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STRUCTURE OF OPTIMAL STOPPING DOMAINS FOR AMERICAN OPTIONS WITH KNOCK OUT DOMAINS

American options give us the possibility to exercise them at any moment of time up to maturity. An optimal stopping domain for American type options is a domain that, if the underlying price process enters we should exercise the option. A knock out option is a American barrier option of knock out type, but with more general shape structure of the knock out domain. An algorithm for generating the optimal stopping domain for American type knock out options is constructed. Monte Carlo simulation is used to determine the structure of the optimal stopping domain. Results of the structural, and stability of studies are presented for different models of payoff functions and knock out domains.

1. INTRODUCTION

American options give us the possibility to exercise them at any moment of time up to maturity. An optimal stopping domain for American type options is a domain that, if the underlying price process enter it, then the option should be exercised in order to get the maximal payoff. Studies on optimal stopping for Markov type processes were done in Snell (1952), Chow, Robbins and Siegmund (1971), Van Moerbeke (1976), Shiryaev (1978), and Peskir and Shiryaev (2006), optimal stopping for American options in Kukush and Silvestrov (1999, 2004), Jönsson, Kukush and Silvestrov (2004, 2005) and Jönsson (2005). Other papers about optimal stopping and American put options are Jacka (1991) and Salminen (1999).

Similar studies on the structure of optimal stopping domains, as considered in this paper but for American options without knockout domains have been done in Jönsson (2001). In this paper we extend these studies to discrete time knock out American options. An option with knock out

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domain can be seen as a barrier option of knock out type, but with region that knocks out the option of a more general shape than for ordinary barrier options.

Some preceding works about barrier options are Boyle and Lau (1994), Broadie, Glasserman and Kou (1997), Baldi, Caramellino and Ivono (1999) and Lo, Lee and Hui (2003). The use of Monte Carlo simulation in finance was introduced by Boyle (1977). It has been extended in several papers, we like to mention Boyle, Broadie and Glasserman (1997) and Longstaff and Schwartz (2001). A nice survey is presented in the book by Glasserman (2004).

We first describe an algorithm that generates optimal stopping domain for American type knock out options. Monte Carlo simulation is used to determine the structure of the optimal stopping domain. This paper is an extension of Lundgren (2007) and includes a more detailed description of the considered model, proof of the corresponding results about structure of optimal stopping domains, several additional numerical examples of optimal stopping domains for different payoff functions and knock out domains, as well as of the algorithm for estimation of the probabilities of classification errors.

2. AMERICAN OPTIONS WITH KNOCK OUT DOMAINS

We consider a price process to be of Markov type with multiplicative increments

$$S_{x,n}(k+1) = S_{x,n}(k)Y_k, \quad k = n, n+1, \dots, \quad (1)$$

where $Y_k, k = n, n+1, \dots$ are independent non-negative random variables, x and n are starting points for the price process in prices and time, respectively, with the initial value $S_{x,n}(n) = x \in \mathbb{R}_+ = [0, \infty)$.

A payoff is represented by a measurable function $g_n(x) : \mathbb{N} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$. We assume the following integrability condition

$$\mathbb{E}g_k(S_{x,n}(k)) < \infty, \quad 0 \leq n \leq k < \infty, \quad x \in \mathbb{R}_+. \quad (2)$$

Introduce a knockout domain H , and define the rules such that if the underlying process enter the domain, the contract will be worthless. The knock out domain H has the following discrete structure

$$H = \{H_0, H_1, H_2, \dots\}, \quad (3)$$

where $H_n, n = 0, 1, \dots$ are some measurable subsets of \mathbb{R}_+ .

Let us introduce the random time $\tau_{x,n}^H$, the first time when the price process enters the knock out domain,

$$\tau_{x,n}^H = \min\{k \geq n : S_{x,n}(k) \in H_k\}.$$

Let $\mathbf{M}_{x,n}^N$ be the class of all Markov moments τ for the process $S_{x,n}(k)$, $k = n, n+1, \dots$ such that $n \leq \tau \leq N$, where N is a non negative integer number interpreted as an expiration date. Our goal is to maximize the expected payoff

$$\Phi_{x,n}(\tau) = \mathbb{E}e^{-R_\tau} g_\tau(S_{x,n}(\tau)) \chi(\tau < \tau_{x,n}^H) \quad (4)$$

over all Markov moments $\tau \in \mathbf{M}_{x,n}^N$, where $\chi(\tau < \tau_{x,n}^H)$ is the indicator making sure that the time τ is less than the first knock out entry time $\tau_{x,n}^H$, and where $R_n = r_0 + r_1 + \dots + r_n$, $R_0 = 0$ and $r_n \geq 0$ is a non random risk free interest rate between moments n and $n+1$.

The time we are looking for is the optimal stopping time τ_{opt} i.e.,

$$\Phi_{x,n}(\tau_{opt}) = \sup_{\tau \in \mathbf{M}_{x,n}^N} \Phi_{x,n}(\tau) \quad (5)$$

The optimal stopping time for the American knock out options have a structure similar with those for ordinary American options. Define the operator T_n acting on a non-negative measurable function $f(x)$ as

$$T_n f(x) = \mathbb{E}e^{-r_n} f(x \cdot Y_{n+1}) \chi(x \cdot Y_{n+1} \notin H_{n+1}).$$

To determine the optimal stopping moment τ_{opt} the method of backward induction is used, this gives for $n = N-1, N-2, \dots, 0$ the following recursion

$$\begin{cases} \omega_0(x) &= g_N(x) \chi(x \notin H_N) \\ \omega_{N-n}(x) &= \max\{g_n(x), T_n \omega_{N-n-1}(x)\} \chi(x \notin H_n) \end{cases} \quad (6)$$

The set of all asset prices at moment n such that it is optimal strategy to stop form a optimal stopping domain at moments $n = 0, \dots, N$,

$$\Gamma_n = \{x \in \mathbb{R}_+ : g_n(x) = \omega_{N-n}(x)\},$$

consequently the complement to Γ_n form the set

$$\Gamma_n^C = \mathbb{R}_+ \setminus \Gamma_n = \{x \in \mathbb{R}_+ : g_n(x) < \omega_{N-n}(x)\},$$

which is the continuation domain at moment $n = 0, 1, \dots, N$. Thus the optimal stopping domain has the following discrete structure

$$\Gamma = \{\Gamma_0, \Gamma_1, \dots, \Gamma_N\}.$$

The following theorem is an analogue of the general results for optimal stopping theory of Markov processes given in the books Chow, Robbins and Siegmund (1971), Shiryaev (1978), and Peskir and Shiryaev (2006). The slight difference with the classical results mentioned above, is that we need

to take into account the possibility for the price process to enter the knock out domain. This makes the stopping rules and the payoff dependent of the whole past trajectory of the price process.

