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LOCAL LIMIT THEOREM FOR TRIANGULAR ARRAY OF RANDOM VARIABLES

For a triangular array of random variables $\{X_{k,n},\ k = 1, \dots, c_n;\ n \in \mathbb{N}\}$ such that, for every n, the variables $X_{1,n}, \dots, X_{c_n,n}$ are independent and identically distributed, the local limit theorem for the variables $S_n = X_{1,n} + \dots + X_{c_n,n}$ is established.

1. Introduction

Consider the triangular array of random variables $\{X_{k,n},\ k = 1, \dots, c_n;\ n \in \mathbb{N}\}$ such that, for every n, the variables $X_{1,n}, \dots, X_{c_n,n}$ are independent and identically distributed, and consider the variables

$$S_n = X_{1,n} + \dots + X_{c_n,n}, \quad n \in \mathbb{N}. \tag{1}$$

We suppose the following condition to hold true:

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \sup_{1\le k\le c_n} P\{|X_{k,n}| \ge \varepsilon\} = 0 \tag{2}$$

Condition (2) is a standard condition of the asymptotic negligibility for a single summand in (1). Recall that a measure M on $\mathbb{R}$ is called canonical if $M(I) < \infty$ for every finite interval I and

$$M^+(x) = \int_x^{+\infty} \frac{1}{y^2}\, M(dy) < +\infty, \quad M^-(x) = \int_{-\infty}^{-x} \frac{1}{y^2}\, M(dy) < +\infty, \quad x > 0.$$

Recall also that, by definition, a sequence of canonical measures $\{M_n\}$ converges to a canonical measure properly (notation $M_n \to M$) if $M_n(I) \to M(I)$ for every finite interval I and $M_n^+(x) \to M^+(x), M_n^-(x) \to M^-(x)$ for every $x > 0$. Under (2), the following necessary and sufficient condition for the sequence $\{S_n\}$ to converge in distribution is well known ([1], Chapter XVII, §2).

Theorem 1. *Let $M_n(dx) = c_n x^2 dF_n(x), b_n = \int_{\mathbb{R}} \sin x\, F_n(dx)$, where F_n is the distribution function for $X_{n,1}$. If the array $\{X_{k,n}\}$ satisfies condition (2), then the sums S_n converge weakly to some r.v. S iff*

$$M_n \to M \text{ and } b_n \to b, \quad n \to +\infty \tag{3}$$

for some $b \in \mathbb{R}$ and a canonical measure M. Under (3), the characteristic function Φ of the variable S is given by the formula

$$\Phi(z) = \exp\left\{ ibz - \frac{1}{2} z^2 M(\{0\}) + \int_{\mathbb{R}\setminus\{0\}} \frac{e^{izx} - 1 - izx}{x^2} M(dx) \right\}, \quad z \in \mathbb{R}. \tag{4}$$

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In this paper, we study the following question. Let conditions (2) and (3) hold and, thus, let the limit theorem for the distributions of S_n hold true, that is, $S_n \Rightarrow S$. What can be said about the *local limit theorem* in such a situation? In other words, what conditions should be imposed on the distributions of $\{X_{k,n}\}$ in order to provide the convergence (in some sense) of the distribution densities for S_n?

In an important particular case, the complete answer to this question is given by the well-known *Gnedenko's theorem* ([2], Chapter IV, §3). Suppose that, in our notation, $c_n = n$ and the array $\{X_{k,n}\}$ has the form

$$X_{k,n} = \frac{\xi_{k,n} - a_n}{d_n}, \tag{5}$$

where all the random variables $\{\xi_{k,n}\}$ are identically distributed (in fact, in such a case, $S_n \overset{d}{=} \frac{\sum_{k=1}^{n}(\xi_k - a_n)}{d_n}$ with i.i.d. $\{\xi_k\}$). The variable S has a stable distribution, and the Gnedenko's theorem states that the distribution densities for S_n converge uniformly to that for S iff S_{n_0} possesses a bounded distribution density for some $n_0 \in \mathbb{N}$.

