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HOMOGENEOUS MARKOV CHAINS IN COMPACT SPACES

For homogeneous Markov chains in a compact and locally compact spaces, the ergodic properties are investigated, using the notions of topological recurrence and connections

1. Introduction

Let X be a metric compact space with a distance $d(x, x'), x, x' \in X$. Denote, by $C(X)$, the set of continuous functions $f : X \to R$. We use the same notation to the Banach space $C(X)$ with the norm

$$||f|| = \sup_{x \in X} |f(x)|.$$

We investigate ergodic properties of a discrete time homogeneous Markov process $\{\xi_n, n \geq 0\}$ in X with transition probability for one step $P(x, B), x \in X, B \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the Borelian σ -algebra in X. We assume that the transition probability satisfies the Feller condition, this means that the function

$$Tf(x) = \int f(z)P(x, dz)$$

is a continuous linear operator $C(X) \to C(X)$.

The main result which will be used in our investigation of ergodic properties of the Markov chains is the weak compactness of the set of all probability measures on $\mathcal{B}(X)$, and the main tool is the topological recurrence and connections.

The application of the weak compactness to the investigation of the ergodicity of dynamical systems in compact phase spaces was proposed by N.M. Krylov and N.N. Bogolubov [1], who developed a method of construction of invariant measures which is used here. In the work of the same authors [2], a variant of the ergodic theorem for a Markov chain (which is treated as a random dynamic system) in the compact space is proved.

The investigation of the ergodicity of Markov chains founded by Krylov and Bogolubov was extended by M.V. Bebutov [3], who formulated and proved the main theorem on the ergodicity of a Markov chain in the compact space, in which all ergodic invariant measures were described. But the proof of the result is not so rigorous as one needs in the modern considerations. In this article, we try to correct the proof using the notions of topological recurrence and connections which were introduced by A.N. Kolmogorov in [4] for countable Markov chains.

In this article, we will prove that any topologically recurrent state is represented as a union of topologically connected subsets and obtain the representation of the ergodic measure for each subset of this kind.

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We consider Markov chains in a locally compact space too. In this case, the set of topologically recurrent states may be empty, so the Markov chain is transient. If the set of topologically recurrent states is not empty, then it is represented as a union of topologically connected subsets, and there exists an invariant measure on any topologically connected subset which is unique to a multiplicative constant (if the space is not a compact, this measure may be infinite). In any way, the Birkhoff's ratio theorem is valid.

For a transient Markov chain $\{\xi_n, n \geq 1\}$, we prove the existence of a finite limit

$$\lim E_x \sum_{k \leq n} 1_{\{\xi_k \in B\}} = Q(x, B)$$

for any bounded $B \in \mathcal{B}$.

2. Topological recurrence and connections

Definition. A point $x \in X$ is said to be topologically recurrent to a Markov chain $\{\xi_n, n \geq 0\}$ if, for any $a > 0$, the relation

$$P_x(\sum_n 1_{\{\xi_n \in B_a(x)\}} = \infty) = 1$$

is fulfilled. Here, P_x is the conditional distribution of the discrete-time stochastic process $\{\xi_n, n \geq 0\}$ under the condition $\xi_0 = x$, $B_a(x)$ is the ball in X of radius a with the center at a point x.

Definition. Let x be a topologically recurrent state. It is topologically connected to a state $y \in X$ if the relation

$$\sum_n P_x(\xi_n \in B_a(y)) > 0$$

is fulfilled for any $a > 0$.

Lemma 1. *The set of topologically recurrent points is not empty.*

Proof. If this set is empty, then, for any $x \in X$, there exists $a(x) > 0$ for which the following relation is fulfilled:

$$P_x(\sum_n 1_{\{\xi_n \in B^o_{a(x)}(x)\}} < \infty / \{\xi_n, n \geq 0\} = 1.$$

Here, $B^o_a(x)$ is the open ball of radius a with the center at a point x. This implies the relation

$$X \subset \bigcup_{x \in X} B^o_{a(x)}(x).$$

It follows from the compactness of the set X that there exists a finite sequence $\{x_k, k \leq l\}$ for which

$$X \subset \bigcup_{k \leq l} B^o_{a(x_k)}(x_k),$$

so

$$\sum_n 1_{\{\xi_n \in X\}} < \infty.$$

This is impossible. Lemma 1 is proved.

Introduce an auxiliary homogeneous Markov chain in the space X depending on a parameter $0 < \lambda < 1$

$$\{x^\lambda_k, k \geq 0\}$$

with transition probability

$$P^\lambda(x,B) = (1-\lambda)\sum_{k>0}\lambda^{k-1}P_x(\xi_k \in B).$$

It satisfies the Feller condition. The Markov chain $\{\xi_k^\lambda, k \ge 0\}$ may be represented by the Markov chain $\{\xi_k, k \ge 0\}$. For this, introduce the sequence of independent identically distributed integer-valued random variables $\{\theta_k, k \ge 1\}$ with the distribution

$$P(\theta_k = m) = (1-\lambda)\lambda^{m-1},\ m \ge 1,$$

and set

$$\zeta_0 = 0,\ \zeta_n = \sum_{k\le n}\theta_k.$$

Then the stochastic processes $\{\xi_n^\lambda, n \ge 0\}$ and

$$\{\xi_{\zeta_n}, n \ge 0\}$$

have the same distribution.