Theorem 1. *The optimal stopping time maximizing (4) is given for every $x \in \mathbb{R}_+$, $n = 0, \dots, N$ by*

$$\tau_{opt} = \min\{k \geq n : S_{x,n}(k) \in \Gamma_k\} \wedge N \quad (7)$$

and

$$\Phi_{x,n}(\tau_{opt}) = \omega_N(x). \quad (8)$$

Proof. It is divided into two parts. The first part is to prove that the reward function $e^{-Rn}\omega_{N-n}(x)$ is an upper bound for the functional $\Phi_{x,n}(\tau)$ for any Markov moment $\tau \in \mathbf{M}_{x,n}^N$. Secondly we prove that this upper bound is archived if the stopping time $\tau = \tau_{opt}$ is taken according to formula (7).

Let $\tau_{x,n}$ be some Markov moment from the class $\mathbf{M}_{x,n}^N$ and define a random reward corresponding to stopping at moment $\tau_{x,n}$ as

$$\xi_{x,n}(\tau_{x,n}) = e^{-R\tau_{x,n}} g_{\tau_{x,n}}(S_{x,n}(\tau_{x,n})) \chi(\tau_{x,n} < \tau_{x,n}^H).$$

We start by showing that the upper bound for $\mathbb{E}\xi_{x,n}(\tau_{x,n})$ is given by $e^{-Rn}\omega_{N-n}(x)$. This will be shown by the following recursive procedure.

For moment $n = N$ we have $\omega_0(x) = g_N(x)\chi(x \notin H_N)$. On the other hand, in this case, $\tau_{x,N} = N$ for any Markov moment $\tau_{x,N} \in \mathbf{M}_{x,N}^N$ and $s_{x,N}(N) = x$. Therefore the expected reward

$$\mathbb{E}\xi_{x,N}(\tau_{x,N}) = e^{-RN} g_N(x)\chi(x \notin H_N) = e^{-RN}\omega_0(x).$$

This equality holds also for the Markov moment τ_{opt} .

For moment $n = N - 1$ the recursion (6) yields, $\omega_1(x) = \max\{g_{N-1}(x), T_{N-1}\omega_0(x)\}\chi(x \notin H_{N-1})$, and the random reward is given by

$$\begin{aligned} \xi_{x,N-1}(\tau_{x,N-1}) &= e^{-R(N-1)} g_{N-1}(x)\chi(x \notin H_{N-1})\chi(\tau_{x,N-1} = N-1) + \\ &+ e^{-RN} g_N(s_{x,N-1}(N))\chi(x \notin H_N)\chi(\tau_{x,N-1} = N). \end{aligned}$$

By taking the expectation and using the Markov property, which makes event $\{s_{x,N-1}(N) \in A\}$ and $\{\tau_{x,N-1} > N-1\}$ independent, and the structure of the recursion (6) we have

$$\begin{aligned} \mathbb{E}\xi_{x,N-1}(\tau_{x,N-1}) &= e^{-R(N-1)} g_{N-1}(x)\chi(x \notin H_{N-1})\mathbb{P}\{\tau_{x,N-1} = N-1\} + \\ &+ e^{-R(N-1)} \int_0^\infty e^{-rN} g_N(y)\chi(x \notin H_{N-1}) \\ &\cdot \chi(y \notin H_N)\mathbb{P}\{s_{x,N-1}(N) \in dy, \tau_{x,N-1} > N-1\} = \\ &= e^{-R(N-1)} g_{N-1}(x)\chi(x \notin H_{N-1})\mathbb{P}\{\tau_{x,N-1} = N-1\} + \end{aligned}$$

$$\begin{aligned}
& + e^{-R_{N-1}} \int_0^\infty e^{-r_N} g_N(y) \chi(x \notin H_{N-1}) \\
& \cdot \chi(y \notin H_N) \mathbb{P}\{\tau_{x,N-1} > N-1\} \mathbb{P}\{s_{x,N-1}(N) \in dy\} = \\
& = e^{-R_{N-1}} g_{N-1}(x) \chi(x \notin H_{N-1}) \mathbb{P}\{\tau_{x,N-1} = N-1\} + \\
& + e^{-R_{N-1}} T_{N-1} \omega_0(x) \mathbb{P}\{\tau_{x,N-1} > N-1\} \leq e^{-R_{N-1}} \omega_1(x).
\end{aligned}$$

By following the same recurrent procedure we get, for $n = 0, 1, \dots, N$ and $x \in \mathbb{R}_+$.

$$\Phi_{x,n}(\tau_{x,n}) = \mathbb{E} \xi_{x,n}(\tau_{x,n}) \leq e^{-R_n} \omega_{N-n}(x)$$

At moment $n = N-1$ we see that the optimal stopping time τ_{opt} defined in equation (7) has two possible values, $\tau_{opt} = N-1$ if $x \in \Gamma_{N-1}$ and $\tau_{opt} = N$ otherwise. That is $\mathbb{P}\{\tau_{opt} = N-1\} = \chi(x \in \Gamma_{N-1})$ and $\mathbb{P}\{\tau_{opt} = N\} = \chi(x \notin \Gamma_{N-1})$. By using these facts and the form of the recurrent formula (6) it is seen that τ_{opt} will give the expected reward

$$\begin{aligned}
\mathbb{E} \xi_{x,N-1}(\tau_{opt}) & = e^{-R_{N-1}} g_{N-1}(x) \chi(x \notin H_{N-1}) \chi(x \in \Gamma_{N-1}) + \\
& + e^{-R_{N-1}} T_{N-1} \omega_0(x) \chi(x \notin \Gamma_{N-1}) = e^{-R_{N-1}} \omega_1(x).
\end{aligned}$$

By follow this procedure in a recurrent way backwards we see that (7) will give the maximum reward, we get, for every $n = 0, 1 \dots N$, $x \in X$,

$$\Phi_{x,n}(\tau_{opt}) = e^{-R_n} \omega_{N-n}(x).$$

The proof above is given to make this paper self readable. There exists an alternative approach to prove Theorem 1. One can try to embed the model in the Markov process setting and then to follow the classical proof given in Shiryaev (1978).

This actually can be done by introducing a two dimensional Markov process $Z_{x,n}(k) = (S_{x,n}(k), I_{x,n}(k))$ where the additional component indicate by non-knock out is be given by $I_{x,n}(k) = \prod_{i=n}^k \chi(S_{x,i} \notin H_i)$. \square

3. PAYOFF FUNCTIONS

Theorem 1 is used to construct and analyze the algorithm for finding optimal stopping domains.

In our experimental studies we restrict our studies to put options, the result for call options are similar.

We also assume that the starting time is 0 and assume that the model is homogenous in time. That is $r_n = r$, $g_n(x) = g(x)$ and Y_k , $k = 0, 1, \dots$ are i.i.d random variables.

A put option is a contract that gives the holder the right but not the obligation to sell an asset to a predetermined price K know as the strike price. The right to sell expires after a predetermined expiration time N ,

called maturity. Since there is a right but not an obligation to exercise the option. The holder needs to pay a premium p for the option. But since the holder always pays the premium p for the option, the optimal stopping time and optimal stopping domain will not be affected of the option price p . So we may let $p = 0$ for this studies of optimal stopping domains. In this paper the option is of American type, then the holder has the right to exercise the option at any moment of time $0 \leq \tau \leq N$. The function that determines the value of the contract is called the *payoff function*. A payoff function is a function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$. In this paper we consider several different types of payoff functions. The first type we consider is a payoff function.

$$g(x) = a[K - x]_+ = \begin{cases} 0 & \text{if } x \geq K, \\ a(K - x) & \text{if } x < K, \end{cases} \quad (9)$$

where a will determine the slope of the payoff function. If a is equal to one, the payoff function corresponds to the payoff for the standard put option.

