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THE DECOMPOSITION OF A SOLUTION OF THE QUASILINEAR STOCHASTIC PARABOLIC EQUATION WITH WEAK SOURCE

We obtain conditions which guarantee the existence of a decomposition of a solution of the quasilinear stochastic parabolic equation with a weak source.

1. Definitions, notations, formulation of the problem.

On a complete probability space $(\Omega, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, \mathbf{P})$, we consider the Cauchy problem

$$du(t,x) = au_{xx}(t,x)dt + b|u(t,x)|^{\gamma-1}u(t,x)dw(t), \tag{1}$$

$$t \in [0;\tau(\omega)), x \in R^1, u(0,x) = u_0(x) \geq 0.$$

Here: $w(t)$ is a standard $(\mathbf{F}_t)_{t\geq 0}$ - adapted Wiener process in R^1; a, b are positive constants, $\gamma \in (0;1)$; $\tau(\omega)$ is a stopping time; $u_0(x)$ is a nonrandom nonnegative function, $u_0 \in \mathbf{W} = W_2^1(R^1) \cap L_{\gamma+1}(R^1)$.

The alphabetic subindex by a function symbol means a partial derivative in the sense of distributions with respect to appropriate variable.

We will use the next notations.

$W_2^1(R^1)$ is the usual Sobolev space consisting of functions having the first derivative (in the sense of distributions) square integrable on R^1, and the space $L_\alpha(R^1)$ consists of functions integrable on R^1 with degree α,

$$p = \|v\|^2 = \int |v(y)|^2 dy, z = \|v_y\|^2 = \int |v_y(y)|^2 dy, q = |||v|||^{\gamma+1} = \int |v(y)|^{\gamma+1} dy.$$

$$f(t;u) = \frac{1}{2}\|u(t,\bullet)\|^2 - \frac{1}{2}\|u(0,\bullet)\|^2 + \frac{a}{2}\int_0^t \|u_x(s,\bullet)\|^2 ds - \frac{b}{\gamma+1}\int_0^t |||u(s,\bullet)|||^{\gamma+1} dw(s).$$

$$M = \left(\frac{(\gamma+1)(3-\gamma)B}{2b}\right)^{\frac{1}{\gamma-1}}, N = 2A - (\gamma-2)B^2, \mu = \frac{2aM^{2(\gamma-1)}}{(\gamma-3)N}.$$

Definition 1.

A stochastic process $u(t,x)$ is the solution of problem (1) if there exists a stopping time $\tau(\omega)$ such that $\forall T \geq 0$

$$u \in L_1(\Omega; C([0;T\wedge\tau(\omega)); L_2(R^1))) \cap L_2([0;T\wedge\tau(\omega)) \times \Omega; W_2^1(R^1) \cap L_{\gamma+1}(R^1))$$

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and $\forall g \in \mathbf{W}$, $\forall t \geq 0$ *the following equality holds with probability 1:*

$$\int u(t \wedge \tau, x)g(x)dx - \int u_0(x)g(x)dx =$$

$$= -a \int_0^{t\wedge\tau} \int u_x(s,x)g_x(x)dxds + b \int_0^{t\wedge\tau} \int |u(s,x)|^{\gamma-1}u(s,x)g(x)dxdw(s).$$

Definition 2. *A stochastic process* $u(t,x)$ *admits a decomposition, if it can be written down in such a form:* $u(t,x) = r(t)v\left(xr^m(t)\right)$. *Here,* $r(t)$ *is a nonnegative stochastic process,* $m \in R^1$, $v \in W_2^1(R^1) \cap L_{\gamma+1}(R^1)$.

The process $r(t)$ is called an amplitude, the function $v(y)$ is called the space form of the process $u(t,x)$. The process $x_f(t) = \min\{x > 0 : v(xr^m(t)) = 0\}$ is called a front point.

We introduce: $\tau_0(\omega) = \inf\{t \geq 0 : r(t) = 0\}$, $\tau_\infty(\omega) = \inf\{t \geq 0 : r(t) \notin R^1\}$, $\tau(\omega) = \min\{\tau_0(\omega), \tau_\infty(\omega)\}$.

Remark 1. In this work, we are based on the ideas of work [1].

Remark 2. In the deterministic theory of partial differential equations with power nonlinearities, the main space form of the solutions is a nonnegative even function which decreases on $y \geq 0$, and its derivative is equal to zero if $y = 0$ ([2, p.173-174]). In this work, we consider the same space forms.

Formulation of the problem. Suppose that the parameters of problem (1) satisfy the previous conditions. We will obtain conditions which guarantee the existence of a decomposition of the solution of problem (1).

2. Main results.

Theorem 1. *Suppose that the following conditions are satisfied.*

1) $\gamma \in (0;1), N \neq 0, B > 0$.

