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GENERALIZED BSDES FOR TIME INHOMOGENEOUS LÉVY PROCESSES UNDER NON-DETERMINISTIC LIPSCHITZ COEFFICIENT

In this paper, we study the generalized backward stochastic differential equations driven by inhomogeneous Lévy processes (GBSDELs in short). We establish the existence and uniqueness of solution by using Picard's iteration setting under non-deterministic Lipschitz and monotone condition.

1. Introduction

It is well known that the general linear case of backward stochastic differential equations in the Brownian framework has been introduced by Bismut [3]. Nonetheless, the first study presenting a systematic treatment of non-linear BSDEs is the seminal paper of Pardoux and Peng [9]. Later, Pardoux and Zhang [11] introduced a new class of BSDE's, which involves the integral with respect to a continuous increasing process. Precisely, given a data (ξ, f, g) of the progressively measurable processes f and g, and the square integrable random variable ξ, they proved the existence and uniqueness of an adapted process (Y, Z) solution of the following equation:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s)dC_s - \int_t^T Z_s dB_s,$$

where $(C_t)_{t \leq T}$ is a continuous real valued increasing process. These equations also provide a probabilistic formula for the viscosity solution of a system of partial differential equations (PDEs in short) with a non-linear Neumann boundary condition by introducing a class of generalized BSDEs. Following this way, El Otmani [5] extends this class of equations of [11] to generalized BSDE driven by a homogeneous Lévy process under Lipschitz generator. Later there have been several extensions namely [1, 6]. Note that, their motivation consists in providing the link between generalized BSDE driven by a homogeneous Lévy process and a class of partial differential integral equations with Neumann boundary condition.

In another context, El Karoui and Huang [7] considered the so-called non-deterministic Lipschitz condition, where the generator is Lipschitz continuous in (y, z) but with constants which are actually random processes themselves. Many works have discussed this subject as well as [2, 12].

The main motivation of our work is that the assumptions on the driver that we consider here are much involved in partial differential equations. We propose a model aimed at extending the usage of model [11] (see p.551) with non-deterministic coefficients:

$$X_t^x = x + \int_0^t b_s(X_s^x)ds + \int_0^t \sigma_s(X_s^x)dB_s + \int_0^t \triangledown\varphi(X_s^x)dK_s^x, \quad t \in [0, T]. \tag{1}$$

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where $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$ and $(X_t^x, K_t^x)_{t\leq T}$ is a pair of progressively measurable continuous processes such that K^x is increasing process. Since the parameters b_t and σ_t are supposed neither deterministic nor bounded, we cannot apply the existing results of BSDEs under Lipschitz generator. Consequently, we are going to enhance the conditions on the generator by considering the non-deterministic Lipschitz condition. Furthermore, we go a step further towards generality and consider the time-inhomogeneous Lévy processes. They are usually as easy to handle as models driven by Lévy processes but allow for additional flexibility.

Motivated by the above contributions, we mainly prove, in this paper, the existence and uniqueness of solutions of generalized backward stochastic differential equations driven by the non-homogeneous Lévy process. By applying the Picard's iteration, we are going to construct the unique solution of generalized BSDEL where the coefficients f is non-deterministic Lipschitz and the coefficient g satisfies the monotone condition.

This paper is structured as follows: In Section 2 the notations and several results which are significant for our analysis are introduced. In section 3, we will prove our main result which is the existence and uniqueness of a solution to generalized BSDELs in the non-homogeneous case under non-deterministic Lipschitz coefficient. Section 4 is devoted to proving a priori estimates for the considered class of generalized BSDELs.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space on which a real-valued inhomogeneous Lévy process $(X_t)_{t\in[0,T]}$ with càdlàg paths is defined. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the right-continuous filtration generated by X: $(\mathcal{F}_t = \sigma\{X_s; s \leq t\})$ and assume that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. The non-homogenous Lévy process X is characterized by:

$$\mathbb{E}\left(\exp^{iuX_t}\right) = \exp\int_0^t \left(iub_s - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} e^{iux} - 1 - iux\mathbb{I}_{\{|x|\leq 1\}}F_s(dx)\right) ds.$$

Here $b_s \in \mathbb{R}$, $c_s \in \mathbb{R}_+^*$ and F_s is a measure on $\mathbb{R}$ that integrates $(1 \wedge |x|^2)$ and satisfies $F_s(\{0\}) = 0$. Furthermore, we assume that

(*i*) The drift term $b_s \in \mathbb{R}$, the volatility coefficients $c_s > 0$ and the Lévy measure F_s satisfy

$$\int_0^T \left(|b_s| + |c_s| + \int_{\mathbb{R}} (1 \wedge |x|^2) F_s(dx)\right) ds < \infty.$$

(*ii*) There are two constants $M, \varepsilon > 0$ such that

$$\int_0^T \int_{|x|>1} e^{ux} F_s(dx)ds < \infty, \ \forall u \in [-(1+\varepsilon)M, (1+\varepsilon)M].$$

In the sequel, we denote by $X_{t-} = \lim_{s\nearrow t} X_s$ and $\Delta X_t = X_t - X_{t-}$. We define the power jumps of the Lévy process X by

$$X_t^{(1)} = X_t \quad \text{and} \quad X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i, \ i \geq 2.$$

