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LIMIT THEOREMS FOR CONDITIONAL DISTRIBUTIONS OF CRITICAL GALTON-WATSON BRANCHING PROCESSES WITHOUT FINITE VARIANCE

In this paper we consider critical Galton-Watson branching processes $Z_k,\, k \geq 0$ in the case when the number of direct offspring of one particle has infinite variance. Limit theorems for conditional distributions of Z_k are proved.

1. Introduction and notations

Let $\{\xi_{i,j},\, i, j \in \mathbb{N}\}$ be a set of independent, identically distributed and taking non-negative integer values random variables with a generating function $F(s) = Es^{\xi_{ij}}$, $0 \leq s \leq 1$. Let $Z_0, Z_k,\, k \in \mathbb{N}$ be a sequence of random variables defined by recurrent relations

$$Z_0 = 1,\, Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{nj},\, n \in \mathbb{N}. \tag{1}$$

Process (1) is called the Galton-Watson branching process starting from one particle.

The Galton-Watson branching process (1) is called subcritical, critical, and supercritical if $F'(1-) < 1$, $F'(1-) = 1$ or $1 < F'(1-) < \infty$ respectively. The main definitions and properties of Galton-Watson branching processes can be found in [1].

Further we will consider only the critical case, i.e., the case of $F'(1-) = 1$.

Introduce the notation for the iteration of the function $F(s)$:

$$F_0(s) = s,\, F_1(s) = F(s),\, F_n(s) = F(F_{n-1}(s)),\, n \in \mathbb{N}.$$

It is easy to see that $F_n(s) = Es^{Z_n}$.

Introduce the random variable $T = \min(k : Z_k = 0)$ which is the degeneration moment of the branching process. Since the critical Galton-Watson branching process degenerates with probability 1, the probability $P(T < \infty) = 1$.

The study of the asymptotics of the conditional distributions of the random variable Z_n under various conditions on the trajectory of the process, apparently, began with the work of A.M. Yaglom [2], in which it was proved that for the case of $F'(1-) = 1$, $F'''(1-) < \infty$, the conditional distribution of $(1 - F_n(0)) Z_n$ under the condition $Z_n > 0$ weakly converges to the exponential law. In [10], it was shown that for Yaglom's result to be valid, it suffices that $F'(1-) = 1$, $F''(1-) < \infty$.

In [9], Harris noted that in the case of $F'(1-) = 1$, $F'''(1-) < \infty$, the conditional distribution of the random variable $(1 - F_k(0)) Z_k$ under the condition $Z_{k+m} > 0$, when passing to the limit, first as $m \to \infty$, and then, as $k \to \infty$, weakly converges to the distribution function

$$G(x) = 1 - e^{-x} - xe^{-x},\, x \geq 0.$$

The case of $F'(1-) = 1$, $F'''(1-) < \infty$ was also considered in [6], where the weak convergence of the conditional finite-dimensional distributions of the random process $(1 - F_n(0)) Z_{[nt]}$, $0 \leq t \leq 1$ was proved under the conditions $Z_0 = O(n)$, $Z_n > 0$ to the

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corresponding distributions of the diffusion process, here the sign $[a]$ denotes the integer part of the number a. Limit theorems of the Yaglom type were also considered in [11]. In [8], Slack proved an analog of Yaglom's theorem for Galton-Watson branching processes in which the number of offspring of one particle has infinite variance. In [3], Pakes studied the asymptotics of the distribution of the total number of particles $Z_0 + Z_1 + ... + Z_n$ under the condition $Z_{n+m} > 0$. In [4], the asymptotics of the Laplace transform of a random variable $(1 - F_n(0)) Z_n$ at $m \to \infty$ and then at $n \to \infty$ was found (Theorem 4) under the condition $Z_{n+m} > 0$ in the case when the number of offspring of one particle has an infinite variance. In [6], E. Seneta, using the result of G. Szekeres [5], proved that for critical branching processes (without any moment conditions, except for $F'(1-) = 1$), the conditional distribution $P(Z_n = j/T = n + k)$, where $k \in \mathbb{N}$ is a fixed number, has a limit of $u_j(k)$, and $\sum_{j=1}^{\infty} u_j(k) = 1$. In [8], Slack reproved the same result without applying the result of G. Szekeres.