Remark 1. The relation

$$P_x(\lim_n \frac{\sum_{k\le n} f(\xi_k^\lambda)}{\sum_{k\le n} f(\xi_k)} = 1-\lambda) = 1$$

is fulfilled for all $x \in X$ and $f \in C(X)$. The proof follows from the equality

$$\sum_{k\le n} f(\xi_k^\lambda) = \sum_{k\le n}\sum_j 1_{\{\zeta_k = j\}} f(\xi_j),$$

and the Blackwell's renewal theorem which implies the relation

$$\lim_{k\to\infty}\sum_n P(\zeta_n = k) = \frac{1}{E\theta_1}.$$

Definition. A state $x \in X$ is topologically regular if the following conditions are fulfilled:

1) The measure $P^\lambda(a,\cdot)$ is not pure atomic, i.e. the relation

$$\inf\{P^\lambda(x, X\setminus\Lambda) : \Lambda \in (CS)\} > 0$$

is valid, where (CS) is the set of countable subsets of X,

2) Denote, by S_x^*, the support of the measure $P^\lambda(x,\cdot)$, then

$$B_a(x) \subset S_x^*$$

for some $a > 0$.

Lemma 2. *Let x be a topologically recurrent state, and let a topologically regular state y satisfy the relation*

$$P_x^\lambda(\sum_k 1_{\{\xi_k^\lambda \in B_a(y)\}} = \infty) = 1$$

for all $a > 0$.

Then the state y is topologically connected to the state x, and the state y is topologically recurrent.

Proof. Denote

$$C_x = \{z \in X : P^\lambda(z, B_a(x)) > 0, a > 0\}.$$

Let $a > 0$ satisfy the relation

$$B_a(y) \subset S_y^*.$$

The relation

$$P^\lambda(x, B_a(y)) > 0$$

implies the formula

$$C_x \cap B_a \neq \emptyset$$

because, if it is wrong, the Markov process starting at the state x will not return in any neighborhood of x after it visited the ball $B_a(y)$. So

$$P^\lambda(y, B_a(x)) > 0$$

for all $a > 0$. Lemma 2 is proved.

Invariant sets

Definition. A set $S \in \mathcal{B}(X)$ is called an invariant set for the Markov chain $\{\xi_n, n \geq 0\}$ if the relation

$$P(\xi_1 \in S/\xi_0 = x) = 1$$

holds for all $x \in S$.

Remark 2. Let S be an invariant set for the Markov chain $\{\xi_n, n \geq 0\}$, and let $\hat{S}$ be a closure of the set S.

Then the set $\hat{S}$ is an invariant to the same Markov chain.

The proof of this statement is based on the Feller property of the transition probability.

Theorem 1. *Let $\{\xi_n, n \geq 0\}$ have the distribution P_x where x is a topologically recurrent and topologically regular state. Introduce the set*

$$S_x = \bigcap_n \text{Closure}\{\xi_k, k \geq n\}.$$

Then

1) S_x is a closed invariant non-random set.

2) Any $y \in S_x$ is a topologically recurrent state. If y is a topologically regular state, then x and y are topologically connected.

3) S_x is the minimal closed invariant set containing the state x.

Proof.

1) It is easy to see that the function

$$F(z, \{\xi_n, n \geq 0\}) = 1_{\{z \in S_x\}}$$

is an invariant function for the Markov chain $\{\xi_n, n \geq 0\}$ because of the relation

$$F(z, \xi_n, n \geq 0\}) = F(z, \{\xi_n, n \geq 1\}).$$

It is known that any invariant function is a function of ξ_0, so S_x is a non-random closed set depending on x only. The last equation implies that the set S_x is an invariant set.

2) Note that the relation $y \in S_x$ implies the relation

$$P_x(\sum_k 1_{\{\xi_k \in B_a(y)\}} = \infty) = 1$$

for all $a > 0$. This implies the topological recurrence of the state y and its topological connection to the state x because of Lemma 2 and the relation of equality $S_x = S_y$ which follows from the relations

$$S_y \subset S_x, \;\; S_x \subset S_y.$$

3) Assume $J \subset S_x, S_x \setminus J \neq \emptyset$ is an invariant closed set. Lemma 1 implies that there exists a recurrent state $z \in J$. Then

$$S_z = S_x, \;\; J \subset S_x \setminus S_z = \emptyset.$$

The theorem is proved.

3. Invariant measures

Let m be a probability measure on $\mathcal{B}(X)$. It is an invariant measure for the transition probability $P(x,B)$ if the relation

$$\int P(x,B)m(dx) = m(B) \tag{1}$$

is fulfilled for any $B \in \mathcal{B}(X)$.

Remark 3. A measure m is invariant iff the relation

$$\int Tg(x)m(dx) = \int g(x)m(dx) \tag{2}$$

is fulfilled for all $g \in C(X)$. This follows from the observation that formula (1) can be rewritten as formula (2) with $g = 1_B$.

Remark 4. Let m be an invariant probability measure. Denote, by P_m, the distribution of the Markov process $\{\xi_k, k \geq 0\}$ if the distribution of ξ_0 is the measure m. Denote

$$S_m = \text{Closure}\{\xi_k, k \geq 1\}.$$

Then S_m is the closed invariant set which is the support of the measure m.