2) $r(t)$ *is the solution of the problem:*

$$dr(t) = Ar^{2\gamma-1}(t)dt + Br^\gamma(t)dw(t), t \in [0;\tau(\omega)), r(0) > 0. \tag{2}$$

3) $v(y)$ *is the solution of the problem:*

$$\mu\left(\frac{p}{q}\right)^2 v_{yy}(y) + (\gamma+1)\frac{p}{q}v^\gamma(y) - v(y) = 0, \tag{3}$$

$$y \in R^1, v(y) \geq 0, v(-y) = v(y), v_y(0) = 0, v(+\infty) = 0.$$

4)

$$u_0(x) = r(0)M\left(\frac{p}{q}\right)^{\frac{1}{\gamma-1}} v\left(xr^{\gamma-1}(0)M^{\gamma-1}\frac{p}{q}\right).$$

Then the solution of problem (1) admits such a form of decomposition:

$$u(t,x) = r(t)M\left(\frac{p}{q}\right)^{\frac{1}{\gamma-1}} v\left(xr^{\gamma-1}(t)M^{\gamma-1}\frac{p}{q}\right).$$

3. Subsidiary results.

We suppose here that the conditions of Theorem 1 are satisfied.

Remark 3. According to Theorem 6 [3, p.246] for all A and B there exists a stopping time $\tau(\omega)$ such that the solution of problem (2) exists and is unique for all $t \in [0; \tau(\omega))$. Moreover, according to [4, p.73], the solution of problem (2) is a nonnegative stochastic process.

Lemma 1. *Let $\delta > 0$. There exists a unique nonnegative generalized solution of the problem:*

$$V_{YY}(Y) + (\gamma+1)\delta|V(Y)|^{\gamma-1}V(Y) - V(Y) = 0, \tag{4}$$

$$Y \geq 0, V_Y(0) = 0, V(0) = V_0 \in (0; ((\gamma+1)\delta)^{\frac{1}{1-\gamma}}).$$

This solution has the following properties.

1. There exists a front point $Y_0 = \inf\{Y \geq 0 : V(Y) \leq 0\} < +\infty$ such that $V(Y) > 0$, if $Y \in [0; Y_0)$ and $V(Y) = 0$, if $Y \geq Y_0$.

2. $V_{YY}(0) < 0$.

3. There exists $\epsilon > 0$ such that $V_{YY} < 0$, if $Y \in (Y_0 - \epsilon; Y_0)$.

Proof. The even function which is positive if $Y \in [0; Y_0)$, equal to the zero if $Y \in [Y_0; +\infty)$ and is the solution of problem (4) for all $Y \in [0; Y_0)$ is named the generalized solution of problem (4)([2, p.173-174]). As $V - (\gamma+1)\delta|V|^{\gamma-1}V$ is a continuous function, then, according to [4, p.89], there exists the solution of problem (4). If $\tilde{V} = V^{-1}$, then

$$\tilde{V}_{YY} = 2\tilde{V}^{-1}\left(\tilde{V}_Y\right)^2 + \delta(\gamma+1)\tilde{V}^{2-\gamma} - \tilde{V}, Y \geq 0, \tilde{V}(0) = V_0^{-1}, \tilde{V}_Y(0) = 0.$$

The right part of this equation is a local Lipschitz function on $\tilde{V} \in [V_0^{-1}; +\infty)$, $\tilde{V}_Y \in (-\infty; +\infty)$. So, the solution of this problem is unique and the solution of problem (4) is unique too. This solution is a continuous function and $V(0) > 0$. This means that either $V(Y) > 0\, \forall Y \geq 0$, or there exists a point $Y_0 > 0$ such that $Y_0 = \inf\{Y \geq 0 : V(Y) \leq 0\} < +\infty$. In the last case, we define $V(Y) = 0$ if $Y \geq Y_0$. In the first case, we obtain a nonnegative classical solution of problem (4). In the second case, we obtain a nonnegative generalized solution of problem (4). The solution of problem (4) is not a constant because if $V(Y) \equiv C$, then $C - (\gamma+1)\delta C^\gamma = 0$ and either $C = 0$, or $C = ((\gamma+1)\delta)^{\frac{1}{1-\gamma}}$. But $0 < V(0) < ((\gamma+1)\delta)^{\frac{1}{1-\gamma}}$. This means that $V(Y)$ is not a constant. If we reduce the order of Eq. (4), we obtain

$$V_Y(Y) = -\sqrt{C_0 + V^2(Y) - 2\delta V^{\gamma+1}(Y)}, \tag{5}$$

where $C_0 = 2\delta V_0^{\gamma+1} - V_0^2 > 0$.