This work is devoted to the study of the conditional distribution Z_k under the condition $k < T \leq k+m$ for the cases when: (a) m is fixed and $k \to \infty$, (b) $m \to \infty$ and then $k \to \infty$, and (c) $k = [nt]$, $m = [(1-t)n]$, $t \in [0,1]$, $n \to \infty$ in the case when the number of offspring of one particle has an infinite variance. It turns out that the imposition of condition (2) for the generating function of the number of offspring of one particle makes it possible to clarify Seneta's theorem [6]. It also turns out that in the case (a), the limit distribution of $P(Z_k = j/k < T \leq k+m)$ (without normalizing the value Z_k) depends on the iterations of the generating function $F(s)$; in other cases, to find the limit, it is necessary to normalize the value Z_k, and limit does not depend on $F(s)$.

2. Statement of the main results

Let Z_k, $k \geq 0$ be the critical Galton-Watson branching process determined recurrently from the relations (1). Introduce the following condition: the generation function $F(s)$ has the form

$$F(s) = s + (1-s)^{1+\alpha} L(1-s), \; 0 \leq s \leq 1, \tag{2}$$

where $\alpha \in (0,1]$ is a fixed number, $L(x)$ is a slowly varying function at zero.

It is known [8] that a random variable with generating function (2) can have an infinite second moment.

The following theorems take place.

Theorem 2.1. *Assume that representation* (2) *holds. Then there is a measure $\mu(j)$, $j \geq 1$ such that for any fixed $k \in \mathbb{N}$ and for all $j \in \mathbb{N}$,*

$$P(Z_n = j/T = n+k) \to \mu(j) \left[F_k^j(0) - F_{k-1}^j(0)\right]$$

as $n \to \infty$, the measure $\mu(j)$, $j \geq 1$ is determined from the relation

$$\lim_{n\to\infty} L^{\frac{1}{\alpha}}(1 - F_n(0)) (\alpha n)^{1+\frac{1}{\alpha}} P_n(i,j) = i\mu(j),$$

where $P_n(i,j) = P(Z_n = j/Z_0 = i)$.

Corollary 2.1. *Assume that representation* (2) *holds. Then for any fixed $k \in \mathbb{N}$,*

$$\Phi_{k,n}(s) = \sum_{j=1}^{\infty} s^j P(Z_n = j/T = n+k) \to U(sF_k(0)) - U(sF_{k-1}(0)), \; 0 \leq s \leq 1$$

as $n \to \infty$, where the function $U(s)$ is the generating function of the measure $\mu(j)$, $j \geq 1$ from the theorem 2.1: $U(s) = \sum_{j=1}^{\infty} \mu(j) s^j$.

Theorem 2.2. *Let the generating function $F(s)$ have the form of (2). Then for all $j \in \mathbb{N}$ and for any fixed number $m \in \mathbb{N}$,*

$$P(Z_n = j/n < T \leq n + m) \to \frac{1}{m}\mu(j) F_m^j(0)$$

as $n \to \infty$, where the measure $\mu(j)$, $j \geq 1$ is from Theorem 2.1.

Corollary 2.2. *Let the generating function $F(s)$ have the form of (2). Then for any fixed number $m \in \mathbb{N}$,*

$$\sum_{j=1}^{\infty} s^j P(Z_n = j/n < T \leq n + m) \to \frac{1}{m} U(sF_m(0)),\ 0 \leq s < 1$$

as $n \to \infty$, where the function $U(s)$ is the generating function of the measure $\mu(j)$, $j \geq 1$ from Theorem 2.1.

