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FUNCTIONAL ITERATED LOGARITHM LAW FOR A WIENER PROCESS

The functional iterated logarithm law for a Wiener process in the Bulinskii form for great and small times is proved.

1. Introduction. Let $w(t)$ be a d-dimensional Wiener process on the probability space $(\Omega, \mathcal{F}, P)$, $t \geq 0$. Introduce the sequence of random processes

$$\xi_n(t) = \frac{w(nt)}{\sqrt{n}\varphi(n)}, \quad n = 3, 4, ... \tag{1}$$

where $\varphi(n)$ is an arbitrary sequence.

In [1], A.V. Bulinskii proved a version of the Strassen iterated logarithm law in which the normalizing function $\varphi(n)$ is an arbitrary monotone increasing function. Let us formulate this result.

Denote, by $C([0,1]; E^d)$, the space of all continuous functions $x(t)$ on the interval $[0,1]$ with values in the Euclidean space $(E^d, |\cdot|)$ and with norm

$$||x|| = \sup_{t\in[0,1]} |x(t)|, \tag{2}$$

Let $K^1([0,1]; E^d)$ be the space of functions $x(t) \in C([0,1]; E^d)$ such that $x(0) = 0$ and $x(t) = \int_0^t \dot{x}(s)ds$ for some function $\dot{x}(\cdot) \in L_2([0,1]; E^d)$. We set

$$||x||_{K^1} = ||\dot{x}||_{L_2([0,1];E^d)}. \tag{3}$$

Following [1], we define Φ as a class of all increasing functions $\varphi(t)$, $t \geq 0$, such that $\lim_{t\to\infty} \varphi(t) = \infty$. Now we introduce a functional

$$J(\varphi, r, c) = \sum_{k=1}^{\infty} \exp\Big\{\frac{-r\varphi^2([c^k])}{2}\Big\}, \quad c > 1, \tag{4}$$

where $[\cdot]$ denotes the integer part of a number. For every $\varphi \in \Phi$, we denote

$$R^2(\varphi) = \inf\{r > 0 : J(\varphi, r, c) < \infty\}, \tag{5}$$

and $R(\varphi) = \infty$ if there exists no $r < \infty$ such that $J(\varphi, r, c) < \infty$. Let us remark that if $J(\varphi, r, c_0) < \infty$ for a certain $c_0 > 1$, then $J(\varphi, r, c) < \infty$ for every $c > 1$.

Let $\mathcal{K}_R = \{x(t) \in K^1, ||x||_{K^1} \leq R^2\}$.

In [1, Theorem 1], the following result was proved.

Bulinskii theorem. For $\varphi \in \Phi$, the limit set of sequence (1), $t \in [0,1]$, coincides with $\mathcal{K}_R$, and $R = R(\varphi)$ is defined in (5).

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In [1, Theorem 3], it was proved that R can be defined as $R = \frac{1}{Q}$, where

$$Q = \varlimsup_{t\to\infty} \frac{\varphi(t)}{\sqrt{2\ln\ln t}}.$$

This theorem yields the usual functional Strassen iterated logarithm law and allows us to study the asymptotic behavior of the process $\xi_n(t)$ for normalizing functions $\varphi(n)$ such that $\varlimsup_{t\to\infty} \frac{\varphi(t)}{\sqrt{2\ln\ln t}} = \infty$.

In the present article, we generalize this result in two directions. First, we consider sequence (1) on the whole time axis and, second, we prove a local version of the Bulinskii theorem.

2. Functional iterated logarithm law on the whole time axis.

Let $C([0,\infty); E^d)$ denote the space of all continuous functions on $[0,\infty)$ with values in E^d. We set

$$\Theta = \Big\{\theta \in C([0,\infty) : E^d) : \theta(0) = 0, \lim_{t\to\infty} \frac{|\theta(t)|}{t} = 0\Big\}.$$