For any $x \in X$ and $n \in N_+$, where N_+ is the set of all integer numbers, introduce probability measures $m_n(x,dz)$ by the relations

$$\int f(z)m_n(x,dz) = \frac{1}{n}\sum_{k<n} T^k f(x), f \in C(X), T^0 f(x) = f(x).$$

Theorem 2. *Assume that all states of S_x are topologically regular.*

Then, for any $z \in S_x$, there exists an invariant measure $m(z,dy)$ satisfying the relation

$$\int f(y)m(z,dy) = \lim_n \int f(y)m_n(z,dy), \quad f \in C(X). \tag{3}$$

These measures have properties

a) for any invariant measure ρ on the set S_x, the relation

$$\int f(z)\rho(dz) = \int (\int f(y)m(z,dy))\rho(dz), \quad f \in C(X)$$

is fulfilled,

b) for any $z \in S_x$, the measure $m(z,\cdot)$ is ergodic.

Proof. Since the sequence of measures $\{m_n(x,dz), n \in N_+\}$ is compact, there exists a subsequence $n_l, n_l \to \infty$ as $l \to \infty$ and a probability measure $m^*(z,dy)$ for which the relation

$$\lim_{l\to\infty} \int f(y)m_{n_l}(z,dy) = \int f(y)m^*(z,dy)$$

holds for all $f \in C(X)$. This implies the relation

$$\int Tf(y)m^*(z,dy) = \lim_{l\to\infty} \int Tf(y)m_{n_l}(z,dy).$$

The relation

$$\int Tf(y)m_n(z,dy) = \frac{1}{n}\sum_{1\leq k\leq n+1} T^k f(z)$$

implies the inequality

$$\sup_{z\in X} |\int Tf(y)m_n(z,dy) - \int f(y)m_n(z,dy)| \le \frac{2||f||}{n},$$

from which we obtain the equality

$$\int Tf(y)m^*(z,dy) = \int f(y)m^*(z,dy).$$

So $m^*(z,dy)$ is an invariant measure for the transition probability $P(z,B)$.

Consider the Markov chain $\{\xi_k, k \ge 0\}$ with ξ_0 having the distribution $m^* = m^*(z,\cdot)$. It is a stationary process with the invariant measure m^*. The Birkhoff ergodic theorem implies that, for all $f \in C$, the relation

$$P_y(\lim \frac{1}{n}\sum_{k\le n} f(\xi_k) = f(\xi_0)) = 1 \tag{4}$$

is fulfilled for almost all y with respect to the measure m^* which is the distribution of ξ_0. In particular, we have the relation

$$\lim_{n\to\infty} E_{m^*}\frac{1}{n}\sum_{k\le n} f(\xi_k) = \int f(y)m^*(dy)$$

which implies formula (3).

Statement a) follows from the formula

$$\int f(z)\rho(dz) = \int Tf(z)\rho(dz) = \int\int f(y)m_n(z,dy)\rho(dz)$$

and formula (3).

To prove statement b), consider the set $IM(S_x)$ of all probability invariant measures on the set S_x. It is a weakly compact convex set in the space $M(X)$ of all finite measures on $\mathcal{B}(X)$. Denote, by $EIM(S_x)$, the set of all extreme invariant measures, they are ergodic. Any measure $m \in IM(S_x)$ is a mixture of ergodic measures,

$$m = \int_{EIM(S_x)} \nu\alpha_m(d\nu),$$

where α_m is a probability measure on the Borelian σ-algebra of the set $EIM(S_x)$. This representation is unique. So formula (3) implies the relation

$$EIM(S_x) = \{m(z,\cdot), z \in S_x\}.$$

4. Ergodicity

The Markov chain $\{\xi_n, n \ge 0\}$ is ergodic if the set $IM(X)$ of all probability invariant measures for the transition probability $P(x,B)$ contains only one element.

Theorem 3. *The Markov chain $\{\xi_n, n \ge 0\}$ is ergodic iff the Markov chain $\{\xi^\lambda_n, n \ge 0\}$ with the transition probability $P^\lambda(x,B)$ is.*

Proof. It follows from the relation $(IM)(X) = (IM)^\lambda(X)$, the last being the set of invariant probability measures for the transition probability P^λ. This follows from the following statement.

Lemma 3. *If a probability measure m is the invariant measure for the transition probability P^λ for some $\lambda_0 \in (0,1)$, then it is an invariant measure for the transition probability P.*

Proof. Introduce operators in the space **MB** of bounded measurable functions $f :\to R$ with the norm

$$||f|| = \sup_{x\in X} |f(x)|,$$

$$Tf(x) = \int f(y)P(x,dy), \ \ T_\lambda f(x) = \int f(y)P_\lambda(x,dy).$$

Then

$$T_\lambda = (1-\lambda)R_\lambda = (1-\lambda)(I-\lambda T)^{-1}.$$

A measure m is invariant for the transition probability P_λ if the relation

$$\int Tf(x)m(dx) = \int f(x)m(dx)$$

holds. The function R_λ is an analytic function of $\lambda, |\lambda| < 1$. The derivatives of this function are represented by the formula

$$D^n_\lambda = \frac{d^n}{d\lambda^n} R_\lambda = (-1)^n n!(R_\lambda)^n. \tag{5}$$