Let us put the Teugels martingales $Y_t^{(i)} = X_t^{(i)} - \mathbb{E}[X_t^{(i)}]$ for all $i \geq 1$. We associate with the non-homogeneous Lévy process $(X_t)_{0\leq t\leq T}$ the family of processes $(H^{(i)})_{i\geq 1}$ defined by $H_t^{(i)} = \sum_{j=1}^i \alpha_{ij} Y_t^{(j)}$. The coefficients α_{ij} correspond to the orthonormalization of the polynomials 1, x, x^2 etc. with respect to the measure $\pi([0,t], dx) = \int_0^t c_s\delta_0(dx)ds + \int_0^t \int_{\mathbb{R}} x^2 F_s(dx)ds$. Specifically, the polynomials q_n defined by $q_n(x) = \sum_{k=1}^n \alpha_{nk}x^{k-1}$ are orthonormal with respect to the measure π, i.e.

$$\int_{\mathbb{R}} q_n(x)q_m(x)\pi(dx) = 0 \text{ if } n \neq m.$$

Note that the martingales $H^{(i)}$ are strongly orthogonal and its predictable quadratic variation process is

$$d\langle H^{(i)}, H^{(j)}\rangle_t = |\vartheta_t^{(i)}||\vartheta_t^{(j)}|\delta_{ij}dt,$$

where $(\vartheta_t)_{t\leq T}$ is a deterministic continuous function.

Now, let $\beta > 0$, $\gamma \geq 0$ and $(a_t)_{t\leq T}$ be a non-negative $\mathcal{F}_t$-adapted process. We define the increasing process $A_t := \int_0^t a_s^2 ds$ and denote by $(C_t)_{t\leq T}$ the one-dimensional continuous, increasing, $\mathcal{F}_t$-progressively measurable process satisfying $C_0 = 0$. Next, we introduce the following spaces:

- $\mathcal{L}^2_{\beta,\gamma}$ is the space of $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variables ξ such that

$$\|\xi\|^2_{\mathcal{L}^2_{\beta,\gamma}} = \mathbb{E}\left[e^{\beta A_T + \gamma C_T}|\xi|^2\right] < +\infty.$$

- $\mathcal{S}^2_{\beta,\gamma}$, $\mathcal{S}^{2,a}_{\beta,\gamma}$ and $\mathcal{S}^{2,c}_{\beta,\gamma}$: the spaces of $\mathbb{R}$-valued and $\mathcal{F}_t$-adapted continuous processes $(Y_t)_{t\leq T}$ such that

$$\|Y\|^2_{\mathcal{S}^2_{\beta,\gamma}} = \mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t + \gamma C_t}|Y_t|^2\right] < +\infty;$$

$$\|Y\|^2_{\mathcal{S}^{2,a}_{\beta,\gamma}} = \mathbb{E}\left[\int_0^T e^{\beta A_t + \gamma C_t}|a_t Y_t|^2 dt\right] < +\infty;$$

$$\|Y\|^2_{\mathcal{S}^{2,c}_{\beta,\gamma}} = \mathbb{E}\left[\int_0^T e^{\beta A_t + \gamma C_t}|Y_t|^2 dC_t\right] < +\infty.$$

- $\mathcal{H}^2_{\beta,\gamma}$ is the space of ℓ^2-valued and $\mathcal{F}_t$-progressively measurable processes $(Z_t)_{t\leq T}$ such that

$$\|Z\|^2_{\mathcal{H}^2_{\beta,\gamma}} = \mathbb{E}\int_0^T e^{\beta A_t + \gamma C_t}\|Z_t\vartheta_t\|^2_{\ell^2}dt = \sum_{k=1}^{\infty}\mathbb{E}\int_0^T e^{\beta A_t + \gamma C_t}|Z_t^{(k)}\vartheta_t^{(k)}|^2 dt < +\infty.$$

- $\mathcal{M}^2_{\beta,\gamma} := \mathcal{S}^{2,a}_{\beta,\gamma} \cap \mathcal{S}^{2,c}_{\beta,\gamma}$.
- $\mathcal{B}^2_{\beta,\gamma} := \mathcal{S}^2_{\beta,\gamma} \cap \mathcal{S}^{2,a}_{\beta,\gamma} \cap \mathcal{S}^{2,c}_{\beta,\gamma} \times \mathcal{H}^2_{\beta,\gamma}$ and $\mathcal{B}^2_\beta := \mathcal{B}^2_{\beta,0}$.

Let's get back to our main problem. The parameters b and σ are not bounded in general since they are stochastic. Consequently, the main objective of this paper is to complete the above works and to study the following generalized BSDEL associated with partial differential equation (1) when the noise is given by the non-homogeneous Lévy process:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s)dC_s - \sum_{k=1}^{\infty}\int_t^T Z_s dH_s^{(k)}, \quad t \in [0, T]. \tag{2}$$

A solution of generalized BSDEL is a pair of $\mathcal{F}_t$–progressively measurable processes $(Y, Z) \in \mathcal{B}^2_\beta$ and satisfies (2) such that the data ξ, f, g and C satisfies the following assumptions:

($\mathcal{A}$.1) The terminal value $\xi \in \mathcal{L}^2_{\beta,\gamma}$.