Our purpose is to establish the uniform convergence of the distribution densities for S_n in the general case. The following feature should be taken into consideration. Every infinitely divisible distribution can be obtained as a limiting one in Theorem 1, and there exist infinitely divisible distributions that are singular w.r.t. Lebesgue measure. Thus, conditions imposed on the array $\{X_{k,n}\}$ should at least provide the absolute continuity of the probability law defined by its characteristic function (4). The following sufficient condition is well known (the *Kallenberg condition*, see [3]).

Proposition 1. *Let*

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon^2 \ln \varepsilon^{-1}} \int_{-\varepsilon}^{\varepsilon} M\{dx\} = +\infty.$$

Then the distribution with the characteristic function (4) possesses a density from the class C^∞.

Note that the condition given above is a *sufficient* one while a *necessary and sufficient* condition has not been found yet. Therefore, at this moment, one can hardly expect to provide a local limit theorem for $\{S_n\}$ in a necessary and sufficient form, like in that the Gnedenko's theorem. In this paper, we give one sufficient condition for a local limit theorem to hold true.

2. The main result

We consider a triangular array of the type

$$X_{k,n} = \frac{\xi_{k,n}}{d_n}, \quad k = 1, \dots, n, \tag{5'}$$

where c_n is not necessarily equal to n and $\{\xi_{k,n}\}$ are not supposed to be identically distributed through the whole array. We assume that, for every n, $\xi_{1,n}, \dots, \xi_{c_n,n}$ are i.i.d. variables and that $c_n \to \infty, n \to \infty$. For such an array we suppose conditions (2) and (3) to hold true and thus $S_n \Rightarrow S$.

By G_n and F_n, we denote the distribution functions for $\xi_{1,n}$ and $X_{1,n}$, respectively (obviously, $G_n(x) = F(d_n x), x \in \mathbb{R}$). Denote, by φ_n, the characteristic function for $\xi_{1,n}$.

Next, consider an independent copy $\{\xi'_{k,n}\}$ of the array $\{\xi_{k,n}\}$ and put $\hat{\xi}_{k,n} = \xi_{k,n} - \xi'_{k,n}, \hat{X}_{k,n} = X_{k,n} - X'_{k,n}$. Denote, by $\hat{G}_n$ and $\hat{F}_n$, the distribution functions for $\hat{\xi}_{1,n}$ and $\hat{X}_{1,n}$, respectively, and put $\hat{M}_n(dx) = c_n x^2 d\hat{F}_n(x)$. We claim the following conditions to hold.

A. There exists $m \in \mathbb{N}$ such that $c_n \geq m, n \in \mathbb{N}$, and the sums $\xi_{1,n} + \ldots \xi_{m,n}$ possess densities g_n.

B. There exist $B > \frac{2}{1-\cos 1}, n_0 \in \mathbb{N}, a > 0$ and $\varepsilon_0 > 0$ such that

$$\left(\varepsilon^2 \ln \frac{1}{\varepsilon}\right)^{-1} \int_{-\varepsilon}^{\varepsilon} \hat{M}_n\{dx\} \geq B, \quad \frac{a}{d_n} < \varepsilon < \varepsilon_0, \quad n \geq n_0.$$

C. $\forall \delta > 0 \quad N(\delta) \equiv \sup_{n \in \mathbb{N}, |z| > \delta} |\varphi_n(z)| < 1.$

D. $\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} g_n^2(x)\, dx < +\infty.$

E. $d_n \to +\infty, \dfrac{\ln d_n}{c_n} \to 0, \quad n \to \infty.$

Theorem 2. *Let conditions (2),(3) and* **A** – **E** *hold true. Then the distributions of* S_n, S *possess densities* f_n, f *and*

$$\sup_x |f_n(x) - f(x)| \to 0, \quad n \to \infty.$$

Remarks. 1. Condition **B** is an analogue of the Kallenberg condition. Note that since $\hat{M}_n([-1, 1])$ is finite for every n, this condition can not hold with a equal 0.