Theorem 2.3. *Let the generating function $F(s)$ have the form of (2). Then for any $\lambda \geq 0$ and all $t \in (0,1)$,*

$$\lim_{n\to\infty} E\left(e^{-\lambda(1-F_n(0))Z_{[nt]}} \Big/ Z_{[nt]} > 0,\ Z_n = 0\right) = \psi_t(\lambda),$$

where

$$\psi_t(\lambda) = \frac{1}{1 - t^{\frac{1}{\alpha}}}\left\{1 - \lambda t^{\frac{1}{\alpha}}\left(1 + \frac{1}{\lambda(1-t)^{\frac{1}{\alpha}}}\right) \cdot \left[1 + t\lambda^{\alpha}\left(1 + \frac{1}{\lambda(1-t)^{\frac{1}{\alpha}}}\right)^{\alpha}\right]^{-\frac{1}{\alpha}}\right\}.$$

Corollary 2.3. *If $F''(1-) < \infty$, then*

$$\lim_{n\to\infty} E\left(e^{-\lambda(1-F_n(0))Z_{[nt]}} \Big/ Z_{[nt]} > 0,\ Z_n = 0\right) = \frac{1}{1 + t(1-t)\lambda}$$

Corollary 2.4. *If $F''(1-) < \infty$, then*

$$P\left((1 - F_n(0)) Z_{[nt]} < x/Z_{[nt]} > 0,\ Z_n = 0\right) \to 1 - e^{-\frac{x}{t(1-t)}},\ x \geq 0.$$

3. Auxiliary result

In what follows, when proving the main results, we need the following results.

Theorem 3.1. *Let the generating function $F(s)$ have the form of (2). Then*

$$E\left(e^{-\lambda(1-F_n(0))Z_n} \Big/ Z_n > 0\right) \to 1 - \lambda(1+\lambda^{\alpha})^{-\frac{1}{\alpha}} \quad \text{as } n \to \infty.$$

Lemma 3.1. *Let the generating function $F(s)$ have the form of (2), both a_n, b_n be sequences of positive numbers tending to zero as $n \to \infty$, for which there are positive constants K_1, K_2 such that for sufficiently large n,*

$$0 < K_1 < \frac{a_n}{b_n} < K_2 < \infty.$$

Then

$$\frac{L(a_n)}{L(b_n)} \to 1 \quad \text{as } n \to \infty.$$

Lemma 3.2. *Let the generating function $F(s)$ have the form of (2). Then*

$$(1 - F_n(0))^{\alpha} L(1 - F_n(0)) \sim \frac{1}{\alpha n},\ n \to \infty.$$

Lemma 3.3. *Let the generating function $F(s)$ have the form of (2). Then*

$$F_n(0) - F_{n-1}(0) \sim (n\alpha)^{-\left(1+\frac{1}{\alpha}\right)} L^{-\frac{1}{\alpha}}(1 - F_n(0)) \quad \text{as } n \to \infty.$$

Lemma 3.4. *Let the generating function $F(s)$ have the form of (2). Then for any $i, j \in \mathbb{N}$, there exists a limit*

$$\lim_{n\to\infty} L^{\frac{1}{\alpha}}(1-F_n(0))(\alpha n)^{1+\frac{1}{\alpha}} P_n(i,j) = i\mu(j),$$

where the measure $\mu(j)$, $j \in \mathbb{N}$ is such that

$$\sum \mu(i) P_1(i,j) = \mu(j),\ j \geq 1,$$

$$\sum_{i=1}^{\infty} \mu(i) p_0^i = 1.$$

Moreover,

$$\mu(1)+\mu(2)+...+\mu(n) \sim \frac{n^\alpha}{\alpha(\alpha+1)L\left(\frac{1}{n}\right)}.$$

Here $P_n(i,j) = P(Z_n = j/Z_0 = i)$, $p_0 = P(Z_1 = 0)$, $\Gamma(x)$ is the gamma function.

The results, contained in Theorem 3.1 and Lemmas 3.1-3.4, are obtained in [8].