On this space, we define the metric

$$||\theta||_\Theta = \sup_{t\geq 0} \frac{|\,\theta(t)|}{1+t}. \tag{6}$$

Then $(\Theta, ||\cdot||_\Theta)$ is a separable Banach space. Let $H^1([0,\infty); E^d)$ be the space of functions $\theta \in \Theta$ such that $\theta(t) = \int_0^t \dot\theta(s)ds$ for some $\dot\theta \in L_2([0,\infty); E^d)$. We define

$$||\theta||_{H^1} = ||\dot\theta||_{L_2([0,\infty);E^d)}. \tag{7}$$

Lemma. There exists the bijective correspondence between the spaces $K^1([0,1]; E^d)$ and $H^1([0,\infty); E^d)$. If $f \in K^1([0,1]; R^d)$, then

$$g(t) = \int_0^t \dot f\Big(\frac{s}{s+1}\Big)\frac{1}{1+s}ds \in H^1([0,\infty); E^d). \tag{8}$$

Conversely, if $g \in H^1([0,\infty); R^d)$, then

$$f(t) = \int_0^t \frac{1}{1-s}\dot g\Big(\frac{s}{1-s}\Big)ds \in K^1([0,1]; E^d). \tag{9}$$

Furthermore, $||g||_{H^1([0.\infty);E^d)} = ||f||_{K^1([0,1];E^d)}$.

Proof. Let $f \in K^1([0,1]; E^d)$. It is necessary to prove that, for the function $g(x)$ from (8),

$$\lim_{t\to\infty} \frac{|g(t)|}{t} = 0. \tag{10}$$

According to the Cauchy–Buniakowski inequality,

$$|g(t)|^2 = \Big|\int_0^{\frac{t}{t+1}} \dot f(s)\frac{1}{1-s}ds\Big|^2 \leq ||f||^2_{K^1} \int_0^{\frac{t}{t+1}} \frac{1}{(1-s)^2}ds = t||f||^2_{K^1}.$$

This yields (10). It is easy to verify the equality of norms. Lemma is proved.

In what follows, we will use the Schilder theorem [2, Theorem 1.3.27] on large deviations of a Wiener process with small variance. For convenience, we present its formulation. Define

$$I(\psi) = \begin{cases} 0, & \text{if } \quad \psi \notin H^1([0,\infty); E^d), \\ \dfrac{1}{2}||\psi||_{H^1}, & \text{if } \quad \psi \in H^1([0,\infty); E^d). \end{cases}$$

Schilder theorem. Let $W_\epsilon(\cdot)$ be the measure corresponding to a process $\epsilon w(t)$ on the space Θ with a Borel σ-algebra $\mathcal{B}$. Then

a) for any closed set $A \in \mathcal{B}$,

$$\overline{\lim_{\epsilon\to 0}}\, \epsilon^2 W_\epsilon(A) \le -\inf\{I(\psi); \psi \in A\}; \tag{11}$$

b)for any open set $B \in \mathcal{B}$,

$$\underline{\lim_{\epsilon\to 0}}\, \epsilon^2 W_\epsilon(B) \ge -\inf\{I(\psi); \psi \in B\}. \tag{12}$$

The functional $I(\psi)$ is called the action functional for a family $W_\epsilon(\cdot)$. For any $a < \infty$, the set $\{\psi : I(\psi) \le a\}$ is closed in the space $(\Theta, \mathcal{B})$.

Introduce a class of functions $\mathcal{L}_R$:

$$\mathcal{L}_R = \{\theta(t) \in H^1 : ||\theta||_{H^1([0,\infty);E^d)} \le R^2\}, \tag{13}$$

where $R = R(\varphi)$ is defined in (5).

Theorem 1. For $\varphi \in \Phi$ with probability 1, the set of limit points of sequence (1) is $\mathcal{L}_R$, where R is defined in (5).

Proof. We prove the theorem in three standard steps. Let us denote $n_k = [c^k]$, $z_k(t) = \xi_{n_k}(t)$.

Step 1). We need to prove that, for every $R^2 < \infty$, every $c > 1$, and every $\delta > 0$, there exists a positive integer k_0 such that, for every $k > k_0$,

$$\rho(z_k, \mathcal{L}_R) < \delta. \tag{14}$$

We set $N_\delta = \{\psi : \rho(\psi, \mathcal{L}_R) \ge \delta\}$. Then there exists $\eta > 0$ such that

$$\inf_{\psi\in N_\delta} I(\psi) \ge \frac{R^2}{2} + \eta.$$

As the set N_δ is closed, relation (11) yields

$$P\Big\{z_k \in N_\delta\Big\} \le \exp\Big\{-\varphi^2(c^k)\Big(\frac{R^2}{2} + \eta\Big)\Big\}$$

By the definition of the number R, we have $\sum_k P\Big\{z_k \in N_\delta\Big\} < \infty$. By applying the Borel–Cantelli lemma, we get (14). Step 1) is proved.