On the right part of this equation, we write a single "-" as, according to Remark 2, the space form must be a decreasing function on $Y \geq 0$. According to (5), $V_Y \to -\sqrt{C_0} < 0$, if $V \to 0$. This means that there exists a point $Y_0 \in (0; +\infty)$ such, that $V(Y_0) = 0$. Hence, there exists a generalized solution of problem (4) which is a nonnegative compactly supported function.

Let us determine the type of convexity for $V(Y)$ in neighborhoods of the points $Y = 0$ and $Y = Y_0$. Let us rewrite Eq. (4) in the next form: $V_{YY}(Y) = V(Y) - (\gamma+1)\delta V^\gamma(Y)$. Then $V_{YY}(0) = V_0^\gamma\left(V_0^{1-\gamma} - (\gamma+1)\delta\right) < 0$, because $V(0) < ((\gamma+1)\delta)^{\frac{1}{1-\gamma}}$ and $\gamma \in (0; 1)$. In a neighborhood of the point Y_0, the values of the function $V(Y)$ are near zero. Therefore, in this neighborhood, the inequality $V_{YY}(Y) = V(Y)^\gamma\left(V^{1-\gamma}(Y) - (\gamma+1)\delta\right) < 0$ is correct. Lemma 1 is proved.

Lemma 2. *If* $0 < V_0 < ((\gamma+1)\delta)^{\frac{1}{1-\gamma}}$*, then the solution of problem (4) is bounded above by*

$$\hat{V}(Y) = \begin{cases} V_0 \cos \hat{L} Y, & 0 \le Y \le \frac{\pi}{2\hat{L}}, \\ 0, & Y > \frac{\pi}{2\hat{L}}, \end{cases}$$

here $\hat{L} = \sqrt{\delta(\gamma+1)V_0^{\gamma-1} - 1}$.

Proof. The positive part of $\hat{V}(y)$ is the solution of the problem:

$$\hat{V}_Y(Y) = -\hat{L}\sqrt{V_0^2 - \hat{V}^2(Y)}, Y \ge 0, \hat{V}(0) = V_0. \tag{6}$$

We will prove that $\hat{L}^2(V_0^2 - V^2) - C_0 - V^2 + 2\delta V^{\gamma+1} \le 0$ if $V \in [0; V_0]$. If $V = 0$, then $\delta(\gamma+1)V_0^{\gamma+1} - V_0^2 \le 2\delta V_0^{\gamma+1} - V_0^2$, because $0 < \gamma < 1$. If $V = V_0$, then both parts of the inequality are equal to zero. The derivative of the left part of the inequality is equal to $2\delta(\gamma+1)V^{\gamma}\left(1 - \left(\frac{V}{V_0}\right)^{1-\gamma}\right) \ge 0$. Hence, the left part of the inequality is an increasing function, and its maximum is in $V = V_0$. But, in this case, the parts of the inequality are equal to each other. Therefore, the left part of the inequality is less or equal to its right part, and the right part of Eq. (6) is more or equal to the right part of Eq. (5). Then, according to Theorem 3 [5, p.39], the solution of problem (4) is bounded above by the solution of problem (6). Lemma 2 is proved.

Lemma 3. *If* $0 < V_0 < ((\gamma+1)\delta)^{\frac{1}{1-\gamma}}$*, then the solution of problem (4) is bounded below by*

$$\check{V}(Y) = \begin{cases} V_0 \cos \check{L} Y, & 0 \le Y \le \frac{\pi}{2\check{L}}, \\ 0, & Y > \frac{\pi}{2\check{L}}, \end{cases}$$

here $\check{L} = \sqrt{2\delta V_0^{\gamma-1} - 1}$.

Proof. The positive part of $\check{V}(y)$ is the solution of the problem:

$$\check{V}_Y(Y) = -\check{L}\sqrt{V_0^2 - \check{V}^2(Y)}, Y \ge 0, \check{V}(0) = V_0. \tag{7}$$

We will prove that $C_0 + V^2 - 2\delta V^{\gamma+1} - \check{L}^2(V_0^2 - V^2) \le 0$, if $V \in [0; V_0]$. If $V = V_0$ or $V = 0$, then both parts of the inequality are equal to zero.

$$\left(C_0 + V^2 - 2\delta V^{\gamma+1} - \check{L}^2(V_0^2 - V^2)\right)_V = 2\delta V^{\gamma}\left(2\left(\frac{V}{V_0}\right)^{1-\gamma} - (\gamma+1)\right) = 0.$$

The solutions of this equation are: $V_1 = 0$ and $V_2 = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} V_0 < V_0$. So, V_2 is a point of minimum. Then, $V = V_0$ is a point of maximum. Hence, the right part of Eq. (7) is less or equal to the right part of Eq. (5). Therefore, according to Theorem 3 [5, p.39], the solution of problem (4) is bounded below by the solution of problem (7). Lemma 3 is proved.