Lemma 3.5. *Let the generating function $F(s)$ have the form of (2). Then for any fixed number $k \in \mathbb{N}$,*

$$\frac{L(1-F_{n+k}(0))}{L(1-F_n(0))} \to 1 \ \text{ as } \ n \to \infty.$$

Lemma 3.6. *Let the generating function $F(s)$ have the form of (2). Then for any fixed positive integer number k,*

$$\frac{F_{n+k}(0)-F_{n+k-1}(0)}{F_n(0)-F_{n-1}(0)} \to 1 \ \text{ as } \ n \to \infty.$$

Theorem 3.2. *Let the generating function $F(s)$ have the form of (2). Then*

$$E\left(e^{-\lambda(1-F_n(0))Z_{[nt]}}/Z_n>0\right) \to \varphi_\alpha(t,\lambda) = \frac{1}{t^{\frac{1}{\alpha}}}\left\{1-\lambda t^{\frac{1}{\alpha}}\left[1+t\lambda^\alpha\right]^{-\frac{1}{\alpha}}\right\} -$$

$$-\frac{1}{t^{\frac{1}{\alpha}}}\left\{1-\lambda t^{\frac{1}{\alpha}}\left(1+\frac{1}{\lambda(1-t)^{\frac{1}{\alpha}}}\right)\times\left[1+t\lambda^\alpha\left(1+\frac{1}{\lambda(1-t)^{\frac{1}{\alpha}}}\right)^\alpha\right]^{-\frac{1}{\alpha}}\right\}.$$

Theorem 3.2 is given in [12].

Proof of Lemma 3.5. It is clear that

$$\frac{L(1-F_{n+k}(0))}{L(1-F_n(0))} = \prod_{i=0}^{k-1}\frac{L(1-F_{n+i+1}(0))}{L(1-F_{n+i}(0))}. \tag{3}$$

Taking into account the property of generating functions and (2), we have

$$\frac{1-F_{n+i+1}(0)}{1-F_{n+i}(0)} = \frac{1-F(F_{n+i}(0))}{1-F_{n+i}(0)} = 1-(1-F_{n+i}(0))^\alpha L(1-F_{n+i}(0)) \to 1 \tag{4}$$

as $n \to \infty$. Then for any $\varepsilon > 0$, there is a number $\mathbb{N}$ such that

$$1-\varepsilon < \frac{1-F_{n+i+1}(0)}{1-F_{n+i}(0)} < 1+\varepsilon$$

for all $n \geq \mathbb{N}$. We choose the value of ε so that $1-\varepsilon > 0$. Then according to Lemma 3.1,

$$\frac{L(1-F_{n+i+1}(0))}{L(1-F_{n+i}(0))} \to 1 \ \text{ as } \ n \to \infty. \tag{5}$$

The assertion of Lemma 3.5 follows from the last ratio and from (3). □

Proof of Lemma 3.6. Since

$$\frac{F_{n+k}(0) - F_{n+k-1}(0)}{F_n(0) - F_{n-1}(0)} = \prod_{i=0}^{k-1} \frac{F_{n+i+1}(0) - F_{n+i}(0)}{F_{n+i}(0) - F_{n+i-1}(0)},$$

the assertion Lemma 3.6 immediately follows from Lemma 3.3 and the relation (5). □

4. Proof of the main results

Proof of Theorem 2.1. Given that the sequence Z_k, $k \geq 0$ is a homogeneous Markov chain, as well as the relation

$$\{T = n+k\} = \{Z_{n+k-1} > 0,\, Z_{n+k} = 0\} = \{Z_{n+k} = 0\} \setminus \{Z_{n+k-1} = 0\}, \tag{6}$$

we have

$$P(Z_n = j/T = n+k) =$$

$$= \frac{P(Z_n=j/Z_0=1)[P(Z_{n+k}=0/Z_n=j) - P(Z_{n+k-1}=0/Z_n=j)]}{F_{n+k}(0) - F_{n+k-1}(0)} =$$

$$= \frac{P_n(1,j)\left[F_k^j(0) - F_{k-1}^j(0)\right]}{F_{n+k}(0) - F_{n+k-1}(0)}. \tag{7}$$