($\mathcal{A}$.2) The maps $f : \Omega \times [0, T] \times \mathbb{R} \times \ell^2 \to \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ are such that

(1) For all $(y, z) \in \mathbb{R} \times \ell^2$, $f(., y, z)$ is $\mathcal{F}$–progressively measurable and

$$\mathbb{E}\int_0^T e^{\beta A_s + \gamma C_s}\left|\frac{f(s, 0, 0)}{a_s}\right|^2 ds < +\infty.$$

(2) There exists $\mathcal{F}_t$-adapted processes $(p_t)_{t\leq T}$ and $(q_t)_{t\leq T}$ with values in $\mathbb{R}^+$ such that For all $t \in [0, T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \ell^2$

$$|f(t, y, z) - f(t, y', z')| \leq p_t|y - y'| + q_t\|(z - z')\vartheta\|_{\ell^2}.$$

(3) For all $t \in [0,T]$, the function $y \to g(t,y)$ is continuous a.s. Furthermore, for all $y \in \mathbb{R}$, $g(.,y)$ is $\mathcal{F}$–progressively measurable and there exists $\theta < 0$ and $\kappa > 0$ such that, for all $t \in [0,T]$ and $y, y' \in \mathbb{R}$

$$(y-y')(g(t,y)-g(t,y')) \leq \theta|y-y'|^2 \text{ and } |g(t,y)| \leq \psi(t)+\kappa|y|,$$

where the adapted process $(\psi(t))_{t\leq T}$ with values in $[1,+\infty[$ such that

$$\mathbb{E}\int_0^T e^{\beta A_t+\gamma C_t}|\psi(t)|^2 dC_t < +\infty.$$

(4) There exists $\epsilon > 0$ such that $a_t^2 = p_t + q_t^2 \geq \epsilon, \ \forall\, t \in [0,T]$.

3. Existence and uniqueness result of generalized BSDEL

Let us begin with the following uniqueness result for generalized BSDEL (2) under the previous assumptions:

Proposition 3.1. *Under the assumptions $(\mathcal{A}.1)$ and $(\mathcal{A}.2)$, the generalized BSDEL (2) has at most one solution.*

Proof. Let (Y,Z) and (Y',Z') be two solutions of generalized BSDEL (2). We apply Itô's formula to $e^{\beta A_t}|Y_t - Y_t'|^2$ for $t \in [0,T]$ to obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{\beta A_t}|Y_t - Y_t'|^2\right] + \beta\mathbb{E}\int_t^T e^{\beta A_s}|Y_s - Y_s'|^2 dA_s + \mathbb{E}\int_t^T e^{\beta A_s}\|(Z_s - Z_s')\vartheta_s\|_{\ell^2}^2 ds\\
&\qquad = 2\mathbb{E}\int_t^T e^{\beta A_s}(Y_s - Y_s')\left(f(s,Y_s,Z_s) - f(s,Y_s',Z_s')\right) ds\\
&\qquad\quad + 2\mathbb{E}\int_t^T e^{\beta A_s}(Y_s - Y_s')\left(g(s,Y_s) - g(s,Y_s')\right) dC_s\\
&\qquad \leq 2\mathbb{E}\int_t^T e^{\beta A_s} p_s|Y_s - Y_s'|^2 ds + 2\mathbb{E}\int_t^T e^{\beta A_s} q_s|Y_s - Y_s'|\|(Z_s - Z_s')\vartheta_s\|_{\ell^2} ds\\
&\qquad\quad + 2\theta\mathbb{E}\int_t^T e^{\beta A_s}|Y_s - Y_s'|^2 dC_s\\
&\qquad \leq 2\mathbb{E}\int_t^T e^{\beta A_s}|Y_s - Y_s'|^2 dA_s + \frac{1}{2}\mathbb{E}\int_t^T e^{\beta A_s}\|(Z_s - Z_s')\vartheta_s\|_{\ell^2}^2 ds.
\end{aligned}$$

For $\beta > 2$, we conclude that $Y = Y'$ and $Z = Z'$. □

Let us state the main result of this paper:

Theorem 3.1. *Under the assumptions $(\mathcal{A}.1)$ and $(\mathcal{A}.2)$, the generalized BSDEL (2) has a unique solution.*

We now prove the existence result under an additional assumption. We suppose that g is κ-Lipschitz, i.e. for all $t \in [0,T]$ and $(y,y') \in \mathbb{R}^2$:

$(\mathcal{H}.1)$ $|g(t,y) - g(t,y')| \leq \kappa|y-y'|$.

Theorem 3.2. *Under the assumptions $(\mathcal{A}.1), (\mathcal{A}.2)$ and $(\mathcal{H}.1)$, there exists at most one progressively measurable process (Y,Z) solution of the generalized BSDEL (2).*

Proof. We first consider the special case when the generator does not depend on (y,z), i.e.

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s)dC_s - \sum_{k=1}^{\infty}\int_t^T Z_s^{(k)} dH_s^{(k)}, \quad t \in [0,T].$$

Using Schwartz inequality one has

$$\begin{aligned}
&\mathbb{E}\left[\xi+\int_0^T f(s)ds+\int_0^T g(s)dC_s\right]^2\\
&\leq 3\left\{\mathbb{E}\left[|\xi|^2\right]+\frac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{f(s)}{a_s}\right|^2 ds\right]+\frac{1}{\gamma}\mathbb{E}\left[\int_0^T e^{\gamma C_s}|g(s)|^2 dC_s\right]\right\}<+\infty.
\end{aligned}$$

The martingale representation theorem for time-inhomogeneous Lévy processes (see El Jamali and El Otmani [8]) implies that there exists a unique predictable process Z, verifying $\mathbb{E}\int_0^T |Z_s|^2 ds<+\infty$, such that

$$\xi+\int_0^T f(s)ds+\int_0^T g(s)dC_s=\mathbb{E}\left[\xi+\int_0^T f(s)ds+\int_0^T g(s)dC_s\right]+\sum_{k=1}^{\infty}\int_0^T Z_s^{(k)}dH_s^{(k)}.$$

