pq^2 + q(p^2+q^2)} \\ P_{2,12} = Pr(\sigma(0) = 2|\sigma(1) = 1, \sigma(2) = 2) &= \frac{q(p^2+q^2)}{2pq^2 + q(p^2+q^2)} \end{aligned}$$

and

$$P_{1,21} = Pr(\sigma(0)=1|\sigma(1)=2,\sigma(2)=1) = \frac{q(p^2+q^2)}{q(p^2+q^2)+2p^2q}$$
$$P_{2,21} = Pr(\sigma(0)=2|\sigma(1)=2,\sigma(2)=1) = \frac{2p^2q}{q(p^2+q^2)+2p^2q}$$
$$P_{1,22} = Pr(\sigma(0)=1|\sigma(1)=2,\sigma(2)=2) = \frac{2pq^2}{2pq^2+p(p^2+q^2)}$$
$$P_{2,22} = Pr(\sigma(0)=2|\sigma(1)=2,\sigma(2)=2) = \frac{p(p^2+q^2)}{2pq^2+p(p^2+q^2)}$$

In the general case we can find the conditional Gibbs measure $Pr(\sigma(0)=i_0|\sigma(1)=i_1,\sigma(2)=i_2)$ as follows

$$Pr(\sigma(0)=1|\sigma(1)=i_1,\sigma(2)=i_2) = \frac{p_{1i_1}p^{(2)}_{1i_2}}{p_{1i_1}p^{(2)}_{1i_2}+p_{2i_1}p^{(2)}_{2i_2}}, \tag{8}$$

and

$$Pr(\sigma(0)=2|\sigma(1)=i_1,\sigma(2)=i_2) = \frac{p_{2i_1}p^{(2)}_{2i_2}}{p_{1i_1}p^{(2)}_{1i_2}+p_{2i_1}p^{(2)}_{2i_2}}, \tag{9}$$

Thus we have proved the following statement

Theorem 6.1. *For Markov chain generated by Hamiltonian (2) with $(p,q)\in F\cup P$ one can specify the order-3 tensor $\mathcal{P}=[p_{i_1i_2i_3}]$ using equalities (8) and (9).*

7. Conclusion

In this paper we present Markov chains with memory 2 generated by Hamiltonian with competing interactions and describe all limit distributions.

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