Step 2). For $R^2(\varphi) < \infty$, we need to prove that every limit point of the sequence $\xi_n(t)$ is an element of $\mathcal{L}_R$. If $n = n_k$, thos follows from step 1). Let now $n \in [n_k, n_{k+1}]$. Denote $\psi(n) = \sqrt{n}\varphi(n)$. Since the function $\psi(n)$ is nondecreasing, we can write

$$\frac{1}{\psi(n)} = \frac{\alpha_{nk}}{\psi(n_k)} + \frac{\beta_{nk}}{\psi(n_{k+1})}, \tag{15}$$

where $\alpha_{nk} \ge 0$, $\beta_{nk} \ge 0$ and $\alpha_{nk} + \beta_{nk} = 1$. Set $\tilde{z}_{nk}(t) = \alpha_{nk} z_k(t) + \beta_{nk} z_{k+1}(t)$. We note that, for large k, the functions $\tilde{z}_{nk} \in \{f : \rho(f, \mathcal{L}_R) < \delta\}$. This follows from the fact that if the functions $x(t), y(t) \in \mathcal{L}_R$ and $\alpha, \beta \ge 0$, $\alpha + \beta = 1$, then $\alpha x(t) + \beta y(t) \in \mathcal{L}_R$. The assertion of step 2) will be proved if we prove the following estimate: for every $\delta > 0$, there exist a number $c > 1$ and a positive integer k_0 such that, for every $k > k_0$,

$$\sup_{t\ge 0} \frac{1}{1+t} \sup_{n\in[n_k,n_{k+1}]} |\xi_n(t) - \tilde{z}_{nk}(t)| < \delta \tag{16}$$

with probability 1. From (15), we have

$$\xi_n(t) = \frac{w(n_k \frac{n}{n_k} t)}{\psi(n)} = z_k\Big(\frac{n}{n_k} t\Big)\frac{\psi(n_k)}{\psi(n)} = \alpha_{nk} z_k\Big(\frac{n}{n_k} t\Big) + \beta_{nk} z_{k+1}\Big(\frac{n}{n_{k+1}} t\Big).$$

Hence,

$$|\xi_n(t) - \tilde{z}_{nk}(t)| \le \Big|z_k(t) - z_k\Big(\frac{n}{n_k}t\Big)\Big| + \Big|z_{k+1}(t) - z_{k+1}\Big(\frac{n}{n_{k+1}}t\Big)\Big| \tag{17}$$

Then

$$\sup_{n\in[n_k,n_{k+1}]}\Big|z_k(t) - z_k\Big(\frac{n}{n_k}t\Big)\Big| \le \sup_{s\in[t,ct]}|z_k(t) - z_k(s)|. \tag{18}$$

Similarly,

$$\sup_{n\in[n_k,n_{k+1}]}\Big|z_{k+1}(t) - z_{k+1}\Big(\frac{n}{n_{k+1}}t\Big)\Big| \le \sup_{s\in[\frac{t}{c},t]}|z_{k+1}(t) - z_{k+1}(s)|. \tag{19}$$

For an arbitrary fixed c, we introduce the sets L_δ and M_δ:

$$L_\delta = \Big\{f(t)\in\Theta : \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}|f(s) - f(t)| \ge \delta\Big\},$$

$$M_\delta = \Big\{f(t)\in\Theta : \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[\frac{t}{c},t]}|f(s) - f(t)| \ge \delta\Big\}$$

From (17)–(19), we get

$$P\Big\{\sup_{t\ge 0}\frac{1}{1+t}\sup_{n\in[n_k,n_{k+1}]}|\xi_n(t) - \tilde{z}_{nk}(t)| \ge \delta\Big\} \le P\{z_k\in L_{\frac{\delta}{2}}\} + P\{z_{k+1}\in M_{\frac{\delta}{2}}\}. \tag{20}$$