This formula implies that the measure m is invariant for the operator $D^{\lambda_0}_n$, i.e.

$$\int D^{\lambda_0}_n f(x)m(dx) = \int f(x)mdx), \quad f \in \mathbf{BM}.$$

The Taylor's formula implies the relation

$$R_\lambda = R_{\lambda_0} + \sum_{n>0} (n!)^{-1} D^n_{\lambda_0}(\lambda-\lambda_0)^n. \tag{6}$$

So the measure m is invariant for all

$$P^\lambda, \quad |\lambda| < 1.$$

It follows from the formula

$$\int f(x)m(dx) = \int P_\lambda f(x)m(dx) = \sum_{k>0} \int T^k f(x)m(dx)$$

that $\int T^k f(x)m(dx) = \int f(x)m(dx)$ for all $k > 0, f \in \mathbf{MB}$. Lemma 3 is proved.

Assume that any state $x \in X$ is topologically recurrent and topologically regular, and any state x is topologically connected to all states $y \in X$.

Lemma 4. *Let a function $f \in C(X)$ satisfy the condition*

$$\sup\{f(x) - f(y) : x \in X, y \in X\} > 0.$$

Then the inequality

$$\inf\{Tf(x) : x \in X\} > \inf\{f(x) : x \in X\}$$

is valid.

Proof. It suffices to consider the case

$$f \geq 0, \inf\{f(x) : x \in X\} = 0.$$

The open set

$$\{x : f(x) > 0\}$$

is not the empty set so $Tf(x) > 0$ because all states are topologically connected. Compactness of the set X implies the relation

$$\inf\{Tf(x) : x \in X\} > 0.$$

Corollary 1. *If a function f satisfies the condition of Lemma 3, then the relations*

$$\inf_{x\in X} T^{n+1}f(x) > \inf_{x\in X} T^n f(x), n \in \mathbf{N}_+,$$

$$\inf_{x\in X} T_\lambda^{n+1}f(x) > \inf_{x\in X} T_\lambda^n f(x), n \in \mathcal{N}_+$$

are fulfilled for any $\lambda \in (0, 1)$.

Denote

$$M_\lambda(f) = \sup_n \inf_{x\in X} T_\lambda^n f(x).$$

Then $M_\lambda(f) > 0$ for all $f \in C{*}_+(X)$ where $C_+^* = \{f \in C_+ :: \sup\{f(x) - f(x') > 0\}$.

Introduce the condition

SUC

$$\inf\{M_-(f) : f \in C_+^*(X), \sup\{\int f(x)m(dx) : m \in IM(X)\} = 1\} = \delta > 0.$$

A Markov chain satisfying this condition is called strong uniform connected.

Theorem 4. *The Markov chain $\{\xi_n, n \geq 0\}$ is ergodic iff condition* **SUC** *is fulfilled.*

Proof. Assume that the Markov chain $\{\xi_n, n \geq 0\}$ is ergodic with ergodic probability measure m. Then the relation

$$\lim_n S_n f(x) = \int f(z)m(dz), \quad f \in C(X), \quad S_n f(x) = \frac{1}{n}\sum_{1\leq k\leq n} T^k f(x).$$

is fulfilled for almost all $x \in X$ with respect to the measure m.

Prove the formula

$$\lim_n \sup_{x\in X} |S_n f(x) - \int f(z)m(dz)| = 0, f \in C(X), S_n f(x) = \sum_{1\leq k\leq n} T^k f(x). \tag{7}$$

If formula (7) is not true for some function f, then there exist sequences

$$\{z_k \in X, k \geq 1\}, \{n_k \in \mathbf{N}_+, n_k \uparrow +\infty\}$$

satisfying the inequality

$$|S_{n_k} f(z_k) - \int f(z)m(dz)| > \rho > 0.$$

We can assume that, for any $g \in C(X)$, there exists

$$\lim_k S_{n_k} g(z_k) = \int g(z)m^*(dz)$$

where m^* is a probability measure. It is easy to check that m^* is an invariant measure. In addition,

$$|\int f(z)m^*(dz) - \int f(z)m(dz)| \geq \rho,$$

so $m^* \neq m$. This contradicts the ergodicity of the Markov chain.

Formula (7) implies condition **SUC** because of Remark 1.

Let condition **SUC** be fulfilled, and let $m_k, k = 1, 2$ be two different probability ergodic distributions for the Markov chain.

Then $m_1 \perp m_2$ and, for any $\rho > 0$, there exist the closed sets $F_k, k = 1, 2$, satisfying the conditions

$$F_1 \cap F_2 = \emptyset, \;\; m_k(F_k) > 1 - \rho, k = 1, 2.$$

Let a function

$$f_\rho : X \to R_+$$

satisfy the conditions

$$f_\rho \in C_+(X), \;\; f_\rho(x) \le 1, \;\; f_\rho(x) = 0, x \in F_2, \;\; e_\rho(x) = 1, x \in F_1.$$

Then the inequalities

$$\int f_\rho(x) m_1(dx) > 1 - \rho, \;\; \int f_\rho(x) m_2(dx) < \rho$$

are valid. If

$$\rho = \frac{\delta}{1 + \delta}$$

where δ is from condition **SUC**, then

$$\int (1 - \rho)^{-1} f_\rho(x) m_1(dx) > 1$$

so $M_-(f_\rho) > \delta$ and

$$\int S_n f_\rho(x) m_2(dx) > \delta(1 - \rho) = \rho$$

for all sufficiently large n. On the other hand,

$$\int S_n f_\rho m_2(dx) = \int f_\rho(x) m_2(dx) < \rho.$$

We obtain a contradiction. Theorem 4 is proved.