2. Due to a simple estimate $\int_{\mathbb{R}} g_n^2(x)\, dx \leq \sup_{x \in \mathbb{R}} g_n(x)$, condition **D** is provided by the condition

D′. $\sup_{n \in \mathbb{N}, x \in \mathbb{R}} g_n(x) < +\infty.$

If the distribution of $\xi_{1,n}$ does not depend on n, then conditions **A** and **D′** coincide with the condition of the Gnedenko's theorem that S_m possesses a bounded density for some m. Thus, conditions **A** and **D** are, in fact, an appropriate n-dependent version of the condition of the Gnedenko's theorem.

3. Condition **C** is an n-dependent version of the so-called *Cramer's (C)-condition* ([2], Chapter III, §3). Since its straightforward verification is not a trivial problem, we provide a sufficient condition for it.

Proposition 2. *Suppose that there exist a non-empty interval* $I = (u, v)$ *and a constant* $C > 0$ *such that*

$$g_n(x) \geq C, \quad x \in (u, v), n \in \mathbb{N}. \tag{6}$$

Then **C** *holds true.*

3. Proofs

We start the proof of Theorem 2 with the following estimate for the characteristic function of S. Denote $\Psi(z) = ibz - \frac{1}{2} z^2 M(\{0\}) + \int_{\mathbb{R} \setminus \{0\}} \frac{e^{izx} - 1 - izx}{x^2} M(dx), z \in \mathbb{R}$.

Lemma 1. *Under conditions (2),(3) and* **B**,**E**,

$$\lim \inf_{|z| \to \infty} \frac{\mathrm{Re}\Psi(z)}{\ln |z|} \leq -B \frac{(1 - \cos 1)}{2}. \tag{7}$$

Proof. Denote by $\hat{\Phi}$ the characteristic function of the variable $\hat{S} = S - S'$, where S' is an independent copy of S. It is easy to see that the array $\{\hat{X}_{k,n}\}$ satisfies condition (2) and, under condition (3), $\hat{S}_n = \hat{X}_{1,n} + \cdots + \hat{X}_{c_n,n} \Rightarrow S - S' = \hat{S}$. Thus, condition (3) also holds true for $\hat{b}_n = 0$ and $\hat{M}_n$, i.e. $\hat{M}_n \to \hat{M}$ for some canonical measure $\hat{M}$ and $\hat{\Phi}$ possesses the representation

$$\hat{\Phi}(z) = e^{\hat{\Psi}(z)}, \quad \hat{\Psi}(z) = -\frac{1}{2} z^2 \hat{M}(\{0\}) + \int_{\mathbb{R} \setminus \{0\}} \frac{e^{izx} - 1 - izx}{x^2} \hat{M}(dx), \quad z \in \mathbb{R}.$$

Since

$$Ee^{iz\hat{S}} = Ee^{izS}Ee^{-izS'} = |Ee^{izS}|^2, \tag{8}$$

we have that $\mathrm{Re}\Psi(z) = \frac{1}{2}\hat{\Psi}(z)$ and, in order to prove (7), it is sufficient to prove that

$$\lim \inf_{\ln|z|\to\infty} \frac{\hat{\Psi}(z)}{|z|} \le -B(1-\cos 1). \tag{9}$$

We have

$$\hat{\Psi}(z) = \mathrm{Re}\hat{\Psi}(z) = -\frac{1}{2}z^2\hat{M}(\{0\}) + \int_{\mathbb{R}\setminus\{0\}} \frac{\cos zx - 1}{x^2}\hat{M}(dx), \quad z \in \mathbb{R},$$

and therefore (9) holds true immediately as soon as $\hat{M}(\{0\}) > 0$. Thus, we exclude this case from consideration and suppose further that $\hat{M}(\{0\}) = 0$.