We then consider the piecewise linear payoff function. That payoff has two different strike price $0 < K_2 < K_1$ and two different slopes $a_1, a_2 \geq 0$, that is the scale parameters for the price intervals $[K_1, K_2)$ and $[K_2, 0]$, respectively,

$$g(x) = \begin{cases} 0 & \text{if } x > K_1, \\ a_1(K_1 - x) & \text{if } K_1 \geq x > K_2, \\ a_1(K_1 - K_2) + a_2(K_2 - x) & \text{if } x \leq K_2. \end{cases} \quad (10)$$

The piecewise payoff function can be replicated by a portfolio of two standard put options, first with strike price K_1 and portfolio weight a_1 and the second with strike price K_2 and portfolio weight \tilde{a}_2 .

Next payoff function considered is the stepwise payoff function. It will yield a constant payoff for a given interval of price in underlying. In this paper a two step $B_1 < B_2$, stepwise payoff function is considered. The function has the form

$$g(x) = \begin{cases} 0 & \text{if } x \in [K_1, \infty), \\ B_1 & \text{if } x \in [K_2, K_1), \\ B_2 & \text{if } x \in [0, K_2). \end{cases} \quad (11)$$

The stepwise payoff function can be replicated by a portfolio of two digital options with strike prices K_1 and K_2 and with weights B_1 and \tilde{B}_2 .

The next payoff function considered is the quadratic put. The payoff functions has one strike price K , a scale parameter a and is given by

$$g(x) = \begin{cases} 0 & \text{if } x > K, \\ a(K - x)^2 & \text{if } x \leq K. \end{cases} \quad (12)$$

We also consider the logarithmic payoff function. The payoff function also have a scale parameter a , a strike price K , and is given by

$$g(x) = \begin{cases} 0 & \text{if } x \geq K, \\ a \log(K - x - 1) & \text{if } x < K. \end{cases} \quad (13)$$

4. THE MONTE CARLO ALGORITHM

The algorithm presented below is similar with those described in Jönsson (2001) for usual American options. But with the difference that the algorithm below takes into account effects coming from knock out domain.

First define an upper and a lower boundary for the stock prices. Denote s_u as the upper level of stock prices and s_l as the lower level. Then define all prices by $s_{n,j} = s_l + j\Delta$, $j = 0, 1, \dots, J$, where $s_{n,j}$ is the price at moment n with level j . Note that $s_{n,0} = s_l$, define Δ in such way that $s_{n,J} = s_u$ where $n = 0, 1, \dots, N$ denote each moment of time. So define a mesh in time and stock prices with discrete points $(n, s_{n,j})$. In this study of optimal stopping domains. Note that the algorithm will generate is a quasi-optimal stopping domain, not the exact true optimal stopping domain. Further we will refer to the quasi-optimal stopping domain as just stopping domain.

the algorithm starts from expiration date N and for each point $(n, s_{n,j})$ on the grid compare the profit we make by exercising the option at time n with keeping the option until time $n + 1$. Investigate for each moment $n = 0, 1, \dots, N$ if the stock price are in the knock out domain $s_{n,j} \in H_n$, and also if the stock price between two adjacent points $(n, s_{n,j})$ and $(n, s_{n,j+1})$ belongs to the stopping domain or not. That is done by constructing an interval

$$I_{n,j} = \left[s_{n,j} - \Delta/2, s_{n,j} + \Delta/2 \right)$$

and if $I_{n,j} \cap \Gamma_n \neq \emptyset$ say that $s_{n,j} \in \Gamma_n$. Also note that since the option matures at $n = N$ all prices at that moment belong to the stopping domain, that is $\Gamma_N = [s_l, s_u]$. We use backward induction and next consider the moment $n = N - 1$, at this moment we have the choice to exercise the option at the moment $n = N - 1$ or to keep holding the contract until the moment $n = N$. So for each j we make the comparison if $g(s_{N-1,j}) \geq T\omega_0(s_{N-1,j})$ then $I_{n-1,j} \cap \Gamma_{n-1,j} \neq \emptyset$. So for the moment $n = N - 1$ the stopping domain is given by,

$$\Gamma_{N-1} = \bigcup_{j: g(s_{N-1,j}) \geq T\omega_0(s_{N-1,j})} I_{N-1,j}.$$

Monte Carlo simulation is used to determine $T\omega_0(s_{N-1,j})$. For simplicity we introduce the short notation $S_{s_j,n}^{(i)}(k) = S^{(i)}(k)$, where $\{S_{x,n}^{(i)}(k), k \geq n\}$ are independent realisations of the price process $S_{x,n}(k)$, $k \geq n$. For M

independent simulations the expected continuation profit will be given by

$$\widehat{T}_{N-1}^{(M)}g(s_{N-1,j}) = \frac{1}{M} \sum_{i=1}^M e^{-r} g(S^{(i)}(N)) \chi(S^{(i)}(N) \notin H_N)$$

The approximated stopping domain will be given by

$$\widehat{\Gamma}_{N-1} = \bigcup_{j: g(s_{n-1,j}) \geq \widehat{T}_{N-1}^{(M)}g(s_{N-1,j})} I_{N-1,j}. \quad (14)$$

For moment $n = N - 2$ and each j the comparison $g(s_{N-2,j}) \geq T\omega_0(s_{N-2,j})$ is made, and then $I_{n-1,j} \subset \Gamma_{n-1,j}$. For the moment $n = N - 2$ the optimal stopping domain is given as in (14). At the moment when we determine the continuation profit, we also need to consider the possibility of entering the stopping domain and having an early exercise at the moment $n = N - 1$. So, the already estimated structure of the stopping domain is considered in the backward procedure. We also need to consider the possibility of entering the knock out domain and getting a zero payoff. The continuation profit is determined by

$$\begin{aligned} \widehat{T}_{N-2}^{(M)}g(s_{N-2,j}) = & \\ & \frac{1}{M} \sum_{i=1}^M (e^{-r} g(S^{(i)}(N-1)) \chi(S^{(i)}(N-1) \in \widehat{\Gamma}_{N-1}) \\ & \chi(S^{(i)}(N-1) \notin H_{N-1}) + e^{-2r} g(S^{(i)}(N)) \\ & \chi(S^{(i)}(N-1) \notin \widehat{\Gamma}_{N-1}) \chi(S^{(i)}(N-1) \notin H_{N-1}) \chi(S^{(i)}(N) \notin H_N). \end{aligned}$$

For every moment $n = 0, 1, \dots, N - 3$ we have to determine the continuation profit $T\omega_{N-n-1}(s_{n,j})$ of every stock price $s_{n,j}$, $j = 0, 1, \dots, J$ on the grid, and use the fact that we know the structure of the stopping domain for each moment of time $n + 1, n + 2, \dots, N - 1, N$ and that we know the structure of the knock out domain when making the estimates.