Corollary. *If* Y_0 *is a front point of the solution of problem (4), then*

$$\frac{\pi}{2\check{L}} \le Y_0 \le \frac{\pi}{2\hat{L}}.$$

Lemma 4. *Let $\delta > 0$. There exists a unique nonnegative generalized solution of the problem:*

$$V_{YY}(Y) - (\gamma+1)\delta|V(Y)|^{\gamma-1}V(Y) + V(Y) = 0, \tag{8}$$
$$Y \geq 0, V_Y(0) = 0, V(0) = V_0 > (2\delta)^{\frac{1}{1-\gamma}}.$$

This solution has the following properties.

1. There exists a front point $Y_0 = \inf\{Y \geq 0 : V(Y) \leq 0\} < +\infty$ such that $V(Y) > 0$, if $Y \in [0; Y_0)$ and $V(Y) = 0$, if $Y \geq Y_0$.

2. $V_{YY}(0) < 0$.

3. There exists $\epsilon > 0$ such that $V_{YY} > 0$, if $Y \in (Y_0 - \epsilon; Y_0)$.

Proof. The proof of the existence and uniqueness is the same as in Lemma 1. Let us determine the type of convexity for $V(Y)$ in neighborhoods of the points $Y = 0$ and $Y = Y_0$. Let us rewrite the equation (8) in next form: $V_{YY}(Y) = (\gamma+1)\delta V^{\gamma}(Y) - V(Y)$. Then, $V_{YY}(0) = V_0^{\gamma}\left((\gamma+1)\delta - V_0^{1-\gamma}\right) < 0$, because $V(0) > (2\delta)^{\frac{1}{1-\gamma}} > ((\gamma+1)\delta)^{\frac{1}{1-\gamma}}$. In a neighborhood of the point Y_0, the values of the function $V(Y)$ are near zero. Therefore, in this neighborhood, $V_{YY}(Y) = V^{\gamma}(Y)\left((\gamma+1)\delta - V^{1-\gamma}(Y)\right) > 0$. Lemma 4 is proved.

Lemma 5. *If $V_0 > (2\delta)^{\frac{1}{1-\gamma}}$, then the solution of problem (8) is bounded above by*

$$\overset{\frown}{\mathrm{V}}(Y) = \begin{cases} V_0 \cos \overset{\frown}{\mathrm{L}}\, Y, & 0 \leq Y \leq \frac{\pi}{2\overset{\frown}{\mathrm{L}}}, \\ 0, & Y > \frac{\pi}{2\overset{\frown}{\mathrm{L}}}, \end{cases}$$

where $\overset{\frown}{\mathrm{L}} = \sqrt{1 - 2\delta V_0^{\gamma-1}}$.

Proof. If we reduce the order of Eq. (8), we obtain

$$V_Y(Y) = -\sqrt{-C_0 - V^2(Y) + 2\delta V^{\gamma+1}(Y)}. \tag{9}$$

The positive part of $\overset{\frown}{\mathrm{V}}(Y)$ is the solution of the problem:

$$\overset{\frown}{\mathrm{V}}_Y(Y) = -\overset{\frown}{\mathrm{L}}\sqrt{V_0^2 - \overset{\frown}{\mathrm{V}}^2(Y)}, Y \geq 0, \overset{\frown}{\mathrm{V}}(0) = V_0. \tag{10}$$

Now the proof of this lemma is the same as that of Lemma 3.

Lemma 6. *If $V_0 > (2\delta)^{\frac{1}{1-\gamma}}$, then the solution of problem (8) is bounded below by*

$$\overset{\smile}{\mathrm{V}}(Y) = \begin{cases} V_0 - 0.25\overset{\smile}{\mathrm{L}}^2 Y^2, & 0 \leq Y \leq \frac{2\sqrt{V_0}}{\overset{\smile}{\mathrm{L}}}, \\ 0, & Y > \frac{2\sqrt{V_0}}{\overset{\smile}{\mathrm{L}}}, \end{cases}$$

where $\overset{\smile}{\mathrm{L}} = \sqrt{2V_0^{\gamma}\left(V_0^{1-\gamma} - (\gamma+1)\delta\right)}$.

Proof. The positive part of $\overset{\smile}{\mathrm{V}}(y)$ is the solution of the problem:

$$\overset{\smile}{\mathrm{V}}_Y(Y) = -\overset{\smile}{\mathrm{L}}\sqrt{V_0 - \overset{\smile}{\mathrm{V}}(Y)}, Y \geq 0, \overset{\smile}{\mathrm{V}}(0) = V_0. \tag{11}$$