Conditions **B** and **E** together with the proper convergence $\hat{M}_n \to \hat{M}$ provide that

$$\left(\varepsilon^2 \ln\frac{1}{\varepsilon}\right)^{-1} \int_{-\varepsilon}^{\varepsilon} \hat{M}\{dx\} \ge B, \quad 0 < \varepsilon < \varepsilon_0. \tag{10}$$

Now (9) follows from the given below estimate, that, in fact, is the main point in the proof of the sufficiency of the Kallenberg condition. Elementary calculations show that $\frac{\cos y - 1}{y^2} \le \cos 1 - 1, |y| \le 1$. Then, for $|z| > \varepsilon_0^{-1}$,

$$\frac{\hat{\Psi}(z)}{\ln|z|} = \frac{1}{\ln|z|}\int_{-\infty}^{+\infty} \frac{\cos zx - 1}{x^2}\,\hat{M}\{dx\} \le \frac{1}{\ln|z|}\int_{-|z|^{-1}}^{|z|^{-1}} \frac{\cos zx - 1}{x^2}\,\hat{M}\{dx\} \le$$

$$\le -(1-\cos 1)\cdot\frac{z^2}{\ln|z|}\int_{-|z|^{-1}}^{|z|^{-1}} \hat{M}\{dx\} \le -(1-\cos 1)\cdot\frac{z^2}{\ln|z|}\cdot B\cdot(z^{-2}\ln|z|) = -B(1-\cos 1). \tag{11}$$

Here, in the last inequality, we have used (10) with $\varepsilon = |z|^{-1}$. Lemma 1 is proved.

Corollary. The characteristic function $\Phi = e^{\Psi}$ belongs to $L_1(\mathbb{R}, dx)$, and therefore the law of S possesses a continuous density $f(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-izx}\Phi(z)\,dz, x \in \mathbb{R}$.

Proof of Theorem 2. Let us introduce some notations. Without loss of generality, we assume that $m = 1$. Denote, by $\varphi_n, \Phi_n, \hat{\vartheta}_n, \hat{\Phi}_n$, the characteristic functions for the variables $\xi_{1,n}, S_n, \hat{X}_{1,n}, \hat{S}_n$, respectively. Condition **D** implies that, for $c_n > 1$, the law of S_n possesses a continuous density $f_n(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-izx}\Phi_n(z)\,dz, x \in \mathbb{R}$ and thus

$$\begin{aligned}
\Delta_n \equiv 2\pi \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| &\le \int_{\mathbb{R}} |\Phi_n(z) - \Phi(z)|\,dz \\
&\le \int_{-A}^{A} |\,\Phi_n(z) - \Phi(z)\,|\,dz + \int_{|z|\ge A} |\,\Phi(z)\,|\,dz + \int_{|z|\ge \delta d_n} |\,\Phi_n(z)\,|\,dz \\
&\quad + \int_{A\le|z|\le\delta d_n} |\,\Phi_n(z)\,|\,dz \\
&= I_1(n,A) + I_2(A) + I_3(n,\delta) + I_4(n,\delta,A),
\end{aligned}$$

where $A, \delta > 0$ are an arbitrary constants. Let us investigate integrals $I_1 - I_4$ separately.

I. Since $S_n \Rightarrow S,\ n \to \infty,\ \Phi_n(\cdot) \to \Phi(\cdot)$ uniformly on every bounded interval. Thus, for every fixed $A > 0$, $I_1(n,A) \to 0, n \to +\infty$.

II. Since $\Phi \in L_1(\mathbb{R}, dx)$ due to Lemma 1, $I_2(A) \to 0, A \to +\infty$.

III. We have $|\Phi_n(z)| = |\varphi_n(d_n^{-1}z)|^{c_n}$, and therefore

$$I_3(n,\delta) = \int_{|z|\ge\delta d_n} |\,\Phi_n(z)\,|\,dz = \int_{|z|\ge\delta d_n} |\varphi_n(d_n^{-1}z)|^{c_n}\,dz = d_n\int_{|v|\ge\delta} |\varphi_n(v)|^{c_n}\,dv \le$$

$$\le d_n N(\delta)^{c_n-2}\int_{\mathbb{R}} |\varphi_n(v)|^2\,dv = (2\pi)^{-1} d_n N(\delta)^{c_n-2}\int_{\mathbb{R}} |g_n(x)|^2\,dx.$$

Thus, conditions **C**, **D** and **E** provide that, for every fixed $\delta>0$, $I_3(n,\delta)\to 0, n\to+\infty$.