5. CLASSIFICATION ERRORS

When we study the algorithm we study the probabilities of classification errors. We will consider two types of classification errors. But there are two more errors occurring, one due to the fact that we create an interval around each point in the mesh $I_{n,j}$, and say that the interval belongs to Γ_n , or not. In worst case this can lead to that we classify almost entire $I_{n,j}$ wrong. The second error occurring is due to the fact that we using already estimated structure of the stopping domain. Since we only have an estimate of the stopping domain used in the estimator $\widehat{T}_n^{(M)}g(s_{n,j})$, the estimator will be biased. But in this paper analysis of these both errors are neglected.

First we have the classification error when the algorithm indicates that the stock price $s_{n,j}$ belongs to the stopping domain $g(s_{n,j}) \geq \widehat{T}_n^{(M)}g(s_{n,j})$,

but instead belongs to the continuation domain $g(s_{n,j}) < T\omega_{N-n-1}(s_{n,j})$. From the central limit theorem we get that the probability $p_{n,j}$ is

$$\begin{aligned} p_{n,j} &= \mathbb{P}\left\{\widehat{T}_n^{(M)} g(s_{n,j}) < g(s_{n,j})\right\} \\ &= \mathbb{P}\left\{\frac{\widehat{T}_n^{(M)} g(s_{n,j}) - T\omega_{N-n-1}(s_{n,j})}{\sigma_{n,j}} \sqrt{M} < \frac{g(s_{n,j}) - T\omega_{N-n-1}(s_{n,j})}{\sigma_{n,j}} \sqrt{M}\right\} \\ &\simeq 1 - \Phi\left(\frac{T\omega_{N-n-1}(s_{n,j}) - g(s_{n,j})}{\sigma_{n,j}} \sqrt{M}\right) \end{aligned}$$

where the standard deviation of the estimate $\widehat{T}_n^{(M)}$ is given by $\sigma_{n,j}/\sqrt{M}$ and $\sigma_{n,j}$ is the standard deviation of one component.

The second type of classification error we can make is when the algorithm indicates that the price does not belong to the stopping domain $g(s_{n,j}) < \widehat{T}_n^{(M)}(s_{n,j})$, but the stock price does $g(s_{n,j}) \geq T\omega_{N-n-1}(s_{n,j})$, we get by the central limit theorem the probability of make such kind of classification error to be

$$\begin{aligned} q_{n,j} &= \mathbb{P}\left\{\widehat{T}_n^{(M)} g(s_{n,j}) > g(s_{n,j})\right\} \\ &= \mathbb{P}\left\{\frac{\widehat{T}_n^{(M)} g(s_{n,j}) - T\omega_{N-n-1}(s_{n,j})}{\sigma_{n,j}} \sqrt{M} > \frac{g(s_{n,j}) - T\omega_{N-n-1}(s_{n,j})}{\sigma_{n,j}} \sqrt{M}\right\} \\ &\simeq 1 - \Phi\left(\frac{T\omega_{N-n-1}(s_{n,j}) - g(s_{n,j})}{\sigma_{n,j}} \sqrt{M}\right) \end{aligned}$$

So the probability of making such error $q_{n,j}$ are the same as for the first type of error $p_{n,j}$. One can use a second moment estimate similar with $\widehat{T}_n^{(M)} g(s_{n,j})$ to get a good estimate of $\sigma_{n,j}$. Define the estimate of the second moment of the moment $n = N - 1$ as

$$\tilde{T}_{N-1}^{(M)} = \frac{1}{M} \sum_{i=1}^M e^{-2r} g^2(S^{(i)}(N)) \chi(S^{(i)}(N) \notin H_N).$$

For moments $n = 0, 1, \dots, N - 2$ the formulas are similar. Then the estimate of the variance is $\tilde{\sigma}_{n,j}^2 = \tilde{T}_n^{(M)} g(s_{n,j}) - (\widehat{T}_n^{(M)} g(s_{n,j}))^2$. Let $E_{n,j}^{\text{cont}} = \max(E_{n,j}^{\text{cont}}, s_{n,j})$ be the continuation profit. Then define the operator L and the dimensionless measure of the variance $d_{n,j}^2$ as

$$L_{n,j} = \frac{E_{n,j}^{\text{cont}} - g(s_{n,j})}{g(s_{n,j})}, \quad d_{n,j}^2 = \frac{\sigma_{n,j}^2}{g(s_{n,j})^2}.$$

Note that if $L_{n,j} \leq 0$ the strategy is to exercise. Then the probability of making classification error is given by

$$p_{n,j} \simeq 1 - \Phi\left(\sqrt{M} \frac{|L_{n,j}|}{d_{n,j}}\right).$$

$s_{N-1,j}$	$g(s_{N-1,j})$	H_N	$\tilde{E}_{N-1,j}^{\text{cont}}$	$\tilde{\sigma}_{N-1,j}^2$	$\tilde{L}_{N-1,j}$	$\tilde{d}_{N-1,j}^2$
99.00	1.00	85.00	1.21216	1.37511	0.21216	1.37511
98.00	2.00	85.00	2.03980	1.85832	0.01990	0.46458
97.00	3.00	85.00	2.98171	2.06349	-0.00610	0.22928
96.00	4.00	85.00	3.96526	2.09163	-0.00869	0.13073
95.00	5.00	85.00	4.95841	2.06295	-0.00832	0.08252
94.00	6.00	85.00	5.95165	2.01901	-0.00806	0.05608
93.00	7.00	85.00	6.94492	1.97731	-0.00787	0.04035
92.00	8.00	85.00	7.93844	1.93460	-0.00770	0.03023
91.00	9.00	85.00	8.93252	1.89355	-0.00750	0.02338
90.00	10.00	85.00	9.92549	1.85841	-0.00745	0.01858
89.00	11.00	85.00	10.90246	1.93013	-0.00887	0.01595
88.00	12.00	85.00	11.74688	3.17346	-0.02109	0.02204
87.00	13.00	85.00	11.94892	10.83933	-0.08085	0.06414
86.00	14.00	85.00	10.46743	32.00194	-0.25233	0.16328

Table 1: Estimated values of the expected continuation profit $\tilde{E}_{N-1,j}^{\text{cont}}$, the variance $\tilde{\sigma}_{N-1,j}^2$ and for the measures $\tilde{L}_{N-1,j}$, $\tilde{d}_{N-1,j}^2$. Note that the measure $\tilde{L}_{N-1,j}$ changes sign when entering the optimal stopping domain.

Let us estimate the classification errors described above and let's consider moments $n = N - 1$ and $n = N - 2$. The reason why moments so early in the recursion are studied are, because of when using the stopping domain for estimation of the price of the option contract, the price is more sensitive to perturbations of the stopping domain close to maturity. We now use Monte Carlo simulation to determine an estimate of $\tilde{E}_{n,j}^{\text{cont}}$.