Now, from the last relation, applying Lemma 3.3 and Lemma 3.4, we have

$$P(Z_n = j/T = n+k) \sim \mu(j)\left[F_k^j(0) - F_{k-1}^j(0)\right] \times$$

$$\times \left(1 + \tfrac{k}{n}\right)^{1+\frac{1}{\alpha}} \times \left[\tfrac{L(1-F_{n+k}(0))}{L(1-F_n(0))}\right]^{\frac{1}{\alpha}}.$$

We conclude from this and Lemma 3.5 that

$$P(Z_n = j/T = n+k) \sim \mu(j)\left[F_k^j(0) - F_{k-1}^j(0)\right]$$

as $n \to \infty$, Q.E.D. □

The proof of Corollary 2.1 immediately follows from Theorem 2.1 and from the continuity theorem for generating functions.

Proof of Theorem 2.2. Taking into account (3), we have

$$P(Z_n = j/n < T \leq n+m) = \frac{1}{\sum_{k=1}^{m} P(T=n+k)} \times \sum_{k=1}^{m} P(Z_n = j,\, T = n+k) =$$

$$= \frac{1}{\sum_{k=1}^{m}[F_{n+k}(0) - F_{n+k-1}(0)]} \times \sum_{k=1}^{m} P(Z_n = j/T = n+k)\left[F_{n+k}(0) - F_{n+k-1}(0)\right].$$

Taking this into account and Lemma 3.6, applying Theorem 2.1, we have

$$P(Z_n = j/n < T \leq n+m) \sim \frac{1}{m}\mu(j)\sum_{k=1}^{m}\left[F_k^j(0) - F_{k-1}^j(0)\right] = \frac{1}{m}\mu(j)F_m^j(0)$$

as $n \to \infty$, Q.E.D. □

The proof of Corollary 2.2 immediately follows from Theorem 2.2 and from the continuity theorem for generating functions.

Proof of Theorem 2.3. We have

$$E\left(e^{-\lambda(1-F_n(0))Z_{[nt]}}/Z_{[nt]}>0, Z_n=0\right)=\frac{F_n\left(e^{-\lambda(1-F_{[nt]}(0))\frac{1-F_n(0)}{1-F_{[nt]}(0)}Z_{[nt]}}\right)-F_n(0)}{1-F_{[nt]}(0)}\times$$

$$\times\frac{1-F_{[nt]}(0)}{F_n(0)-F_{[nt]}(0)}-E\left(e^{-\lambda(1-F_n(0))Z_{[nt]}}/Z_n>0\right)\times\frac{1-F_n(0)}{F_n(0)-F_{[nt]}(0)}=$$

$$=\frac{F_n\left(e^{-\lambda(1-F_{[nt]}(0))t^{\frac{1}{\alpha}}Z_{[nt]}}\right)-F_n(0)}{1-F_{[nt]}(0)}\times\frac{1-F_{[nt]}(0)}{F_n(0)-F_{[nt]}(0)}-$$

$$-E\left(e^{-\lambda(1-F_n(0))Z_{[nt]}}/Z_n>0\right)\times\frac{1-F_n(0)}{F_n(0)-F_{[nt]}(0)}+$$

$$+\frac{F_n\left(e^{-\lambda(1-F_{[nt]}(0))\frac{1-F_n(0)}{1-F_{[nt]}(0)}Z_{[nt]}}\right)-F_n\left(e^{-\lambda(1-F_{[nt]}(0))t^{\frac{1}{\alpha}}Z_{[nt]}}\right)}{1-F_{[nt]}(0)}\times\frac{1-F_{[nt]}(0)}{F_n(0)-F_{[nt]}(0)}. \tag{8}$$

By virtue of (4), for sufficiently large i,

$$\frac{1-F_{i+1}(0)}{1-F_i(0)}>1-\frac{2}{\alpha i}.$$

Then for sufficiently large n,

$$\frac{1-F_n(0)}{1-F_{[nt]}(0)}=\prod_{i=[nt]}^{n}\frac{1-F_{i+1}(0)}{1-F_i(0)}>\prod_{i=[nt]}^{n}\left(1-\frac{2}{\alpha i}\right)\sim$$

$$\sim\left(1-\frac{2}{\alpha nt}\right)^{n(1-t)}\sim e^{-\frac{2(1-t)}{\alpha t}}. \tag{9}$$