The sets L_δ and M_δ are closed in Θ for every $c<\infty$. We prove this assertion, for example, only for the set L_δ. Let $f_n(t)\in L_\delta$ and $\lim_{n\to\infty}||f_n - f||_\Theta = 0$. Then the inequalities

$$\begin{aligned}\delta &\le \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}|f_n(s) - f_n(t)| \le \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}|f_n(s) - f(s)| + \\ &+ \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}|f(s) - f(t)| + \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}|f_n(t) - f_n(t)| \le \\ &\le \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}(1+s)\frac{|f_n(s) - f(s)|}{1+s} + \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}|f(s) - f(t)| + \\ &+ ||f_n - f||_\Theta \le c||f_n - f||_\Theta + \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}|f(s) - f(t)| + ||f_n - f||_\Theta\end{aligned}$$

yield

$$\delta \le (1+c)||f_n - f||_\Theta + \sup_{t\ge 0}\frac{1}{1+t}\sup_{s\in[t,ct]}|f(s) - f(t)|.$$

Passing to the limit in the last inequality as $n\to\infty$, we obtain $f\in L_\delta$.

Applying (11), we have

$$P\{z_k\in L_{\frac{\delta}{2}}\} \le \exp\{-\varphi^2(n_k)\inf_{f\in L_\delta} I(f)\}. \tag{21}$$

Since

$$\sup_{s\in[t,ct]}|f(s) - f(t)|^2 = \sup_{s\in[t,ct]}\Big|\int_t^s \dot{f}(u)du\Big|^2 \le (c-1)tI(f),$$

we get

$$\frac{\delta^2}{4} \le \sup_{t\ge 0}\frac{1}{(1+t)^2}\sup_{s\in[t,ct]}|f(s) - f(t)|^2 \le \frac{c-1}{4}I(f)$$

for $f\in L_{\frac{\delta}{2}}$. If we choose $c = 1 + \frac{2\delta^2}{R^2}$, then

$$\inf_{f\in L_\delta} I(f) \ge \frac{R^2}{2}. \tag{22}$$

Relations (21) and (22) and the definition of R^2 yield

$$\sum_k P\{z_k \in L_{\frac{\delta}{2}}\} \le \sum_k \exp\Big\{-\varphi^2(n_k)\frac{R^2}{2}\Big\} < \infty. \tag{23}$$

In an analogous way, we can prove for the same c that

$$\sum_k P\{z_{k+1} \in M_{\frac{\delta}{2}}\} \le \sum_k \exp\Big\{-\varphi^2(n_k)\frac{R^2}{2}\Big\} < \infty. \tag{24}$$

From (20), (23), (24) and the Borel–Cantelli lemma, we get (16). Thus, step 2) is proved.

Step 3). In order to complete the proof of Theorem 1, we need to prove that if $R^2 \le \infty$, then every $g \in \mathcal{L}_r$ is a limit point of z_k for $r < R$. That is, there exists a sequence n_k such that, with probability 1,

$$\lim_{k\to\infty} \rho(\xi_{n_k}, g) = 0. \tag{25}$$

Applying the Itô formula to the Wiener process $w(t)$ and to the function $f(s,x) = \frac{n}{n-s}(x,a)$ $s \in [0, \frac{tn}{t+n}]$, $n > 0$, a is an arbitrary vector from E^d, we have

$$\frac{t+n}{n}\Big(w\Big(\frac{tn}{t+n}\Big), a\Big) = \int_0^{\frac{tn}{t+n}} \frac{n}{(n-s)^2}(w(s),a)\,ds + \int_0^{\frac{tn}{t+n}} \frac{n}{n-s}\,(dw(s),a).$$

Whence we get

$$\begin{aligned}\int_0^{\frac{tn}{t+n}} \frac{n}{n-s}\,dw(s) &= \frac{t+n}{n}w\Big(\frac{tn}{t+n}\Big) - \int_0^{\frac{tn}{t+n}} \frac{n}{(n-s)^2}w(s)\,ds = \\ &= \frac{t+n}{n}w\Big(\frac{tn}{t+n}\Big) - \frac{1}{n}\int_0^t w\Big(\frac{vn}{v+n}\Big)\,ds.\end{aligned} \tag{26}$$