5. Irregular Markov chains

A Markov chain is irregular if the transition probability of the chain does not satisfy the condition of topological regularity. We consider some examples of this kind proposed by Professor A.M. Kulik (Kyiv Institute of Mathematics).

Example 1. Let $X = \mathbb{N} \cup \{\infty\}$ with $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$ $(\frac{1}{\infty} \equiv 0)$. Define the transition probability P of the Markov chain in X by

$$P(x, A) = Q(x, A \cap \mathbb{N}) \mathbb{I}_{x \in \mathbb{I}}\mathrm{N} + \delta_\infty \mathbb{I}_{x = \infty},$$

where Q is a transition probability of some ergodic Markov chain in $\mathbb{N}$ such that the ergodic measure for this chain is supported by the whole $\mathbb{N}$. Then the following features hold.

a) All states from the set $X \setminus \{\infty\}$ are topologically recurrent, topologically regular, and topologically connected to one another,

b) The state ∞ is absorbing and is not topologically connected to any $x \in X \setminus \{\infty\}$, while any $x \in X \setminus \{\infty\}$ is topologically connected to ∞,

c) There exist two ergodic distributions: one on the set $X \setminus \{\infty\}$ and another on the singlet $\{\infty\}$.

Example 2 Let $X = \{0, 1\}^\infty$, i.e. points of the set are represented in the form

$$x = \{[x]_k, k \ge 1\}, [x]_k \in \{0, 1\},$$

and the distance is determined by the formula

$$d(x, y) = \sum_{k \ge 1} 2^{-k} |[x]_k - [y]_k|.$$

Let the transition probability from a state x to a state y be given by the formula

$$P(x,y) = \begin{cases} \frac{2}{3}, & [y]_k = [x]_{k+1}, k \geq 1 \\ \frac{1}{6}, & [y]_k = [x]_{k-1}, k \geq 2, [y]_1 = 0 \\ \frac{1}{6}, & [y]_k = [x]_{k-1}, k \geq 2, [y]_1 = 1 \end{cases}.$$

For this Markov chain, all states are topologically recurrent and pairwise topologically connected. For any $x \in X$, the set

$$I_x = \bigcup_{m \in \mathbb{N}} \{y \in X : \sum_k \mathbb{1}_{[x]_k \neq [y]_{m+k}} < +\infty\}$$

is a countable invariant set. If x is periodic (i.e., $\exists m : [x]_k = [x]_{k+m}, k \in \mathbb{N}$), then there exists a unique invariant measure ρ_x on the set I_x. The peculiarity of this statement is the fact: the closure of the set I_x (i.e., the set S_x) coincides with X for any x, while there exists a wide variety of (mutually singular) ergodic measures on X.

Consider a general result related to irregular Markov chains.

Theorem 5. *Consider a Markov chain in X with transition probability $P(x, A), x \in X, A \in \mathcal{B}(X)$ satisfying the conditions*

1) the Feller condition,

2) for any $x \in X$, the support of the measure $P^\lambda(x, \cdot)$ is the set $\Lambda_x \in (CCS)$ where (CCS) is the set of all compact countable subsets of X,

3) any $x \in X$ is topologically recurrent.

Then, for any $x \in X$, the set Λ_x is a closed invariant set, and there exists a unique ergodic measure ρ_x on the set Λ_x.

The proof follows from Theorem 2.

6. Markov chains in locally compact spaces

In this section, we denote, by X, a locally compact metric space with a distance $d(x, x')$. Consider a Markov chain $\{\xi_n, n \geq 0\}$ in X. Assume that the transition probability $P(x, B), x \in X, B \in \mathcal{B}(X)$ of the chain satisfies the C_0 Feller condition, i.e.

$$Tf(x) = \int f(z)P(x, dz) \in C_0(X), f \in C_0(X),$$

where

$$C_0(X) = \{f \in C(X) : \lim_{x \to \infty} f(x) = 0\}$$

and $C(X)$ is the Banach space of all continuous bounded fumctions $f : X \to R$.

Introduce the C_0-weak convergence of measures on a σ-algebra $\mathcal{B}(X)$, for which the sequence of measures $\{m_n, n \geq 1\}$ is convergent to a measure m if the relation

$$\lim_n \int f(x)m_n(dx) = \int f(x)m(dx)$$

is fulfilled for any $f \in C_0(X)$.

It is known that the set $M(X)$ of all finite measures on $\mathcal{B}(X)$ is a locally compact set with respect to the C_0-weak convergence. That is, for any bounded sequence of measures $\{m_n, n \geq 1\}$, there exists a C_0-weakly convergent subsequence $\{m_{n_k}.n_k \in \mathcal{N}, n_k \to \infty\}$. This implies the existence of invariant measures for the Markov chain $\{\xi_n, n \geq 1\}$.