In all simulations as was mentioned in section 3, we have assumed that the underlying process follows a geometrical random walk, i.e as in equation (1) where Y_n are described by $Y_n = e^{\mu + \sigma W_n}$, and W_n is a sequence of independent random variables having standard normal distribution. We will consider a constant yearly interest rate of $r = 4\%$ a strike price of $K = 100$, $\Delta = 1.0$ and a barrier given by $H_n = \{x : x \geq 85\}$. We use $M = 10^7$ simulations for an underlying process with a yearly drift of $\mu = 0.0$ and yearly volatility of $\sigma = 0.24$. It is seen in Table 1 that when $s_{N-1,j} = 97$ the measure $\tilde{L}_{N-1,j}$ is first time negative. From that we can conclude that the strategy to exercise the option at that moment is more profitable than to continue to hold the option until moment N . We also see that the variance increases as the underlying price approaches the barrier. Note also that $\tilde{E}_{N-1,j}^{\text{cont}}$ for $s_{N-1,j} = 86$ is less than for $s_{N-1,j} = 87$, this is because of the large probability of crossing the barrier.

In Table 2 we show the probability of making classification errors for different number of simulations. It is seen that the probability of making classification error has the greatest value around $s_{N-1,j} = 97$, and from Table 1 it is known that it is the boundary of the stopping domain. Note also that the probability of making such errors are relatively large close to the boundary even when the number of simulations are as large as $M = 1 \cdot 10^6$. So the number of simulations should be large close to the boundary.

$s_{N-1,j}$	$p_{N-1,j}$				
	$M=5 \cdot 10^4$	$M=1 \cdot 10^5$	$M=1 \cdot 10^6$	$M=1 \cdot 10^7$	$M=1 \cdot 10^8$
99.00	0.0	0.0	0.0	0.0	0.0
98.00	$1.11 \cdot 10^{-16}$	0.0	0.0	0.0	0.0
97.00	0.3292	0.2929	0.1451	0.0468	$7.074 \cdot 10^{-12}$
96.00	0.3249	0.2618	$4.758 \cdot 10^{-10}$	0.0	0.0
95.00	0.2766	0.0739	$3.597 \cdot 10^{-13}$	0.0	0.0
94.00	0.1662	0.0305	$1.188 \cdot 10^{-14}$	0.0	0.0
93.00	0.0672	0.0128	$3.186 \cdot 10^{-14}$	0.0	0.0
92.00	0.008088	0.0344	$1.155 \cdot 10^{-14}$	0.0	0.0
91.00	0.0379	0.001539	$6.661 \cdot 10^{-16}$	0.0	0.0
90.00	0.001421	0.003090	$1.110 \cdot 10^{-16}$	0.0	0.0
89.00	$5.026 \cdot 10^{-05}$	$7.286 \cdot 10^{-09}$	0.0	0.0	0.0
88.00	0.0	0.0	0.0	0.0	0.0

Table 2: The probability of classification error $p_{N-1,j}$, note that the probability of making classification errors at the boundary of the stopping domain is large and has preferred size first for $M = 10^8$ simulations.

Lets now move back in time to the moment $n = N - 2$ with all other parameters the same. First it is noted that parameter $\tilde{L}_{N-2,j}$ become neg-

$s_{N-2,j}$	$g(s_{N-2,j})$	H_{N-1}	$\tilde{E}_{N-2,j}^{\text{cont}}$	$\tilde{\sigma}_{N-2,j}^2$	$\tilde{L}_{N-2,j}$	$\tilde{d}_{N-2,j}^2$
99.00	1.00	85.00	1.42556	2.18098	0.42556	2.18098
98.00	2.00	85.00	2.17862	2.65320	0.08931	0.66330
97.00	3.00	85.00	3.05612	2.69609	0.01871	0.29957
96.00	4.00	85.00	4.00748	2.45158	0.00187	0.15322
95.00	5.00	85.00	4.99254	2.19546	-0.00149	0.08782
94.00	6.00	85.00	5.98865	2.05237	-0.00189	0.05701
93.00	7.00	85.00	6.98787	1.98238	-0.00173	0.04046
92.00	8.00	85.00	7.98657	1.94025	-0.00168	0.03032
91.00	9.00	85.00	8.97966	1.94184	-0.00226	0.02397
90.00	10.00	85.00	9.93983	2.19356	-0.00602	0.02194
89.00	11.00	85.00	10.7710	3.60404	-0.02082	0.02979
88.00	12.00	85.00	11.1982	8.96621	-0.06681	0.06227
87.00	13.00	85.00	10.6576	22.6948	-0.18019	0.13429
86.00	14.00	85.00	8.54428	41.6160	-0.38969	0.21233

Table 3: Estimated values of the expected continuation profit $\tilde{E}_{N-2,j}^{\text{cont}}$, the variance $\tilde{\sigma}_{N-2,j}^2$ and for the measures $\tilde{L}_{N-2,j}$, $\tilde{d}_{N-2,j}^2$. Note that the measure $\tilde{L}_{N-2,j}$ changes sign when entering the optimal stopping domain.

ative first when $s_{N-2,j} = 95$ compared with $s_{N-1,j} = 97$ in Table 1. So the strategy to continue is more profitable compare to continue first when the underlying asset starts at 95 for moment $N - 2$ compared with 97 for moment $N - 1$. This effect can be explained by that the probability of the underlying assets to go down is greater with two days to maturity than with one day left. The same effects that was seen in Table 1 for $\tilde{E}_{N-1,j}^{\text{cont}}$ is also seen here for $\tilde{E}_{N-2,j}^{\text{cont}}$, the estimate decreases close to the barrier, but in this case for $s_{N-2,j} = 87$ instead of $s_{N-1,j} = 86$ as in Table 1. The estimated variance also increases rapidly close to the barrier, it is five times higher for

$s_{N-2,j} = 86$ than $s_{N-2,j} = 88$.

Finally in Table 4 the probabilities of making classification errors are given for moment $n = N - 2$. It is again seen that the greatest probability of making such errors are close to the boundary of the optimal stopping domain. We also see that in order to have a good accuracy the number of simulations is less than when moment $n = N - 1$ is considered. At this moment to get a good estimate $M = 1 \cdot 10^6$ number of simulations will give a good estimate. However the biases discussed earlier will at smaller n be greater and make the total probability of classification errors greater than for moment $n = N - 1$.