IV. We have $|\Phi_n(z)|^2 = |\hat\Phi_n(z)| = |\hat\vartheta_n(z)|^{c_n}$. Denote $\hat\psi_n(z) = c_n(\hat\vartheta_n(z)-1)$; recall that the distribution of $\hat X_{1,n}$ is symmetric, and therefore the function $\hat\vartheta_n$ is real-valued. Then, using the elementary inequality $\ln(1+x)\le x, x>-1$, we obtain that

$$|\Phi_n(z)|^2 \le e^{\hat\psi_n(z)} = \exp\left[c_n\int_{\mathbb{R}} (\cos zx - 1)d\hat F_n(x)\right].$$

Now we use condition **B** with $\varepsilon = |z|^{-1}$ and repeat estimate (11):

$$\begin{aligned}|\,\Phi_n(z)\,|^2 &\le \exp\left[-(1-\cos 1)\,z^2\int_{-|z|^{-1}}^{|z|^{-1}} c_n y^2\,d\hat F_n(x)\right]\\ &\le \exp\left[-(1-\cos 1)\,B\,\ln|z|\right] = z^{-(1-\cos 1)\,B}\end{aligned}$$

for $n\ge n_0, \frac{1}{\varepsilon_0}\le |z|\le \frac{d_n}{a}$. Thus, if $\delta<\frac{1}{a}$ and $A>\frac{1}{\varepsilon_0}$, then

$$I_4(n,\delta,A)\le \int_{|z|\ge A} z^{-\frac{(1-\cos 1)\,B}{2}}\,dz \to 0,\quad A\to+\infty,\quad n\ge n_0.$$

Now, let $\varepsilon>0$ be fixed. Take $\delta=\frac{1}{2a}$ and choose A such that $I_4(n,\delta,A)<\frac{\varepsilon}{4}$ for every $n\ge n_0$ and $I_2(A)<\frac{\varepsilon}{4}$. Then choose $N\ge n_0$ such that, for every $n\ge N$, $I_1(n,A)<\frac{\varepsilon}{4}$ and $I_3(n,\delta)<\frac{\varepsilon}{4}$. Under such a choice,

$$\Delta_n < \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4} = \varepsilon,\quad n\ge N.$$

Theorem 2 is proved.

Proof of Proposition 2. We have

$$\begin{aligned}\varphi_n(z) &= \int_{\mathbb{R}} e^{izx} g_n(x)\,dx\\ &= \int_u^v Ce^{izx}\,dx + \left(\int_u^v \left(e^{izx}g_n(x)-C\right)\,dx + \int_{\mathbb{R}\setminus[u,v]} e^{izx}g_n(x)\,dx\right)\\ &= C(v-u)\varrho(z) + \Upsilon_n(z),\end{aligned}$$

where ϱ is the characteristic function of the uniform distribution on $[u,v]$. Since $g_n(x)\ge C$ on $[u,v]$, we have that

$$|\Upsilon_n(z)| \le \int_u^v \left(g_n(x)-C\right)\,dx + \int_{\mathbb{R}\setminus[u,v]} g_n(x)\,dx = 1 - C(v-u).$$

On the other hand, the uniform distribution satisfies Cramer's (C)-condition:

$$\sup_{|z|>\delta}|\varrho(z)|<1\quad\text{for any } \delta>0.$$

This implies that

$$N(\delta) = \sup_{n\in\mathbb{N},|z|>\delta}|\varphi_n(z)| \le 1 - C(v-u) + C(v-u)\sup_{|z|>\delta}|\varrho(z)| < 1,\quad \delta>0.$$

Proposition a is proved.

4. An example

In this section, we give one example for the statement of Theorem 2. Let M be the canonical measure of the form $M\{dx\} = \dfrac{x}{e^x - e^{-x}}dx$. If the characteristic function Φ of a random variable S is given by formula (4) with this measure M and $b = 0$, then S has a distribution density f of the form $f(x) = \dfrac{1}{\pi \mathrm{ch} x}$ (the law of S is called *the hyperbolic cosine distribution*).

Denote $\Pi_t\{dx\} = \dfrac{1}{x^2}\mathbf{I}_{|x|>t}M\{dx\}$. Define the sequence $\{t_n\}$ in such a way that $\Pi_{t_n}\{\mathbb{R}\} = n$. Since $\lim\limits_{x\to 0}\dfrac{e^x - e^{-x}}{2x} = 1$, one can show that $t_n \sim 1/n,\ n \to \infty$.