Compactly imbedded Markov chains

Denote by Φ the set of the functions $\phi \in C_0(X)$ such that the set $F_\phi = \text{Closure}\{x \in X : \phi(x) > 0\}$ is compact.

Lemma 5. *Let $\phi \in \Phi, 0 \leq \phi \leq 1$. Suppose that every $x \in F_\phi$ is topologically recurrent and, for any $y \in X$, there exists $x \in \{\phi > 0\}$ that is topologically connected to y. Introduce a function*

$$P_\phi(x, A) = E_x \sum_{n \geq 1} \prod_{0 \leq k < n} (1 - \phi(\xi_k))\phi(\xi_n) 1_{\{\xi_n \in A\}}, \quad x \in X, A \in \mathcal{B}(X).$$

Then $P_\phi(x, A)$ is the transition probability of a homogeneous Markov process in X that satisfies the Feller property on the compact set F_ϕ.

Proof. Note that

$$P_\phi(x, X) = E_x \sum_{n \geq 1} \phi(\xi_n) \prod_{k<n} (1 - \phi(\xi_k)) = 1 + E_x \prod_{k \geq 1} (1 - \phi(\xi_k))$$

because of the formula

$$1 - \prod_{k \geq 1} (1 - a_k) = \sum_{n \geq 1} a_n \prod_{k<n} (1 - a_k), 0 < a_k < 1, k \geq 1.$$

In addition,

$$\prod_k (1 - \phi(\xi_k)) \leq \exp\{-\sum_k \phi(\xi_k)\} = 0,$$

because the recurrence condition imposed on the Markov chain $\{\xi_k k \geq 0\}$ implies the relation $\sum_k \phi(\xi_k) = +\infty$ P_y-a.s. for every $y \in X$.

So $P_\phi(x, X) = 1$ for all $x \in X$, and P_ϕ is a transition probability.

To prove the Feller property of the transition probability P_ϕ use the formula

$$\int f(y) P_\phi(x, dy) = E_x \sum_{n \geq 1} (\prod_{k<n} (1 - \phi(\xi_k)))\phi(\xi_n) f(\xi_n).$$

Denote

$$I_n f(x) = E_x (\prod_{k<n} (1 - \phi(\xi_k)))\phi(\xi_n) f(\xi_n).$$

This function is continuous in x because of the Feller property of the transition probability $P(x, A)$. If $f > 0$, then $I_n f(x) > 0$. The series

$$\int f(y) P_\phi(x, dy) = \sum_{n \geq 1} I_n f(x)$$

of non-negative functions on the compact set F_ϕ converges uniformly to a continuous function.

Extended filtrations

Let $\{\theta_k, k \geq 0\}$ be a sequence of independent random variables which uniformly distributed on the interval $(0, 1)$ and are independent on the sequence $\{\xi_k, k \geq 0\}$.

Denote, by $\hat{\mathcal{F}}$, the σ-algebra generated by the random variables

$$\{\xi_k, k \leq n\} \cup \{\theta_k, k \leq n\}.$$

The filtration $\{\hat{\mathcal{F}}_n, n \geq 0\}$ is an extended filtration for the Markov chain $\{\xi_k, k \geq 0\}$.

Lemma 6. *Consider a stopping time with respect to the extended filtration:*

$$\nu^\phi = \inf\{k \in N : \theta_k < \phi(\xi_k)\}.$$

Then the relation

$$P(\xi(\nu^\phi) \in B/\xi_0 = x) = P^\phi(x, B)$$

is valid.

The proof follows from the relations

$$P(\xi(\nu^\phi \in B/\xi_0)$$

$$= \sum_{n\geq 1} = E_x 1_{\{\theta_0 \geq \phi(\xi_0), \cdots \theta_{n-1} \geq \phi(\xi_{n-1}), \theta_n < \phi(\xi_n)\}} 1_{\{\xi_n \in B\}}$$

$$= \sum_{n\geq 1} (\prod_{k<n} (1 - \phi(\xi_k))\phi(\xi_n) 1_{\{\xi_n \in B\}}.$$

Theorem 6. *Introduce a sequence of stopping times with respect to the filtration*

$$\{\hat{\mathcal{F}}_n, n \geq 0\} :$$

$$\nu_0^\phi = \nu^\phi, \ \nu_l^\phi = \inf\{k > \nu_{l-1}^\phi : \theta_k < \phi(\xi_k)\}, \quad l > 0.$$

Then the sequence $\{\xi^\phi = \xi(\nu_l^\phi), l \geq 0\}$ is a homogeneous Markov chain with values in the compact set F_ϕ and the transition probability function $P^\phi(x, A)$

The proof follows from the strong Markov property of the Markov chain $\{\xi_k, k \geq 0\}$ and Lemma 6.