The random process $\eta_n(t) = \int_0^{\frac{tn}{t+n}} \frac{n}{n-s}\,dw(s)$ is a process with independent increments, because it is defined on nonintersecting intervals by independent increments of the Wiener process. By the property of stochastic integrals, we have

$$E\exp\big\{i\lambda(\eta_n(t)-\eta_n(s))\big\} = \exp\Big\{-\frac{\lambda^2}{2}\int_{\frac{sn}{s+n}}^{\frac{tn}{t+n}} \frac{n^2}{(n-u)^2}\,du\Big\} = \exp\Big\{-\frac{\lambda^2}{2}(t-s)\Big\}.$$

Hence, the process $\eta_n(t)$ is also a Wiener process for any n. From (26), we obtain

$$\tilde{\eta}_n(t) = \frac{\eta_n(nt)}{\sqrt{n}\varphi(n)} = (t+1)\frac{w\Big(n\frac{t}{t+1}\Big)}{\sqrt{n}\varphi(n)} - \int_0^t \frac{w\Big(n\frac{s}{s+1}\Big)}{\sqrt{n}\varphi(n)}\,ds. \tag{27}$$

Let a function $g \in \mathcal{L}_r$. Define a function $f(t)$ by (9). By virtue of the lemma, $f \in \mathcal{K}_r$. It follows from (8) that the function $g(t)$ admits the representation

$$g(t) = (t+1)f\Big(\frac{t}{t+1}\Big) - \int_0^t f\Big(\frac{s}{s+1}\Big)ds. \tag{28}$$

From (27) and (28), we get

$$\tilde{\eta}_n(t) - g(t) = (t+1)\left[\frac{w\Big(n\frac{t}{t+1}\Big)}{\sqrt{n}\varphi(n)} - f\Big(\frac{t}{t+1}\Big)\right] - \int_0^t \left[\frac{w\Big(n\frac{s}{s+1}\Big)}{\sqrt{n}\varphi(n)} - f\Big(\frac{s}{s+1}\Big)\right]ds.$$

Whence

$$\frac{|\tilde{\eta}_n(t) - g(t)|}{1+t} \le \left|\frac{w\Big(n\frac{t}{t+1}\Big)}{\sqrt{n}\varphi(n)} - f\Big(\frac{t}{t+1}\Big)\right| + \frac{t}{1+t}\sup_{t\in[0,1]}|\xi_n(t) - f(t)|.$$

Thus,

$$||\tilde{\eta}_n - g||_\Theta \le 2||\xi_n - f||,$$

and

$$\Big\{||\xi_n - f|| < \delta\Big\} \subset \Big\{||\tilde{\eta}_n - g||_\Theta < 2\delta\Big\}. \tag{29}$$

As has been stated above, the law of the process $\eta_n(t)$ is the same as that of the process $w(t)$. Therefore,

$$P\Big\{||\tilde{\eta}_n - g||_\Theta < 2\delta\Big\} = P\Big\{||\xi_n - g||_\Theta < 2\delta\Big\}. \tag{30}$$

Consider the events

$$A_k = \{||\xi_{n_k} - f|| < \delta\},\ B_k = \{||\tilde{\eta}_{n_k} - g||_\Theta < 2\delta\},\ C_k = \{||\xi_{n_k} - g||_\Theta < 2\delta\}$$

By the Bulinskii theorem, there exists a sequence n_k such that, for the processes $\xi_n(s)$ for $s \in [0,1]$ and every δ- neighborhood of the function f :

$$P\{\xi_{n_k} \in (f)_\delta \quad \text{i.o.}\} = 1, \tag{31}$$

where i.o. means "infinitely often". Equality (31) yields

$$P\Big\{\bigcap_{m=1}^{\infty}\bigcup_{l=m}^{\infty} A_l\Big\} = \lim_{m\to\infty} P\Big\{\bigcup_{l=m}^{\infty} A_l\Big\} = 1.$$