Let $Y_t=\mathbb{E}\left[\xi+\int_t^T f(s)ds+\int_t^T g(s)dC_s/\mathcal{F}_t\right]$. Then the process (Y,Z) verifies the generalized BSDEL (2) which coefficients are dependent only on time. On the other hand, we can show that

$$\begin{aligned}
e^{\beta A_t}|Y_t|^2 &= e^{\beta A_t}\mathbb{E}\left[\xi+\int_t^T f(s)ds+\int_t^T g(s)dC_s\big|\mathcal{F}_t\right]^2\\
&\leq 3\mathbb{E}\left[e^{\beta A_T}|\xi|^2+e^{\beta A_t}\left|\int_t^T a_s e^{-\frac{\beta A_s}{2}}e^{\frac{\beta A_s}{2}}\frac{f(s)}{a_s}ds\right|^2\right.\\
&\qquad\left.+e^{\beta A_t}\left|\int_t^T e^{\frac{\beta A_s+\gamma C_s}{2}}e^{-\frac{\beta A_s+\gamma C_s}{2}}g(s)dC_s\right|^2\big|\mathcal{F}_t\right]\\
&\leq 3\mathbb{E}\left[e^{\beta A_T}|\xi|^2+e^{\beta A_t}\int_t^T a_s^2 e^{-\beta A_s}ds\int_t^T e^{\beta A_s}\left|\frac{f(s)}{a_s}\right|^2 ds\right.\\
&\qquad\left.+\int_t^T e^{\beta(A_t-A_s)-\gamma C_s}dC_s\int_t^T e^{\beta A_s+\gamma C_s}|g(s)|^2 dC_s\big|\mathcal{F}_t\right]\\
&\leq 3\mathbb{E}\left[e^{\beta A_T}|\xi|^2+e^{\beta A_t}\int_t^T e^{-\beta A_s}dA_s\int_t^T e^{\beta A_s}\left|\frac{f(s)}{a_s}\right|^2 ds\right.\\
&\qquad\left.+\int_t^T e^{-\gamma C_s}dC_s\int_t^T e^{\beta A_s+\gamma C_s}|g(s)|^2 dC_s\big|\mathcal{F}_t\right]\\
&\leq 3\mathbb{E}\left[e^{\beta A_T}|\xi|^2+\frac{1}{\beta}\int_0^T e^{\beta A_s}\left|\frac{f(s)}{a_s}\right|^2 ds+\frac{1}{\gamma}\int_0^T e^{\beta A_s+\gamma C_s}|g(s)|^2 dC_s\big|\mathcal{F}_t\right].
\end{aligned}$$

By Doob's maximal quadratic inequality, we deduce that $\mathbb{E}\left[\sup_{0\leq t\leq T}e^{\beta A_t}|Y_t|^2\right]<+\infty$. Moreover, using Itô's formula to $e^{\beta A_t}|Y_t|^2$, we have

$$\begin{aligned}
&\beta\mathbb{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s+\mathbb{E}\int_0^T e^{\beta A_s}\|Z_s\vartheta_s\|_{\ell^2}^2 ds\\
&\leq \mathbb{E}\left[e^{\beta A_T}|\xi|^2\right]+2\mathbb{E}\int_0^T e^{\beta A_s}|Y_s||f(s)|ds+2\mathbb{E}\int_0^T e^{\beta A_s}|Y_s||g(s)|dC_s\\
&\qquad+2\mathbb{E}\sup_{0\leq t\leq T}\left|\sum_{k=1}^{\infty}\int_t^T e^{\beta A_s}Y_s Z_s^{(k)}dH_s^{(k)}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] + (\beta-1)\mathbb{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s + \frac{1}{\beta-1}\mathbb{E}\int_0^T e^{\beta A_s}\left|\frac{f(s)}{a_s}\right|^2 ds \\
&\quad + \frac{1}{\gamma}\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|g(s)|^2 dC_s + 2(1+c^2)\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}|Y_t|^2\right] \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^T e^{\beta A_s}\|Z_s\vartheta_s\|^2_{\ell^2} ds.
\end{aligned}
$$

We conclude that the pair (Y, Z) is a solution of generalized BSDEL (2) which coefficients are dependent only on time.

Now, we define the sequence (Y^n, Z^n) as follows: $(Y^0, Z^0) = (0, 0)$ and (Y^{n+1}, Z^{n+1}) is the unique solution of the generalized BSDEL

$$
Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + \int_t^T g(s, Y_s^n)dC_s - \sum_{k=1}^{\infty}\int_t^T Z_s^{(k),n+1}dH_s^{(k)}, \quad t \in [0, T].
$$

We shall prove that (Y^n, Z^n) is a Cauchy sequence in the Banach space $\mathcal{B}^2_{\beta,\gamma}$ with the norm

$$
\|(Y, Z)\|^2_{\mathcal{B}^2_{\beta,\gamma}} := \|Y\|^2_{\mathcal{M}^2_{\beta,\gamma}} + \|Z\|^2_{\mathcal{H}^2_{\beta,\gamma}}.
$$

Note that for $\beta > 3$ and $\gamma > 1 + 2\kappa^2$, we can show that

$$
\begin{aligned}
\sup_{n\geq 0}\|(Y^n, Z^n)\|^2_{\mathcal{B}^2_{\beta,\gamma}} \leq c_{\beta,\gamma}\Bigg\{&\mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right] + \mathbb{E}\int_0^T e^{\beta A_t+\gamma C_t}\left|\frac{f(t,0,0)}{a_t}\right|^2 dt \\
&+\mathbb{E}\int_0^T e^{\beta A_t+\gamma C_t}|\psi(t)|^2 dC_t\Bigg\}.
\end{aligned}
$$