Consider the triangular array $\{X_{k,n}, 1 \le k \le n\}$ with the distribution functions of $X_{1,n}$ being equal to $F_n(x) = \frac{1}{n}\Pi_{t_n}((-\infty, x]), x \in \mathbb{R}, n \ge 1$. It is clear that $X_{1,n}, n \ge 1$ possess the distribution densities

$$p_n(x) = \frac{1}{n}\frac{1}{x(e^x - e^{-x})}\mathbf{I}_{|x|>t_n}, \quad x \in \mathbb{R}, \quad n \ge 1.$$

One can easily verify that conditions (2),(3) are satisfied and therefore $S_n \Rightarrow S$. Now we are going to show that, for the distributions of S_n, the local limit theorem also holds true. In order to do this, we represent the array $\{X_{k,n}\}$ in the form (5′) in such a way that conditions **A** – **E** hold true.

We put $d_n = n$ and $\xi_{k,n} = nX_{k,n}, 1 \le k \le n$. Since in the case under consideration $c_n = n$, **E** holds true. The variables $\xi_{k,n}$ possess the distribution densities

$$\begin{aligned} g_n(x) &= \frac{1}{n}p_n\left(\frac{x}{n}\right) = \frac{1}{n^2}\left(\frac{x}{n}\left(e^{x/n} - e^{-x/n}\right)\right)^{-1}\mathbf{I}_{|x|>nt_n} \\ &= \frac{1}{x}\left(n\left(e^{x/n} - e^{-x/n}\right)\right)^{-1}\mathbf{I}_{|x|>nt_n} \end{aligned}$$

and thus **A** holds true. We have $g_n(x) \le \frac{1}{2x^2}\mathbf{I}_{|x|>nt_n}$, that provides $\mathbf{D}'$ (and therefore **D**).

Next, since $nt_n \to 1,\ n \to \infty$, there exists $d > 0$ such that $nt_n < d, n \in \mathbb{N}$. Then, on the interval $[d+1, d+2]$, the following estimate holds

$$g_n(x) = \frac{1}{x^2}\frac{x/n}{e^{x/n} - e^{-x/n}} \ge \frac{1}{x^2}\frac{x}{e^x - e^{-x}} \ge \frac{1}{xe^x} \ge \frac{1}{(d+2)e^{d+2}}.$$

This provides **C**.

At last, let us verify condition **B**. Take $a > 0$ such that $t_n < \frac{a}{2n}, n \in \mathbb{N}$. Then, for $\varepsilon > \frac{a}{n}$, we have $\varepsilon > 2t_n$ and therefore

$$\begin{aligned} \frac{1}{\varepsilon^2 \ln 1/\varepsilon}\int_{-\varepsilon}^{\varepsilon} M_n\{dx\} &= \frac{2}{\varepsilon^2 \ln 1/\varepsilon}\int_{t_n}^{\varepsilon}\frac{x}{e^x - e^{-x}}dx \\ &\ge \frac{2(\varepsilon - t_n)}{\varepsilon(e^\varepsilon - e^{-\varepsilon})\ln 1/\varepsilon} \ge \frac{1}{(e^\varepsilon - e^{-\varepsilon})\ln 1/\varepsilon}. \end{aligned}$$

Since $(e^\varepsilon - e^{-\varepsilon})\ln 1/\varepsilon \to 0,\ \varepsilon \to 0$, for a given $B > \frac{2}{1-\cos 1}$, there exists ε_0 such that $(e^\varepsilon - e^{-\varepsilon})\ln 1/\varepsilon \le B,\ \varepsilon < \varepsilon_0$. At last, there exists n_0 such that $2a/n < \varepsilon_0, n \ge n_0$. Condition **B** holds true with B, ε_0, n_0 given before.

All the conditions of Theorem 2 are verified, and thus we conclude that, for the sequence $S_n = X_{1,n} + \cdots + X_{n,n}$ given before, the correspondent distribution densities converge uniformly to the density of the hyperbolic cosine distribution.

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