As mentioned above, the estimator $\hat{T}_n^{(M)}g(s_{n,j})$ has biases, due to that it uses already estimated structure of the stopping domain will make that the probability of classification errors increases when n becomes smaller. To

$s_{N-1,j}$	$p_{N-2,j}$				
	$M=5 \cdot 10^4$	$M=1 \cdot 10^5$	$M=1 \cdot 10^6$	$M=1 \cdot 10^7$	$M=1 \cdot 10^8$
99.00	0.0	0.0	0.0	0.0	0.0
98.00	0.0	0.0	0.0	0.0	0.0
97.00	$4.59 \cdot 10^{-14}$	0.0	0.0	0.0	0.0
96.00	0.042724555	0.023568059	$3.14 \cdot 10^{-08}$	0.0	0.0
95.00	0.412963812	0.20353611	$9.92 \cdot 10^{-09}$	0.0	0.0
94.00	0.44102954	0.030851588	$8.83 \cdot 10^{-12}$	0.0	0.0
93.00	0.125290687	$3.03 \cdot 10^{-04}$	$9.61 \cdot 10^{-12}$	0.0	0.0
92.00	0.088964356	$6.66 \cdot 10^{-04}$	0.0	0.0	0.0
91.00	$2.33 \cdot 10^{-04}$	$5.65 \cdot 10^{-09}$	0.0	0.0	0.0
90.00	0.0	0.0	0.0	0.0	0.0
89.00	0.0	0.0	0.0	0.0	0.0
88.00	0.0	0.0	0.0	0.0	0.0

Table 4: The probability of classification error $p_{N-2,j}$, note that the barrier gets estimated at $s_{N-2,j} = 95$ for this setting. Also note that the greatest probability is at the boundary of the stopping domain.

minimize this biases Δ should be chosen small enough, and the number of simulations M close to the boundary should be chosen large to make the structure used accurate.

The choice of Δ will also generate classification errors, due to the construction with intervals $I_{n,j}$. To avoid this type of error the Δ should be chosen to be small enough. From a practical point of view the Δ should be chosen with respect to the accuracy in comparison with the preferred runtime of the algorithm.

As mentioned the probability of making classification errors are closest to the boundary of the stopping domain. When knowing the structure of the stopping domain improvements can be made. To get quicker result and better accuracy, an interval halving method is suggested. So instead of creating a static mesh, it is quicker to, for each n first determine if the point in the middle of the interval $[s_l, s_u]$ belongs to the stopping domain or the

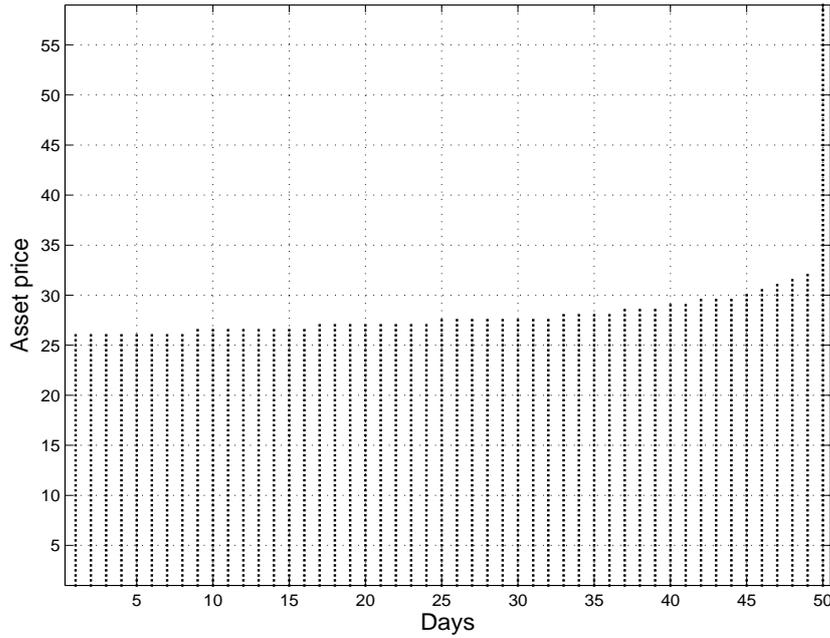


Figure 1: The stopping domain for a standard put payoff

continuation domain. Then by knowing the structure of the stopping domain to the corresponding payoff function the interval can be made smaller to in the end being minimized around the boundary of the stopping domain until the interval $I_{n,j}$ is as small to have a preferred accuracy.

6. EXAMPLES OF OPTIMAL STOPPING DOMAINS

In all examples it is assumed that the price process is following a geometrical random walk, i.e given by (1) where $Y_n = e^{\mu + \sigma W_n}$, having the daily parameters $\mu = 0.0$, $\sigma = 0.05$, and W_n , $n = 1, 2, \dots$ is a sequence of i.i.d normally distributed random variables.

Further we consider a constant yearly interest rate of $r = 1\%$, a maturity $N = 50$ and we use $M = 100000$ simulations on each point in the mesh to evaluate if the point is included into the stopping domain or not. The reason why such unrealistic parameters for μ and σ are studied is that the interesting effects of the stopping domains will get more visible then.

In this section examples of stopping domains for the specified payoff functions presented in Section 3, together with knock out domains will be considered.

We consider the knock out domain of the following form

$$H_n = \{x \geq 0 : x \leq \alpha\}, \quad (15)$$

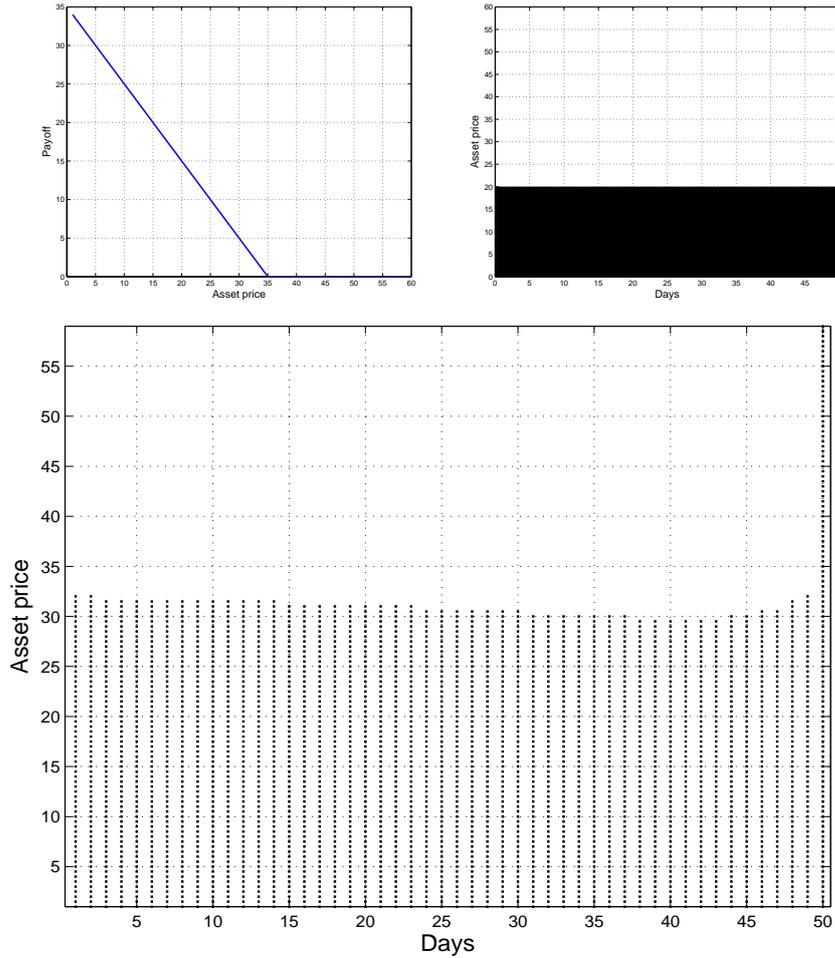


Figure 2: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

where $\alpha \in \mathbb{R}_+$, or

$$H_n = \begin{cases} \emptyset & n \notin [\alpha, \beta], \\ x & c \leq x \leq d, n \in [\alpha, \beta]. \end{cases} \quad (16)$$

For each payoff function defined in section 3, investigation is made about the structure of the stopping domain in combination with different kinds of knock out domains. This section is finished with a general discussion about how the structure is generated.