For $n \geq m \geq 1$, let us put $\eta^{n,m} = \eta^n - \eta^m$ for $\eta = Y, Z$. We apply Itô's formula to obtain

$$
\begin{aligned}
&\beta\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|Y_s^{n+1,m+1}|^2 dA_s + \gamma\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|Y_s^{n+1,m+1}|^2 dC_s \\
&\qquad + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\|Z_s^{n+1,m+1}\vartheta_s\|^2_{\ell^2} ds \\
&\leq 2\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\left|Y_s^{n+1,m+1}\right|\left|f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)\right| ds \\
&\quad + 2\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\left|Y_s^{n+1,m+1}\right|\left|g(s, Y_s^n) - g(s, Y_s^m)\right| dC_s \\
&\leq (\beta-1)\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|Y_s^{n+1,m+1}|^2 dA_s \\
&\quad + \frac{1}{\beta-1}\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\left(|Y_s^{n,m}|^2 dA_s + \|Z_s^{n,m}\vartheta_s\|^2_{\ell^2} ds\right) \\
&\quad + (\gamma-1)\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|Y_s^{n+1,m+1}|^2 dC_s + \frac{\kappa^2}{\gamma-1}\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|Y_s^{n,m}|^2 dC_s.
\end{aligned}
$$

Choosing $\beta > 2$ and $\gamma > 1 + \kappa^2$ then $\bar{\kappa} = \max\{1/(\beta-1); \kappa^2/(\gamma-1)\} \in]0; 1[$. We deduce that

$$
\|(Y^{n,m}, Z^{n,m})\|^2_{\mathcal{B}^2_{\beta,\gamma}} \leq \bar{\kappa}^{m+1}\|(Y^{n-m}, Z^{n-m})\|^2_{\mathcal{B}^2_{\beta,\gamma}} \underset{n,m\to+\infty}{\longrightarrow} 0.
$$

Which implies that (Y^n, Z^n) is a Cauchy sequence and converges to $(Y, Z) \in \mathcal{B}^2_{\beta,\gamma}$ satisfying (2). $\square$

Proof of Theorem 3.1. Let $g_n(t,y) = (\rho_n * g(t,.))(y)$ with $\rho_n : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a sequence of smooth functions which approximates the Dirac measure at 0 and satisfies $\int_{\mathbb{R}} \rho_n(x)dx = 1$ and $\sup_{n\geq 0}\int_{\mathbb{R}} |x|\rho_n(x)dx < +\infty$.
Remark that g_n is uniformly Lipschitz in y :

$$\begin{aligned}
|g_n(t,y) - g_n(t,y)| &= |(\rho_n * g(t,.))(y) - (\rho_n * g(t,.))(y')| \\
&\leq \int_{\mathbb{R}} \rho_n(x)\,|g(t,y-x) - g(t,y'-x)|\,dx \\
&\leq \kappa|y-y'|\int_{\mathbb{R}} \rho_n(x)dx = \kappa|y-y'|.
\end{aligned}$$

Moreover, it satisfies $(\mathcal{A}.2)(3)$. Indeed

$$\begin{aligned}
(y-y')[g_n(t,y) - g_n(t,y)] &= (y-y')[(\rho_n * g(t,.))(y) - (\rho_n * g(t,.))(y')] \\
&\leq \int_{\mathbb{R}} \rho_n(x)((y-x)-(y'-x))(g(t,y-x) - g(t,y'-x))dx \\
&\leq \theta|y-y'|^2\int_{\mathbb{R}} \rho_n(x)dx = \theta|y-y'|^2,
\end{aligned}$$

and

$$\begin{aligned}
|g_n(t,y)| &= |(\rho_n * g(t,.))(y)| \leq \int_{\mathbb{R}} \rho_n(x)|g(t,y-x)|dx \\
&\leq \psi(t)\int_{\mathbb{R}} \rho_n(x)dx + \kappa\int_{\mathbb{R}} \rho_n(x)|y-x|dx \\
&\leq (\psi(t)+\kappa|y|)\int_{\mathbb{R}} \rho_n(x)dx + \kappa\int_{\mathbb{R}} |x|\rho_n(x)dx = \psi(t)+\kappa|y|.
\end{aligned}$$

Hence, there exists a unique solution (y^n, z^n) of the generalized BSDEL

$$y_t^n = \xi + \int_t^T f(s,y_s^n,z_s^n)ds + \int_t^T g_n(s,y_s^n)dC_s - \sum_{k=1}^{\infty}\int_t^T z_s^{(k),n}dH_s^{(k)}.$$

Applying Itô's formula and taking an expectation on both sides, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[e^{\beta A_t+\gamma C_t}|y_t^n|^2\right] + \beta\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|y_s^n|^2dA_s \\
&\quad + \gamma\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|y_s^n|^2dC_s + \mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}\|z_s^n\vartheta_s\|_{\ell^2}^2ds \\
&\quad = \mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right] + 2\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}y_s^n f(s,y_s^n,z_s^n)ds \\
&\qquad + 2\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}y_s^n g_n(s,y_s^n)dC_s \\
&\quad \leq \mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right] + 2\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}y_s^n f(s,0,0)ds \\
&\quad + 2\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}p_s|y_s^n|^2ds + 2\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}q_s|y_s^n|\|z_s^n\vartheta_s\|_{\ell^2}ds \\
&\quad + 2\theta\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|y_s^n|^2dC_s + 2\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}y_s^n g_n(s,0)dC_s \\
&\quad \leq \mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right] + 3\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|y_s^n|^2dA_s
\end{aligned}$$

$$+\frac{1}{2}\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}\|z_s^n\vartheta_s\|_{\ell^2}^2 ds+\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds$$
$$+\frac{\gamma}{2}\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|y_s^n|^2 dC_s+\frac{2}{\gamma}\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|\psi(s)|^2 dC_s.$$