We will prove such an inequality: $H(V) \geq 0$, if $V \in [0; V_0]$. Here $H(V) = \overset{\smile}{\mathrm{L}}^2(V_0 - V) + C_0 - 2\delta V^{\gamma+1} + V^2$. Notice that $H(0) = V_0^{\gamma+1}\left(V_0^{1-\gamma} - 2\gamma\delta\right) > 0$ and $H(V_0) = 0$. Let us demonstrate that $H(V)$ is a decreasing function on $[0; V_0]$. $H_V(V) = 2(\gamma + 1)\delta\left(V_0^{\gamma} - V^{\gamma}\right) + 2(V - V_0)$. Let us prove that $H_V(V) \leq 0$ on $V \in [0; V_0]$. $H_V(0) =$

$2V_0^\gamma\left((\gamma+1)\delta - V_0^{1-\gamma}\right) < 0$, $H_V(V_0) = 0$, $H_{VV}(V) = 2V^{\gamma-1}(V^{1-\gamma} - \gamma(\gamma+1)\delta)$. If $V \in (0; (\gamma(\gamma+1)\delta)^{\frac{1}{1-\gamma}})$, then $H_{VV}(V) \leq 0$. Hence, $H_V(V)$ is a decreasing function on $V \in (0; (\gamma(\gamma+1)\delta)^{\frac{1}{1-\gamma}})$ and $H_V((\gamma(\gamma+1)\delta)^{\frac{1}{1-\gamma}}) < 0$. If $V \in ((\gamma(\gamma+1)\delta)^{\frac{1}{1-\gamma}}; V_0]$, then $H_{VV}(V) > 0$. Hence, $H_V(V)$ is an increasing function on $V \in ((\gamma(\gamma+1)\delta)^{\frac{1}{1-\gamma}}; V_0]$ and $H(V) \leq H(V_0) = 0$. Ultimately, $H_V(V) \leq 0$ on $V \in [0; V_0]$. Therefore, $H(V)$ is a decreasing function on $V \in [0; V_0]$ and $H(V) \geq H(V_0) = 0$ for all $V \in [0; V_0)$. It means that the right part of Eq. (11) is less or equal to the right part of Eq. (9). Then, according to Theorem 3 [5, p.39], the solution of problem (8) is bounded below by the solution of problem (11). Lemma 6 is proved.

Corollary. *If Y_0 is a front point of the solution of problem (8), then*

$$\frac{2\sqrt{V_0}}{\smile{\mathrm{L}}} \leq Y_0 \leq \frac{\pi}{2\,\frown{\mathrm{L}}}.$$

Lemma 7. *For some $v_0 > 0$, there exists a unique generalized solution of the problem:*

$$\mu\left(\frac{p}{q}\right)^2 v_{yy}(y) + (\gamma+1)\frac{p}{q}v^\gamma(y) - v(y) = 0, \tag{12}$$

$$y \geq 0, v_y(0) = 0, v(0) = v_0.$$

This solution is a nonnegative function with bounded support.

Proof. Let $N < 0$, then $\mu > 0$. Denote: $\check{p} = \|\check{V}\|^2, \check{q} = |||\check{V}|||^{\gamma+1}$, $\overline{p} = \|V\|^2, \overline{q} = |||V|||^{\gamma+1}$, $\hat{p} = \|\hat{V}\|^2, \hat{q} = |||\hat{V}|||^{\gamma+1}$. According to Lemmas 2 and 3, such an inequality is correct: $\frac{\check{p}}{\hat{q}} \leq \frac{\overline{p}}{\overline{q}} \leq \frac{\hat{p}}{\check{q}}$. For $\check{V}(Y)$ and $\hat{V}(Y)$, we obtain $\check{p} = \frac{\pi V_0^2}{2\check{L}}$, $\hat{p} = \frac{\pi V_0^2}{2\hat{L}}$, $\check{q} = \frac{\sqrt{\pi}V_0^{\gamma+1}\Gamma(0.5\gamma+1)}{\check{L}\Gamma(0.5\gamma+1.5)}$, $\hat{q} = \frac{\sqrt{\pi}V_0^{\gamma+1}\Gamma(0.5\gamma+1)}{\hat{L}\Gamma(0.5\gamma+1.5)}$. Here, Γ is the gamma-function. Then the inequality

$$\frac{\sqrt{\pi}\hat{L}\Gamma(0.5\gamma+1.5)V_0^{1-\gamma}}{2\check{L}\Gamma(0.5\gamma+1)} \leq \frac{\overline{p}}{\overline{q}} \leq \frac{\sqrt{\pi}\check{L}\Gamma(0.5\gamma+1.5)V_0^{1-\gamma}}{2\hat{L}\Gamma(0.5\gamma+1)}$$

is correct. If V_0 runs from $(2\delta)^{\frac{1}{1-\gamma}}$ to $+\infty$, then the right part runs from 0 to $+\infty$ and the left part runs from Z to $+\infty$. Here, Z is the minimal value of the left part of this inequality. This means that, for some $\delta > 0$, there exists V_0 such that $\frac{\overline{p}}{\overline{q}} = \delta$. Then, Eq. (4) transforms into

$$V_{YY}(Y) + (\gamma+1)\frac{\overline{p}}{\overline{q}}V^\gamma(Y) - V(Y) = 0.$$