Domains generated from standard put payoff function. We will start our investigation from the simplest case, that is the payoff function that corresponds to the standard put option. The standard put option has the following payoff function (9). Through out all simulations in this section we have used a payoff function having parameters $K = 35$ and $a = 1.0$.

The first example given in Figure 1, there is no knock out domain at all

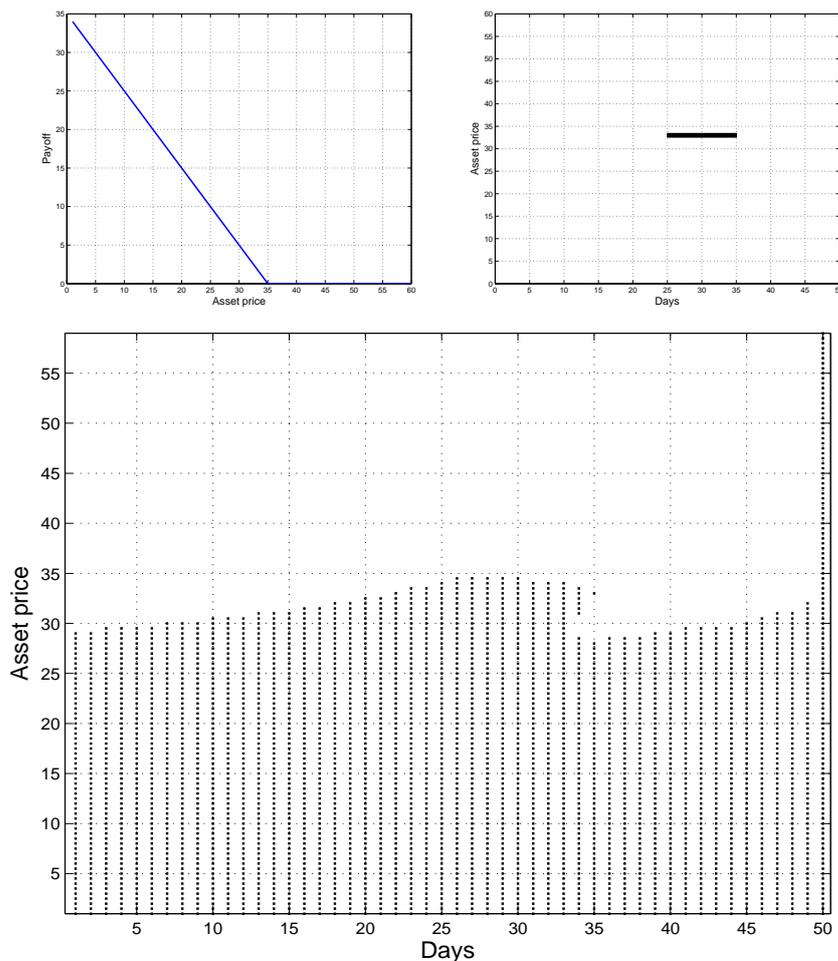


Figure 3: Upper left: The considered payoff function Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

considered. In this case it is seen that the stopping domain is monotonically increasing in time, that because it becomes less profitable to own the option when time decreases. This result will we use as a benchmark for standard payoff function when investigating the influence given by different knock out domains.

The first knock out domain considered is constant in time and is given by equation (15) with parameter $\alpha = 20$. The result is given in Figure 2, and we see that due to the introduced knock out domain the stopping domain is no longer monotonic in time. So the greatest payoff we can receive for this contract is if the underlying process would hit the stopping domain close to day 40 and we would get a payoff of 5.50. The reason of this shape is because of that the probability of hitting the knock out domain decreasing in time which make the continuation strategy more competitive, and whe we will come closer to maturity N the stopping strategy is again more competitive.

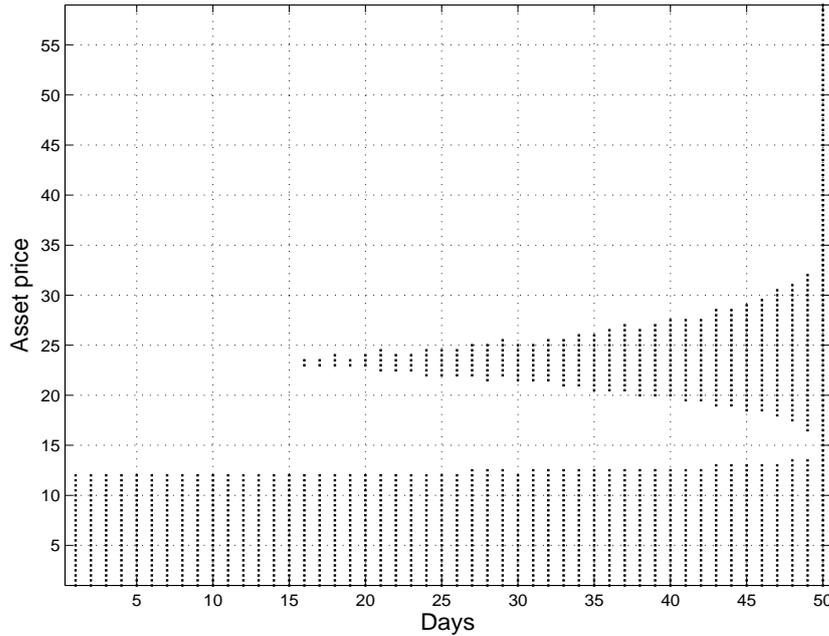


Figure 4: The stopping domain generated by a piecewise linear payoff function having parameters $K_1 = 35$ $K_2 = 15$ and $a_1 = 1.0$ $a_2 = 3.5$

Also note that if our model would be continuous then the probability of crossing the barrier and entering the knock out domain would be zero. Then the stopping domain would be as in Figure 1.

In the third example given in Figure 3, we consider the standard put payoff together with a small knock out domain as a strip described by equation (16) having parameters $c = 25$, $d = 35$, $\alpha = 33.5$ and $\beta = 32.5$. This corresponds to a small box. Here we see that the small box influence the stopping domain all the time until day 35 when there is no more possibility to enter the knock out domain. From this moment the stopping domain has the same shape as for ordinary put payoff function with no knock out domain at all, as in Figure 1. We also note that if the knock out domain are defined as above but a option contract with maturity at day 50 and starting at day 34 with $S_{34} = 30$ is considered. Then we may have to exercise the option if the price goes up and if the price goes down. So the stopping domain for this contract starting from day 34 are the same as for a contract starting at day 0 and both are valid until day 50.