Which implies that

$$\mathbb{E}\left[e^{\beta A_t+\gamma C_t}|y_t^n|^2\right]+(\beta-3)\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|y_s^n|^2 dA_s$$
$$+\frac{\gamma}{2}\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|y_s^n|^2 dC_s+\frac{1}{2}\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}\|z_s^n\vartheta_s\|_{\ell^2}^2 ds$$
$$\leq \mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right]+\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds+\frac{2}{\gamma}\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|\psi(s)|^2 dC_s.$$

Therefore, Burkholder-Davis-Gundy's inequality leads to

$$\sup_{n\geq 0}\left\{\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t+\gamma C_t}|y_t^n|^2\right]+\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|y_s^n|^2 dA_s+\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|y_s^n|^2 dC_s\right.$$
$$\left.+\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\|z_s^n\vartheta_s\|_{\ell^2}^2 ds\right\}$$
$$\leq c\left\{\mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right]+\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds\right.$$
$$\left.+\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|\psi(s)|^2 dC_s\right\}.$$

Defining $\mathcal{U}_t^n = f(t,y_t^n,z_t^n)$ and $\mathcal{V}_t^n = g_n(t,y_t^n)$. We deduce from the above and our assumptions that

$$\sup_{n\geq 0}\mathbb{E}\left[\int_0^T|\mathcal{U}_t^n|^2 dt+\int_0^T|\mathcal{V}_t^n|^2 dC_t\right]\leq\frac{3}{\beta}\mathbb{E}\int_t^T e^{\beta A_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds$$
$$+2\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|\psi(s)|^2 dC_s+\sup_{n\geq 0}\mathbb{E}\left[\frac{3}{\beta}\mathbb{E}\int_t^T e^{\beta A_s}|y_s^n|^2 dA_s\right.$$
$$\left.+\frac{3}{\beta}\mathbb{E}\int_t^T e^{\beta A_s}\|z_s^n\vartheta_s\|_{\ell^2}^2 ds+2\kappa^2\int_0^T e^{\beta A_s+\gamma C_s}|y_s^n|^2 dC_s\right]<+\infty.$$

The sequences $(y^n)_{n\geq 0},(z^n)_{n\geq 0},(\mathcal{U}^n)_{n\geq 0}$ et $(\mathcal{V}^n)_{n\geq 0}$ are bounded. By Bolzano-Weierstrass theorem (Bartle and Sherbet [4], Theorem 3.4.2, p.78), we can extract a convergent subsequence such that

$$(y^{n_k},z^{n_k},\mathcal{U}^{n_k},\mathcal{V}^{n_k})\xrightarrow[k\to+\infty]{}(Y,Z,\mathcal{U},\mathcal{V}).$$

It is then easy to deduce that

$$Y_t=\xi+\int_t^T\mathcal{U}_s ds+\int_t^T\mathcal{V}_s dC_s-\sum_{k=1}^{\infty}\int_t^T Z_s^{(k)}dH_s^{(k)}.$$

Now, to reach our goal, it remains to show that $\mathcal{U}_t=f(t,Y_t,Z_t)$ and $\mathcal{V}_t=g(t,Y_t)$. Let $(\mathcal{P}_t)_{t\leq T},(\mathcal{Q}_t)_{t\leq T}$ and $(\mathcal{X}_t)_{t\leq T}$ be three progressively measurable processes such that

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}(|\mathcal{P}_t|^2+\|\mathcal{Q}_t\vartheta_t\|_{\ell^2}^2)dt\right]<+\infty\quad\text{and}\quad\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathcal{X}_t|^2 dC_t\right]<+\infty.$$

In view of the non-deterministic Lipschitz condition of f and the monotonicity condition of g, we have

$$\begin{aligned}&f(t,y^n_t,z^n_t)-f(t,\mathcal{P}_t,\mathcal{Q}_t)-p_t|y^n_t-\mathcal{P}_t|-q_t\|(z^n_t-\mathcal{Q}_t)\vartheta_t\|_{\ell^2}\\&\leq|f(t,y^n_t,z^n_t)-f(t,\mathcal{P}_t,\mathcal{Q}_t)|-p_t|y^n_t-\mathcal{P}_t|-q_t\|(z^n_t-\mathcal{Q}_t)\vartheta_t\|_{\ell^2}\leq 0,\end{aligned}$$

and $(y^n_t-\mathcal{X}_t)(g_n(t,y^n_t)-g_n(t,\mathcal{X}_t))\leq\theta|y^n_t-\mathcal{X}_t|^2\leq 0$. Therefore

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{\beta A_t}|y^n_t-\mathcal{P}_t|(f(t,y^n_t,z^n_t)-f(t,\mathcal{P}_t,\mathcal{Q}_t)-p_t|y^n_t-\mathcal{P}_t|-q_t\|(z^n_t-\mathcal{Q}_t)\vartheta_t\|_{\ell^2}dt\right]\\+\mathbb{E}\left[\int_0^T e^{\beta A_t}(y^n_t-\mathcal{X}_t)(g_n(t,y^n_t)-g_n(t,\mathcal{X}_t))dC_t\right]\leq 0.\end{aligned}$$

Note that $\mathbb{E}\left[\int_0^T e^{\beta A_t}|g_n(t,\mathcal{X}_t)-g(t,\mathcal{X}_t)|^2dC_t\right]\xrightarrow[n\to+\infty]{}0$, we get

$$\begin{aligned}\limsup_{n\to+\infty}\mathbb{E}\left[\int_0^T e^{\beta A_t}|y^n_t-\mathcal{P}_t|(f(t,y^n_t,z^n_t)-f(t,\mathcal{P}_t,\mathcal{Q}_t)-p_t|y^n_t-\mathcal{P}_t|\right.\\\left.-q_t\|(z^n_t-\mathcal{Q}_t)\vartheta_t\|_{\ell^2})dt\right]+\mathbb{E}\left[\int_0^T e^{\beta A_t}(y^n_t-\mathcal{X}_t)(g_n(t,y^n_t)-g(t,\mathcal{X}_t))dC_t\right]\leq 0.\end{aligned}\tag{3}$$