Let us transform the variables: $Y = \frac{\overline{q}}{\sqrt{\mu p}}y$, $v(y) = V(\frac{\overline{q}}{\sqrt{\mu p}}y)$. We consider that

$$\overline{p} = \frac{\overline{q}}{\sqrt{\mu p}}p, \overline{q} = \frac{\overline{q}}{\sqrt{\mu p}}q, \frac{\overline{p}}{\overline{q}} = \frac{p}{q}.$$

Then, the function $v(y)$ is the solution of Eq. (12). As $V(Y) \geq 0$, $V_Y(0) = 0$, $V(0) = V_0$, then $v(y) \geq 0$, $v_y(0) = 0$, $v(0) = V_0$. Since $V(Y)$ has bounded support, $v(y)$ has bounded support and, for its front point, the inequality

$$\frac{\pi\sqrt{\mu p}}{2\overline{q}\check{L}} \leq y_0 \leq \frac{\pi\sqrt{\mu p}}{2\overline{q}\hat{L}}$$

is correct. Let $N > 0$, then $\mu < 0$. Using Lemmas 5 and 6, we obtain the similar results. Lemma 7 is proved.

Corollary. *The even function, which is the same as the solution of problem (12) on $y \geq 0$, is the solution of problem (3).*

Remark 4. As the functions $r(t)$, $v(y)$, and $V(Y)$ are nonnegative, we will pass to the notations of their moduli.

4. Proof of Theorem 1.

The functional $f(t;u)$ is defined for $(\mathbf{F}_t)_{t\geq 0}$-adapted $u \in L_2([0;T)\times\Omega;\mathbf{W})$. We will prove that $f(t\wedge\tau;u)$ has the Gateaux differential on the subspace $\mathbf{W}$ in mean square, and this differential has the form

$$Df(t\wedge\tau;u) = \int u(t\wedge\tau,x)g(x)dx - \int u(0,x)g(x)dx+$$

$$+a\int_0^{t\wedge\tau}\int u_x(s,x)g_x(x)dxds - b\int_0^{t\wedge\tau}\int |u(s,x)|^{\gamma-1}u(s,x)g(x)dxdw(s). \tag{13}$$

Here, $g \in \mathbf{W}$.

We will calculate the Gateaux differential on the subspace according to [6, p.118]. It is known that the Lebesgue integrals are Gateaux differentiable. Let us prove that the stochastic integral is also Gateaux differentiable on the subspace $\mathbf{W}$ in mean square, and its differential is equal to $(\gamma+1)\int_0^{t\wedge\tau}\int|u(s,x)|^{\gamma-1}u(s,x)g(x)dxdw(s)$. We must prove that

$$\lim_{h\to 0} M\left(\int_0^{t\wedge\tau}\int\frac{|u+hg|^{\gamma+1}-|u|^{\gamma+1}}{h}dxdw(s)-(\gamma+1)\int_0^{t\wedge\tau}\int|u|^{\gamma-1}ugdxdw(s)\right)^2 = 0. \tag{14}$$

Using the mean value theorem, we obtain

$$M\left(\int_0^{t\wedge\tau}\int\left(\frac{|u+gh|^{\gamma+1}-|u|^{\gamma+1}}{h}-(\gamma+1)|u|^{\gamma-1}ug\right)dxdw(s)\right)^2 =$$

$$=(\gamma+1)^2M\int_0^{t\wedge\tau}\left(\int\int_0^1\left(|u+\theta hg|^{\gamma+1}(u+\theta hg)-|u|^{\gamma-1}u\right)d\theta gdx\right)^2ds \leq$$

$$\leq(\gamma+1)^2||g||^2M\int_0^{t\wedge\tau}\int\int_0^1\left(|u+\theta hg|^{\gamma-1}(u+\theta hg)-|u|^{\gamma-1}u\right)^2d\theta dxds. \tag{15}$$

If $|h|\leq 1$, then $\left(|u+\theta hg|^{\gamma-1}(u+\theta hg)-|u|^{\gamma-1}u\right)^2 \leq 2(|u|+|g|)^{2\gamma}+2|u|^{2\gamma}$. According to the Lebesgue theorem [7, p.284], we obtain (14).