By virtue of (29), $A_k \subset B_k$. Hence,

$$P\Big\{\bigcap_{m=1}^{\infty}\bigcup_{l=m}^{\infty} B_l\Big\} = \lim_{m\to\infty} P\Big\{\bigcup_{l=m}^{\infty} B_l\Big\} \ge \lim_{m\to\infty} P\Big\{\bigcup_{l=m}^{\infty} A_l\Big\} = 1.$$

For this reason, the events B_k take place infinitely often, and

$$\lim_{k\to\infty} P(B_k) = 1.$$

¿From this and (30), we get

$$P\Big\{\bigcap_{m=1}^{\infty}\bigcup_{l=m}^{\infty} C_l\Big\} = \lim_{m\to\infty} P\Big\{\bigcup_{l=m}^{\infty} C_l\Big\} \ge \lim_{m\to\infty} P\Big\{C_m\Big\} = \lim_{m\to\infty} P\Big\{B_m\Big\} = 1. \tag{32}$$

It follows from (32) that, with probability 1, the events C_k take place infinitely often. Equality (25) and Theorem 1 are proved.

3. Small-time functional iterated logarithm law.

We will prove a result similar to Theorem 1 for the process

$$w_n(t) = \frac{w(\frac{t}{n})}{\sqrt{\frac{1}{n}\varphi(n)}},\ t \in [0,\infty),\ n \ge 3. \tag{33}$$

To this end, we make use a method from [3]. On the space Θ, we define a time inversion transformation T by the formula

$$(T\theta)(t) = \begin{cases} 0, & t = 0, \\ t\,\theta(\frac{1}{t}), & t > 0. \end{cases}$$

In [3], it is noted that the transformation T is bijective on Θ. Furthermore, T is an isometry from Θ onto Θ and from H^1 onto H^1:

$$||\theta||_\Theta = ||T\theta||_\Theta, \quad ||\theta||_{H^1} = ||T\theta||_{H^1}. \tag{34}$$

Theorem 2. For $\varphi \in \Phi$, the set of limit points of sequence (33) is $\mathcal{L}_R$ with probability 1, where R is defined in (5).

Proof. We denote

$$\tilde{w}(t) = (Tw)(t) = \begin{cases} 0, & t = 0, \\ t\, w(\frac{1}{t}), & t > 0. \end{cases}$$

It is known that $\tilde{w}(t)$ is also a Wiener process. For $t > 0$,

$$\frac{\tilde{w}(nt)}{\sqrt{n}\varphi(n)} = t\frac{w(\frac{1}{t}\frac{1}{n})}{\sqrt{\frac{1}{n}}\varphi(n)}. \tag{35}$$

From (35), we get

$$\sup_{t\geq 0}\frac{1}{1+t}\left|\frac{\tilde{w}(nt)}{\sqrt{n}\varphi(n)} - (Tf)(t)\right| = \sup_{t\geq 0}\frac{1}{1+t}\left|w_n(t) - f(t)\right|. \tag{36}$$

Suppose that there exists a subsequence n_k such that

$$\lim_{n_k\to\infty} ||w_{n_k} - f||_\Theta = 0. \tag{37}$$

From (36), we have

$$\lim_{n_k\to\infty}\sup_{t\geq 0}\frac{1}{1+t}\left|\frac{\tilde{w}(n_k t)}{\sqrt{n}\varphi(n_k)} - (Tf)(t)\right| = 0. \tag{38}$$

According to Theorem 1, we conclude that $(Tf)(t) \in \mathcal{L}_R$. Then relations (34) imply that $f(t) \in \mathcal{L}_R$. Thus, each limit point of the sequence $w_n(t)$ is an element of $\mathcal{L}_R$. Conversely, let a function $f(t) \in \mathcal{L}_R$. Then, by (34), the function $(Tf)(t) \in \mathcal{L}_R$ and, by virtue of Theorem 1, there exists a subsequence n_k such that

$$\lim_{n_k\to\infty}\sup_{t\geq 0}\frac{1}{1+t}\left|\frac{\tilde{w}(n_k t)}{\sqrt{n}\varphi(n_k)} - (Tf)(t)\right| = 0.$$

This relation and (36) yield (37). Theorem 2 is proved.

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