On the other hand, Itô's formula implies that

$$\begin{aligned}|y^n_0|^2&=e^{\beta A_T}|\xi|^2-\beta\int_0^T e^{\beta A_t}|y^n_t|^2dA_t+2\int_0^T e^{\beta A_t}y^n_tf(t,y^n_t,z^n_t)dt\\&+2\int_0^T e^{\beta A_t}y^n_tg_n(t,y^n_t)dC_t-\int_0^T e^{\beta A_t}\|z^n_t\vartheta_t\|^2_{\ell^2}dt-2\sum_{k=1}^{\infty}\int_0^T e^{\beta A_t}y^n_tz^{(k),n}_tdH^{(k)}_t.\end{aligned}$$

Using the fact that $y^n_0\xrightarrow[n\to+\infty]{}Y_0$ in $\mathbb{R}$, and that the mapping $Z\longrightarrow\mathbb{E}\int_0^T e^{\beta A_t}\|Z_t\vartheta_t\|^2_{\ell^2}dt$ is convex and continuous in $\mathcal{H}^2_{\beta,0}$, hence

$$\begin{aligned}&\liminf_{n\to+\infty}2\mathbb{E}\left[\int_0^T e^{\beta A_t}y^n_t\left(f(t,y^n_t,z^n_t)-\frac{\beta}{2}a^2_ty^n_t\right)dt\right]+2\mathbb{E}\left[\int_0^T e^{\beta A_t}y^n_tg_n(t,y^n_t)dC_t\right]\\&\geq|Y_0|^2-\mathbb{E}\left[e^{\beta A_T}|\xi^2|\right]+\mathbb{E}\left[\int_0^T e^{\beta A_t}\|Z_t\vartheta_t\|^2_{\ell^2}dt\right]\\&=2\mathbb{E}\left[\int_0^T e^{\beta A_t}Y_t\left(\mathcal{U}_t-\frac{\beta}{2}a^2_tY_t\right)dt\right]+2\mathbb{E}\left[\int_0^T e^{\beta A_t}Y_t\mathcal{V}_tdC_t\right].\end{aligned}\tag{4}$$

Combining (3) together with (4) yields

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{\beta A_t}|Y_t-\mathcal{P}_t|(\mathcal{U}_t-f(t,\mathcal{P}_t,\mathcal{Q}_t)-p_t|Y_t-\mathcal{P}_t|-q_t\|(Z_t-\mathcal{Q}_t)\vartheta_t\|_{\ell^2})dt\right.\\\left.+\int_0^T e^{\beta A_t}(Y_t-\mathcal{X}_t)(\mathcal{V}_t-g(t,\mathcal{X}_t))dC_t\right]\leq 0.\end{aligned}$$

We choose $\mathcal{P}_t=Y_t-\varepsilon(\mathcal{U}_t-f(t,Y_t,Z_t))$, $\mathcal{Q}_t=Z_t$ and $\mathcal{X}_t=Y_t-\varepsilon(\mathcal{V}_t-g(t,Y_t))$ for $\varepsilon>0$, then divide by ε and let $\varepsilon\to 0$, the following holds

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathcal{U}_t-f(t,Y_t,Z_t)|^2dt+\int_0^T e^{\beta A_t}|\mathcal{V}_t-g(t,Y_t)|^2dC_t\right]\leq 0.$$

Which leads to the conclusion of this section, i.e. (Y, Z) is the solution of generalized BSDEL (2). The proof of theorem (3.1) is now complete. $\square$

4. A priori estimates

In this part, we state the priori estimates on the bounds of the solution (Y, Z) with respect to the data (ξ, f, g, C).

Proposition 4.1. *Assuming that $(\mathcal{A}.1)-(\mathcal{A}.2)$ hold. Let (Y, Z) be a solution of generalized BSDEL with data (ξ, f, g, C), then there exists a constant c depending on $\beta > 3$ and $\gamma > 0$ such that*

$$\begin{aligned}
&\|Y\|^2_{\mathcal{S}^2_{\beta,\gamma}} + \|Y\|^2_{\mathcal{S}^{2,a}_{\beta,\gamma}} + \|Y\|^2_{\mathcal{S}^{2,c}_{\beta,\gamma}} + \|Z\|^2_{\mathcal{H}^2_{\beta,\gamma}} \\
&\qquad \leq c_{\beta,\gamma}\Bigg\{\mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right] + \mathbb{E}\int_0^T e^{\beta A_t+\gamma C_t}\left|\frac{f(t,0,0)}{a_t}\right|^2 dt \\
&\qquad\qquad + \mathbb{E}\int_0^T e^{\beta A_t+\gamma C_t}|\psi(t)|^2 dC_t\Bigg\}.
\end{aligned}$$