Thus, a stationary point for $f(t\wedge\tau;u)$ is a generalized solution of problem (1). Let us prove that there exists a stationary point for $f(t\wedge\tau;u)$ and this stationary point admits a decomposition. By substituting $u(t,x)=r(t)\phi(p,q)v(y)$ into $f(t\wedge\tau;u)$, we obtain

$$f(t\wedge\tau;u)=\frac{1}{2}p\phi^{3-\gamma}(p,q)\left(r^{3-\gamma}(t\wedge\tau)-r^{3-\gamma}(0)\right)+$$

$$+\frac{az}{2}\phi^{\gamma+1}(p,q)\int_0^{t\wedge\tau}r^{\gamma+1}(s)ds-\frac{bq}{\gamma+1}\phi^2(p,q)\int_0^{t\wedge\tau}r^2(s)dw(s).$$

Here, $r(t)$ is the solution of problem (2) with some real A and positive r_0 and B, ϕ is some nonnegative differentiable function, $v(y)$ is some function in $\mathbf{W}$, and $y = xr^{\gamma-1}(t)\phi^{\gamma-1}(p,q)$. Let $\phi(p,q)$ satisfy the equality

$$\frac{1}{2}p\phi^{3-\gamma}(p,q) = \frac{bq}{(\gamma+1)(3-\gamma)B}\phi^2(p,q).$$

Then $\phi(p,q) = M\left(\frac{p}{q}\right)^{\frac{1}{\gamma-1}}$ and

$$f(t\wedge\tau;u) = \frac{p}{2}M^{3-\gamma}\left(\frac{p}{q}\right)^{\frac{3-\gamma}{\gamma-1}}\left(r^{3-\gamma}(t\wedge\tau) - r^{3-\gamma}(0) - (3-\gamma)B\int\limits_0^{t\wedge\tau} r^2(s)dw(s)\right) +$$
$$+\frac{az}{2}M^{\gamma+1}\left(\frac{p}{q}\right)^{\frac{\gamma+1}{\gamma-1}}\int\limits_0^{t\wedge\tau} r^{\gamma+1}(s)ds.$$

According to the Itô formula,

$$r^{3-\gamma}(t\wedge\tau) - r^{3-\gamma}(0) = \frac{(3-\gamma)N}{2}\int\limits_0^{t\wedge\tau} r^{\gamma+1}(s)ds + (3-\gamma)B\int\limits_0^{t\wedge\tau} r^2(s)dw(s)$$

and

$$f(t\wedge\tau;u) = \frac{M^{3-\gamma}(3-\gamma)N}{4}\int\limits_0^{t\wedge\tau} r^{\gamma+1}(s)ds\left[p^{\frac{2}{\gamma-1}}q^{\frac{3-\gamma}{1-\gamma}} - \mu z p^{\frac{\gamma+1}{\gamma-1}}q^{\frac{\gamma+1}{1-\gamma}}\right].$$

As

$$\|u(s,\bullet)\|^2 = r^{3-\gamma}(s)M^{3-\gamma}p^{\frac{2}{\gamma-1}}q^{\frac{\gamma-3}{\gamma-1}}, \|u_x(s,\bullet)\|^2 = r^{\gamma+1}(s)M^{\gamma+1}p^{\frac{\gamma+1}{\gamma-1}}q^{\frac{\gamma+1}{1-\gamma}}z,$$

$$|||u(s,\bullet)|||^{\gamma+1} = r^2(s)M^2p^{\frac{2}{\gamma-1}}q^{\frac{\gamma-3}{\gamma-1}}, \tag{16}$$

then

$$f(t\wedge\tau;u) = \frac{(3-\gamma)N}{4}M^{2(1-\gamma)}\int\limits_0^{t\wedge\tau}\left[\frac{|||u(s,\bullet)|||^{2(\gamma+1)}}{\|u(s,\bullet)\|^2} - \mu\|u_x(s,\bullet)\|^2\right]ds.$$

Let us calculate the Gateaux differential on the subspace $W_2^1(R^1)\cap L_{\gamma+1}(R^1)$:

$$Df(t\wedge\tau;u) = \frac{(3-\gamma)N}{2}M^{2(1-\gamma)}\times$$
$$\times\int\limits_0^{t\wedge\tau}\int\left[\frac{|||u(s,\bullet)|||^{\gamma+1}}{\|u(s,\bullet)\|^2}(\gamma+1)u^\gamma(s,x) - \frac{|||u(s,\bullet)|||^{2(\gamma+1)}}{\|u(s,\bullet)\|^4}u(s,x) + \mu u_{xx}\right]g(x)dxds.$$

Using (16), we obtain

$$Df(t\wedge\tau;u) = \frac{(3-\gamma)N}{4}\left(\frac{p}{q}\right)^{\frac{1}{\gamma-1}}\int\limits_0^{t\wedge\tau} r^{2\gamma-1}(s)ds\times$$
$$\times\int\left[(\gamma+1)\frac{p}{q}v^\gamma(y) - v(y) + \mu\left(\frac{p}{q}\right)^2 v_{yy}(y)\right]g(x)dx.$$

If $v(y)$ is the solution of problem (3), then $Df(t\wedge\tau;u)=0$ and the function

$$u(t,x)=r(t)M\left(\frac{p}{q}\right)^{\frac{1}{\gamma-1}}v\left(xr^{\gamma-1}(t)M^{\gamma-1}\frac{p}{q}\right)$$

is a stationary point for $f(t\wedge\tau;u)$. Thus, $u(t,x)$ is a generalized solution of problem (1) which admits a decomposition. Theorem 1 is proved.