Remark 5. Assume that, for every $\phi \in \Phi$, the Markov chain $\{\xi_k^\phi, k \geq 1\}$ is ergodic with ergodic measure m^ϕ on the σ -algebra $\mathcal{B}(F_\phi)$. Then the Markov chain $\{\xi_n, n \geq 0\}$ is ergodic with ergodic measure m on the σ-algebra $\mathcal{B}(X)$ for which

$$\int g(x)m(dx) = \frac{\int S_\phi(g, x)m^\phi(dx)}{\int S_\phi(1, x)m^\phi(dx)}, \quad \phi \in \Phi,$$

where

$$S_\phi(g, x) = E_x \sum_k g(\xi_k) \prod_{j<k} (1 - \phi(\xi_j))\phi(\xi_k).$$

The proof follows from the relations

$$\frac{\sum_{k<\nu_\phi} g(\xi_k)}{\nu^\phi} = \frac{\sum_{l<n} U_l(g)}{\sum_{l<n} U_l(1)}, \ U_l(g) = \sum g(\xi_j) 1_{\{\nu_{l-1}^\phi < j \leq \nu_l^\phi\}}$$

and the ratio ergodic theorem.

7. Transient Markov chains in a locally compact space

Definition. The Markov chain $\{\xi_n, n \geq 0\}$ in the locally compact space X is called transient iff the condition

$$P_x(\sum_k 1_{\{\xi_k \in B_a(y)\}} < \infty) = 1$$

is fulfilled for all $a > 0, x \in X, y \in X$.

It is easy to check that, for the transient Markov chain $\{\xi_k, k \geq 0\}$, the relation

$$P_x(\sum_k 1_{\{\xi_k \in C\}} < \infty) = 1$$

is fulfilled for any $x \in X$ and a compact set $C \subset X$.

Lemma 7. *The relation*

$$P(\lim_n d(\bar{x}, \xi_n) = \infty) = 1$$

is fulfilled for any $\bar{x} \in X$.

Proof. Set

$$\nu_r = Card\{\xi_k : d(\bar{x}, \xi_k)\}, r = 1, 2, \cdot.$$

Then ν_r are $\mathcal{F}_\infty$ measurable random variables for which the relations

$$P(\nu_1 \le \nu_2 \le \cdot \le \nu_m \le \cdot < \infty) = 1,$$

$$d(\bar{x}, \xi_k) > r, k > \nu_r$$

are fulfilled. Lemma 7 is proved.

Theorem 7. *Let* $C \subset X$ *be a compact set. Then the relation*

$$E_x \sum_k 1_{\{\xi_k \in C\}} < \infty$$

is fulfilled for any $x \in X$.

Proof. Introduce the random variables

$$\rho(C) = \sum_k 1_{\{\xi_k \in C\}}, \ \ \rho_n(C) = \sum_{k \ge n} 1_{\{\xi_k \in C\}},$$

and the stopping times

$$\theta_1 = \min\{k : \xi_k \in C\}, \ \ \theta_1^* = \min\{k > \theta_1 : \xi_k \notin C\},$$

$$\theta_n = \min\{k > \theta_{n-1}^* : \ xi_k \in C\}, \ \ \theta_n^* = \min\{k > \theta_n : \xi_k \notin C\}, n > 1.$$

Note that, for $x \in C$, the relations $P_x(\theta_1 = 0) = 1$ and

$$P_x(\theta_1^* \ge l) \ge E_x g(C, \xi_l)$$

are fulfilled, where $l > 0, l \in N_+$ and

$$g(C, z) = d(z, C) \wedge 1.$$

Since

$$E_x g(C, \xi_l), x \in C, l > 0, l \in N_+$$

is a continuous function, so

$$\inf_{x \in C} E_x g(C, \xi_l)) > 0$$

for some $l > 0, l \in N_+$. The inequality

$$P_x(\theta_1^* \ge l) \ge \alpha, x \in C$$

is fulfilled for some $l \in N_+$ and $\alpha > 0$. This implies the formula

$$\sup_{x \in C} E_x \theta_1^* \le \frac{l}{1 - \alpha}. \tag{8}$$

It is easy to see that the conditional distribution of the random variable $\theta_n^* - \theta_n$ with respect to the σ-algebra $\mathcal{F}_{x_{\theta_n}}$ coincides with the conditional distribution of the random variable θ_1 with respect to the random variable ξ_0 if $\xi_0 = \xi_{\theta_n}$. This follows from the strong Markov property of the Markov chain $\{\xi_k, k \ge 0\}$. So the inequality

$$E_x(\theta_n^* - \theta_n / \mathcal{F}_{\theta_n}) \le \frac{l}{1 - \alpha} \tag{9}$$

is fulfilled for all n for which

$$P_x(\theta_n < \infty) > 0$$

and, in this case,

$$P(\theta_n^* < \infty / \mathcal{F}_{\theta_n}) = 1_{\{\theta_n < \infty\}}.$$

Consider a sequence

$$\eta_n = E(\phi(\rho_{n+1}(C))/\mathcal{F}_n, n > 0,$$

where $\phi(t) = 1 - \exp\{-t\}$. The inequalities

$$\rho_n(C) \geq \rho_{n+1}(C), \ \ \phi(\rho_n(C)) \geq \phi(\rho_{n+1}(C))$$

imply the relation

$$E(\eta_{n+1}/\mathcal{F}_n) \leq \eta_n.$$

So the sequence $\{\eta_n, n \geq 0\}$ is a non-negative supermartingale, and the relation