Proof. Applying Itô's formula (Protter [10], Theorem 33, p.81) to $e^{\beta A_t+\gamma C_t}|Y_t|^2$ yields that

$$\begin{aligned}
&e^{\beta A_t+\gamma C_t}|Y_t|^2 + \beta\int_t^T e^{\beta A_s+\gamma C_s}|Y_s|^2 dA_s \\
&\qquad + \gamma\int_t^T e^{\beta A_s+\gamma C_s}|Y_s|^2 dC_s + \int_t^T e^{\beta A_s+\gamma C_s}\|Z_s\vartheta_s\|^2_{\ell^2} ds \\
&\qquad = e^{\beta A_T+\gamma C_T}|\xi|^2 + 2\int_t^T e^{\beta A_s+\gamma C_s} Y_s f(s, Y_s, Z_s) ds \\
&\qquad + 2\int_t^T e^{\beta A_s+\gamma C_s} Y_s g(s, Y_s) dC_s - 2\sum_{k=1}^{\infty}\int_t^T e^{\beta A_s+\gamma C_s} Y_s Z_s^{(k)} dH_s^{(k)}.
\end{aligned}$$

On the one hand, using the assumption $(\mathcal{A}.2)(2)$ and $(\mathcal{A}.2)(3-4)$ respectively, we obtain

$$\begin{aligned}
2Y_s f(s, Y_s, Z_s) &\leq 2|Y_s||f(s,0,0)| + 2p_s|Y_s|^2 + 2q_s|Y_s|\|Z_s\vartheta_s\|_{\ell^2} \\
&\leq 3a_s^2|Y_s|^2 + \left|\frac{f(s,0,0)}{a_s}\right|^2 + \frac{1}{2}\|Z_s\vartheta_s\|^2_{\ell^2}; \\
2Y_s g(s, Y_s) &\leq 2\theta|Y_s|^2 + 2|Y_s||g(s,0)| \leq \frac{\gamma}{2}|Y_s|^2 + \frac{2}{\gamma}|\psi(s)|^2.
\end{aligned}$$

Consequently, we deduce that

$$\begin{aligned}
(5)\quad &e^{\beta A_t+\gamma C_t}|Y_t|^2 + (\beta-3)\int_t^T e^{\beta A_s+\gamma C_s}|Y_s|^2 dA_s \\
&\qquad + \frac{\gamma}{2}\int_t^T e^{\beta A_s+\gamma C_s}|Y_s|^2 dC_s + \frac{1}{2}\int_t^T e^{\beta A_s+\gamma C_s}\|Z_s\vartheta_s\|^2_{\ell^2} ds \\
&\qquad \leq e^{\beta A_T+\gamma C_T}|\xi|^2 + \int_0^T e^{\beta A_s+\gamma C_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds \\
&\qquad + \frac{2}{\gamma}\int_0^T e^{\beta A_s+\gamma C_s}|\psi(s)|^2 dC_s - 2\sum_{k=1}^{\infty}\int_t^T e^{\beta A_s+\gamma C_s} Y_s Z_s^{(k)} dH_s^{(k)}.
\end{aligned}$$

Thus, taking an expectation in above both sides, yields

$$
\begin{aligned}
(6)\quad & \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|Y_s|^2 dA_s + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|Y_s|^2 dC_s + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\|Z_s\vartheta_s\|^2_{\ell^2} ds \\
& \leq c_{\beta,\gamma}\left\{\mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right] + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds \right. \\
& \qquad \left. + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|\psi(s)|^2 dC_s\right\}.
\end{aligned}
$$

To reach our purpose, we take the supremum over $t \in [0,T]$ in (5), we get that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t+\gamma C_t}|Y_t|^2\right] &\leq \mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right] + \mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds \\
&+ \frac{2}{\gamma}\mathbb{E}\int_t^T e^{\beta A_s+\gamma C_s}|\psi(s)|^2 dC_s + 2\mathbb{E}\sup_{0\leq t\leq T}\left|\sum_{k=1}^{\infty}\int_0^t e^{\beta A_s+\gamma C_s} Y_s Z_s^{(k)} dH_s^{(k)}\right|.
\end{aligned}
$$

From Burkholder-Davis-Gundy's inequality, there exists a positive universal constant c such that

$$
\begin{aligned}
2\mathbb{E}\sup_{0\leq t\leq T}\left|\sum_{k=1}^{\infty}\int_0^t e^{\beta A_s+\gamma C_s} Y_s Z_s^{(k)} dH_s^{(k)}\right| &\leq 2c\mathbb{E}\left[\int_0^T e^{2\beta A_s+2\gamma C_s}|Y_s|^2\|Z_s\vartheta_s\|^2_{\ell^2} ds\right]^{\frac{1}{2}} \\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t+\gamma C_t}|Y_t|^2\right] + 2c^2\mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\|Z_s\vartheta_s\|^2_{\ell^2} ds.
\end{aligned}
$$

Henceforth, we have

$$
\begin{aligned}
(7)\quad & \mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t+\gamma C_t}|Y_t|^2\right] \\
& \leq c_{\beta,\gamma}\left\{\mathbb{E}e^{\beta A_T+\gamma C_T}|\xi|^2 + \mathbb{E}\int_0^T e^{\beta A_t+\gamma C_t}\left|\frac{f(t,0,0)}{a_t}\right|^2 dt \right. \\
& \qquad \left. + \mathbb{E}\int_0^T e^{\beta A_t+\gamma C_t}|\psi(t)|^2 dC_t\right\}.
\end{aligned}
$$

Finally, the desired result is obtained by (6) and (7) :

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t+\gamma C_t}|Y_t|^2\right] + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|Y_s|^2 dA_s \\
& \quad + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|Y_s|^2 dC_s + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\|Z_s\vartheta_s\|^2_{\ell^2} ds \\
& \leq c_{\beta,\gamma}\left\{\mathbb{E}\left[e^{\beta A_T+\gamma C_T}|\xi|^2\right] + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds \right. \\
& \qquad \left. + \mathbb{E}\int_0^T e^{\beta A_s+\gamma C_s}|\psi(s)|^2 dC_s\right\}.
\end{aligned}
$$

□

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