5. Dynamics of the solution.

If

$$u_0(x)=r(0)M\left(\frac{p}{q}\right)^{\frac{1}{\gamma-1}}v\left(xr^{\gamma-1}(0)M^{\gamma-1}\frac{p}{q}\right),$$

then

$$u(t,x)=r(t)M\left(\frac{p}{q}\right)^{\frac{1}{\gamma-1}}v\left(xr^{\gamma-1}(t)M^{\gamma-1}\frac{p}{q}\right).$$

Consider the dynamics for $r(t)$.

Theorem 2. *Let $r(t)$ be the solution of problem (2).*

If $2A<B^2$, then $\mathbf{P}\left\{\lim\limits_{t\to\tau(\omega)} r(t)=0\right\}=1.$

If $2A>B^2$, then $\mathbf{P}\left\{\lim\limits_{t\to\tau(\omega)} r(t)=+\infty\right\}=1.$

Proof. Denote $\lambda=1-2AB^{-2}$. Let $P(r,\epsilon,\beta,\alpha)$ be the solution of the problem

$$0.5B^2r^{2\gamma}P_{rr}+Ar^{2\gamma-1}P_r=0,$$
$$0<\epsilon<r<\beta, P(\epsilon,\epsilon,\beta,\alpha)=\alpha, P(\beta,\epsilon,\beta,\alpha)=1-\alpha, 0\le\alpha\le 1.$$

If $\alpha=1$, then P is the probability of such an event: the process $r(t)$ leaves the interval $(\epsilon;\beta)$ over the left endpoint. If $\alpha=0$, then P is the probability of such an event: the process $r(t)$ leaves the interval $(\epsilon;\beta)$ over the right endpoint.

Let $2A<B^2$, then $P(r,\epsilon,\beta,1)=\frac{r^\lambda-\beta^\lambda}{\epsilon^\lambda-\beta^\lambda}$ and $\lim\limits_{\beta\to+\infty}P(r,\epsilon,\beta,1)=1, \forall r>\epsilon\ge 0$ because $\lambda>0$. This means that $r(t)$ reaches the zero level with probability 1.

Let $2A>B^2$, then $P(r,\epsilon,\beta,0)=\frac{r^\lambda-\epsilon^\lambda}{\beta^\lambda-\epsilon^\lambda}$ and $\lim\limits_{\epsilon\to 0}P(r,\epsilon,\beta,0)=1, \forall r<\beta$. This means that $r(t)$ tends to $+\infty$ with probability 1 if $t\to\tau(\omega)$. Theorem 2 is proved.

Remark 5. If $A=\frac{\gamma}{2}B^2$, then the process

$$r(t)=\left(r^{1-\gamma}(0)+(1-\gamma)Bw(t)\right)^{\frac{1}{1-\gamma}}$$

is the solution of problem (2).

Now we can describe the dynamics of $u(t,x)$.

Let $2A<B^2$. Then, according to Theorem 2, the amplitude and the front point tend to zero. The configuration of the space form depends on the sign of the parameter N. If $2A<(\gamma-2)B^2$, then $\mu>0$ and, for $V_0<\left((\gamma+1)\frac{p}{q}\right)^{\frac{1}{1-\gamma}}$, we obtain the first space form for $v(y)$, according to Lemma 1. If $(\gamma-2)B^2<2A<B^2$, then $\mu<0$ and, for $V_0<\left(2\frac{p}{q}\right)^{\frac{1}{1-\gamma}}$, we obtain the second space form for $v(y)$, according to Lemma 4.

Let $2A>B^2$. Then, according to Theorem 2, the amplitude and the front point tend to infinity. As $\gamma\in(0;1)$, then $\gamma-2<1$, $\mu<0$, and the space form has the second type, according to Lemma 4.

6. Conclusion.

Let us compare the dynamics of the solution of the stochastic problem (1) and the solution of a similar deterministic problem. According to [8, p.221], if the deterministic equation has the source term and $\gamma \in (0;1)$, then its solution tends to infinity on the whole space. The sign of the last term of Eq. (1) is variable, and it is not possible to classify this term neither as the source nor as the absorber. If the drift coefficient of problem (2) is large comparatively to the diffusion coefficient, then the process $u(t,x)$ develops with intensification. If the drift coefficient of problem (2) is small comparatively to the diffusion coefficient, then the process $u(t,x)$ develops in the peaking regime.

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