Because $F_i(0)<F_{i+1}(0)$, it is obvious that

$$\frac{1-F_n(0)}{1-F_{[nt]}(0)}=\prod_{i=[nt]}^{n}\frac{1-F_{i+1}(0)}{1-F_i(0)}<1.$$

This and (9) imply that for sufficiently large n,

$$e^{-\frac{2(1-t)}{\alpha t}}<\frac{1-F_n(0)}{1-F_{[nt]}(0)}<1.$$

Therefore, by virtue of Lemma 3.1,

$$\frac{L(1-F_n(0))}{L\left(1-F_{[nt]}(0)\right)}\to 1 \text{ as } n\to\infty. \tag{10}$$

Now, applying Lemma 3.2 and taking into account (10), we have

$$\frac{1-F_n(0)}{1-F_{[nt]}(0)}\sim t^{\frac{1}{\alpha}}\left[\frac{L\left(1-F_{[nt]}(0)\right)}{L(1-F_n(0))}\right]^{\frac{1}{\alpha}}\to t^{\frac{1}{\alpha}} \text{ as } n\to\infty. \tag{11}$$

By virtue of Lemma 3.2 and relation (11), we have

$$\frac{1-F_{[nt]}(0)}{F_n(0)-F_{[nt]}(0)}=\frac{1-F_{[nt]}(0)}{1-F_{[nt]}(0)-[1-F_n(0)]}=$$

$$=\left(1-\frac{1-F_n(0)}{1-F_{[nt]}(0)}\right)^{-1}\sim\frac{1}{1-t^{\frac{1}{\alpha}}} \text{ as } n\to\infty. \tag{12}$$

and

$$\frac{1-F_n(0)}{F_n(0)-F_{[nt]}(0)} \sim \frac{t^{\frac{1}{\alpha}}}{1-t^{\frac{1}{\alpha}}} \text{ as } n \to \infty. \tag{13}$$

Applying the mean value theorem and the inequality

$$|e^x - e^y| \le |x-y|\,, \quad Re\, x, y \le 0,$$

given that $EZ_k = 1$ for any k, as well as the increase of the function $F_n'(s)$ with respect to s, we have

$$\left| F_n \left(e^{-\lambda\left(1-F_{[nt]}(0)\right)\frac{1-F_n(0)}{1-F_{[nt]}(0)}Z_{[nt]}} \right) - F_n \left(e^{-\lambda\left(1-F_{[nt]}(0)\right)t^{1/\alpha}Z_{[nt]}} \right) \right| \le$$

$$\le \lambda \left(1-F_{[nt]}(0)\right) \left| \frac{1-F_n(0)}{1-F_{[nt]}(0)} - t^{1/\alpha} \right| \tag{14}$$

Now from (8), (11)-(14), and Theorem 3.1, Theorem 3.2, we find

$$E\left(e^{-\lambda(1-F_n(0))Z_{[nt]}} / Z_{[nt]} > 0,\ Z_n = 0\right) \to \frac{1}{1-t^{\frac{1}{\alpha}}} \left\{ 1 - \lambda t^{\frac{1}{\alpha}} \left(1+\lambda^\alpha t\right)^{-\frac{1}{\alpha}} \right\} -$$

$$- \frac{t^{\frac{1}{\alpha}}}{1-t^{\frac{1}{\alpha}}} \varphi_\alpha(t, \lambda) = \psi_t(\lambda)\,.$$

as $n \to \infty$, which completes the proof of Theorem 2.3. $\square$

Proof of Corollary 2.3 and Corollary 2.4. It is clear that if $F''(1-) < \infty$, then $\alpha = 1$ in representation (2). Therefore, Corollary 2.3 immediately follows from Theorem 2.3 in the case of $\alpha = 1$. Corollary 2.4 follows from Corollary 2.3 and from the continuity theorem for generating functions. $\square$

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