(10) $$P_x(\sup_{n \leq N} \eta_n > a) \leq \frac{E_x \eta_N}{a}$$

holds. Note that the sequence $E_x \eta_n$ is decreasing to zero because of the relation

$$P_x(\rho_n(C) \to 0) = 1.$$

Let

$$a = \frac{e-1}{e},$$

and

$$N_0 = \min\{k : E_x \eta_k \leq \frac{a}{2}\}.$$

Then, for any $l \in N_+$ for which $\theta_l^* \geq N_0$ and

$$P_x(\theta_l^* < \infty) > 0,$$

the relation

$$P(\theta_{l+1} < \infty / \mathcal{F}_{\theta_l^*}) \leq \frac{1}{2}.$$

is fulfilled. Note that, for this l and $k \in N_+, k > 0$, the relation

$$P(\theta_{l+k} < \infty / \mathcal{F}_{\theta_l^*}) \leq \frac{1}{2} E(1_{\{\theta_{l+k-1} < \infty\}} / \mathcal{F}_{\theta_l^*}) \leq \frac{1}{2^k}.$$

is fulfilled too. Using the relation

$$\sum_n 1_{\{\xi_n \in C\}} = \sum_{i \geq 0} 1_{\{\theta_i < \infty\}} \sum_k 1_{\{\theta_i \leq k < \theta_i^*\}}$$

we obtain the inequality

$$E_x \rho_(C) \leq \frac{1}{1-\alpha} E_x \sum_{i \geq 0} E_x 1_{\{\theta_i < \infty\}} \leq \frac{1}{1-\alpha}(N_0 + \sum_{k \geq 0} \frac{1}{2^k}) < \infty.$$

Theorem 7 is proved.

Corollary 2. *A function*

(11) $$Q(x, E) = E_x \sum_k 1_{\{\xi_k \in E\}}$$

is determined for all $x \in X$ and

$$E \in \bigcup_{n \in N_+} \mathcal{B}(B_n),$$

where

$$B_n = B_n(\bar{x}), \bar{x}$$

is a point in X.

Remark 6. Assume that the relation

$$\sum_k P_x(\xi_k \in B_a(y)) > 0 \tag{12}$$

is fulfilled for some $y \in X$ and all $a > 0$. Then the formula

$$Q(y, E) = Q(x, E) - P_x(\xi_0 \in E) - P_x(\xi_1 \in E) + P_y(\xi_0 \in E) + P_y(\xi_1 \in E)$$

is valid. The proof follows from the proof of Lemma 2.

Subharmonic functions in R^d

Assume $X = R^d$. For a transient Markov chain $\{\xi_k, k \geq 0\}$, consider the sequence of random variables

$$\eta_n = \min\{|\xi_k| : k \geq n\}, n \geq 0.$$

Theorem 8. *Let formula (10) be fulfilled for all $x \in R^d$ and $y \in R^d$. Then the sequence $\{\eta_n, n \geq 0\}$ satisfies the conditions*

(i) $\eta_{n+1} \geq \eta_n$ for all $n \geq 0$;

(ii) $P_x(\sup_n \eta_n = \infty) = 1$ for all $x \in R^d$;

(iii) there exists a non-negative concave function $g : R_+ \to R_+$ for which $E_x g(\eta_n) < \infty$;

(iv) set

$$h(x) = E_x g(\eta_0),$$

then $\{h(\xi_k), k \geq 0\}$ is a submartingale, so $h(x)$ is a continuous submartingale function.

Proof. Statement (i) follows from the definition of η_n, statement (ii) follows from Theorem 7.

To prove statement (iii), it suffices to prove that, for a subsequence $\{n_k, k \geq 1\}$ with $n_k \uparrow \infty$, there exists a function of the kind mentioned in (iii) for which $E_x g(\eta_{n_k}) < \infty$ for all $k \geq 1$.

The relation

$$\lim_a P_x(|\eta_n| > a) = 0$$

for all $n \in N_+$ implies the existence ofsequences $\{n_k, k \geq 1\}$ and

$$\{p_k, k \geq 1\}, p_k \geq 0, \ \sum_k p_k = 1,$$

for which

$$P_x(\sum_k p_k \eta_{n_k} < \infty) = 1.$$

Denote

$$\zeta = \sum_k p_k \eta_k.$$

Then $P_x(0 \leq \zeta < \infty) = 1$ and there exists a function g satisfying the conditions of statement (iii) for which $E_x g(\zeta) < \infty$, so

$$\infty > E_x g(\zeta) \geq \sum_k p_k E g_{n_k}.$$

To prove statement (iv), consider a sequence

$$\zeta_n = E(g(\eta_n / \mathcal{F}_n), n \geq 0.$$

The inequality $\eta_{n+1} \geq \eta_n$ implies the relation

$$E(\zeta_{n+1}/\mathcal{F}_n = E(g(\eta_{n+1})/\mathcal{F}_n) \geq E(g(\eta_n)/\mathcal{F}_n) = \zeta_n$$

so the sequence $\{\zeta_n, n \geq 0\}$ is a submartingale and

$$\zeta_n = h(\xi_n)$$

because of the strong Markov property of the Markov chain $\{\xi_n, n \geq 0\}$. Theorem 8 is proved.

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