By adding more boxes the stopping domain will have more waves like in Figure 3. If the

boxes are placed in a larger distance from the boundary of the stopping domain given in Figure 1 the stopping domain will not grow together with the stopping domain generated by the knock out box, and a small *island* shaped stopping domain will appear instead. Similar result are presented for quadratic payoff function in Figure 15.

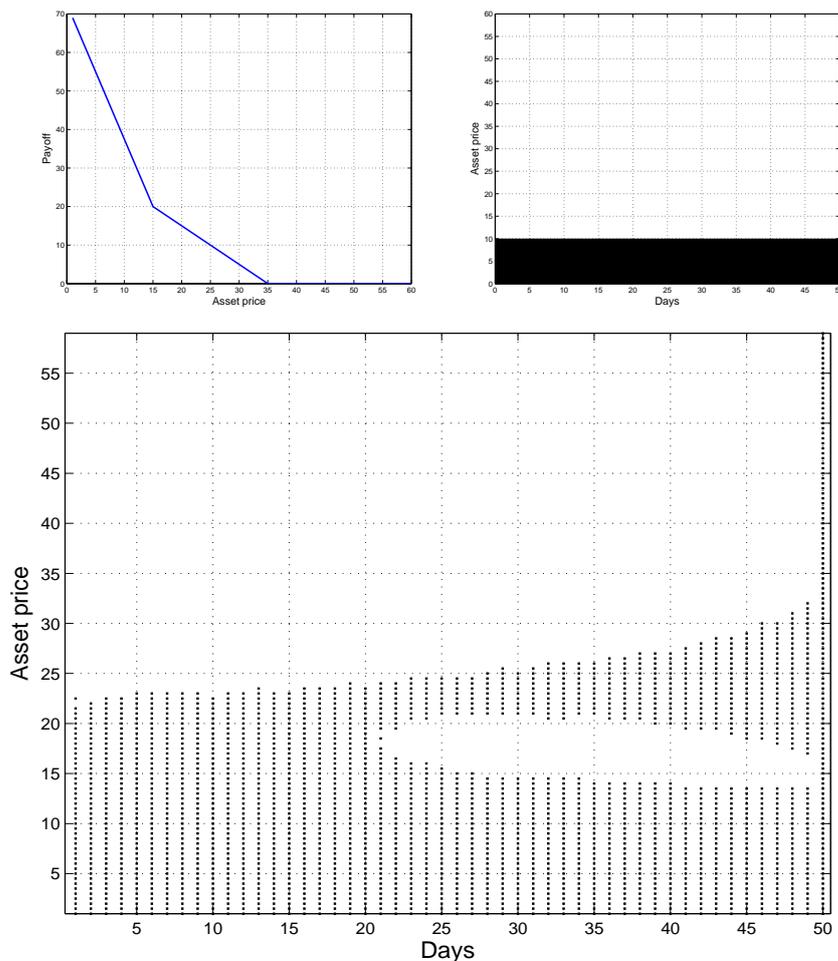


Figure 5: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

Domains generated from piecewise linear put payoff function. We now consider a more complex stopping domain. The piecewise linear payoff function given by equation (10). In this investigation we let the payoff function have parameters $K_1 = 35$ $K_2 = 15$ and slopes $a_1 = 1.0$ $a_2 = 3.5$. This contract corresponds to a portfolio of one standard put option with strike price 35, and 2.5 options with strike price 15, all options having the same maturity fifty days into the future. We again start with a benchmark example having no knock out domain at all. As seen in Figure 4 the result has a multi threshold structure and form a *harbor*. This structure appear because that when the underlying process is close to K_2 but have not yet crossed it, the probability of crossing it in the future is large which makes the continuation strategy more competitive than the stopping strategy, see Jönsson, Kukush and Silvestrov (2004, 2005) and Kukush and Silvestrov (2004) for more intensive studies of this payoff function and the effects

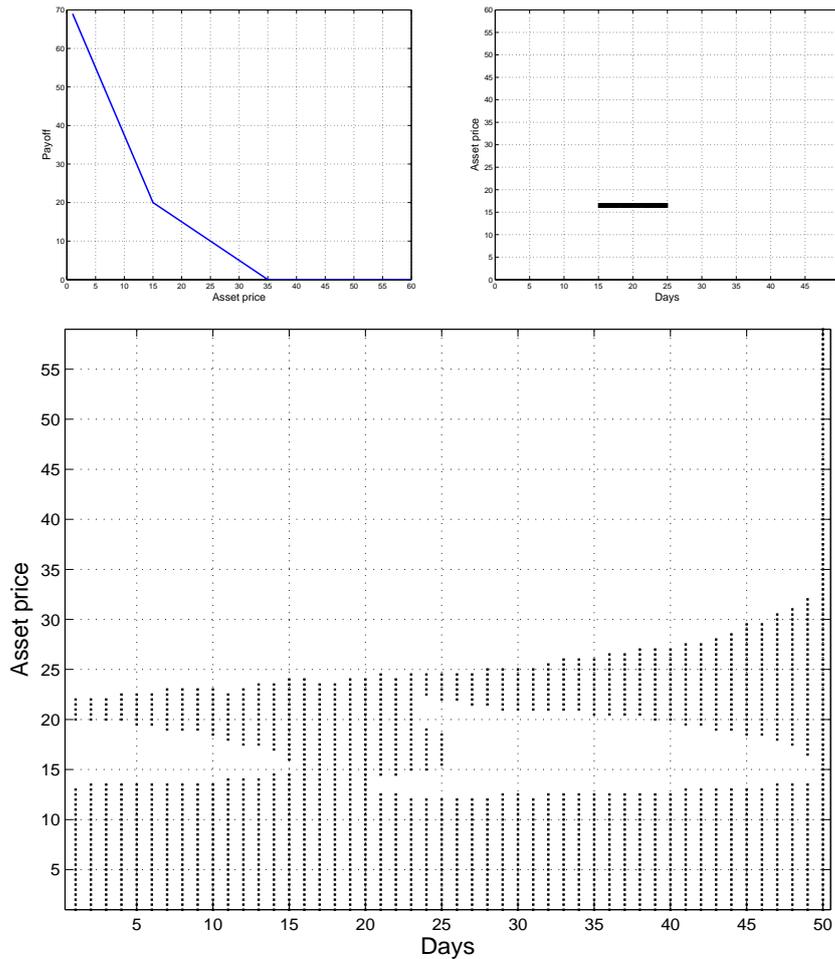


Figure 6: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

generated by it.

Now a knock out domain is introduced, in first example we consider the piecewise payoff function described above together with a standard barrier of down and out type. This knock out domain is given by equation (15) having parameter $\alpha = 10$. The structure of the stopping domain generated by this combination is presented in Figure 5. By comparing the results in Figure 4 with Figure 5, it is seen that the introduced barrier makes the disjoint stopping domain in Figure 4 grow together and form an *inner lake*. The reason why it grows together is due to the probability of getting *knocked out* is large in this area so the stopping strategy becomes more competitive, i.e the expected reward for a price process starting in that region will be less than the reward received if the contract is exercised at that moment. It is again worth noting that if a contract is defined between day 25 and 50, this closed lake is now open depending on which S_0 that contract has.

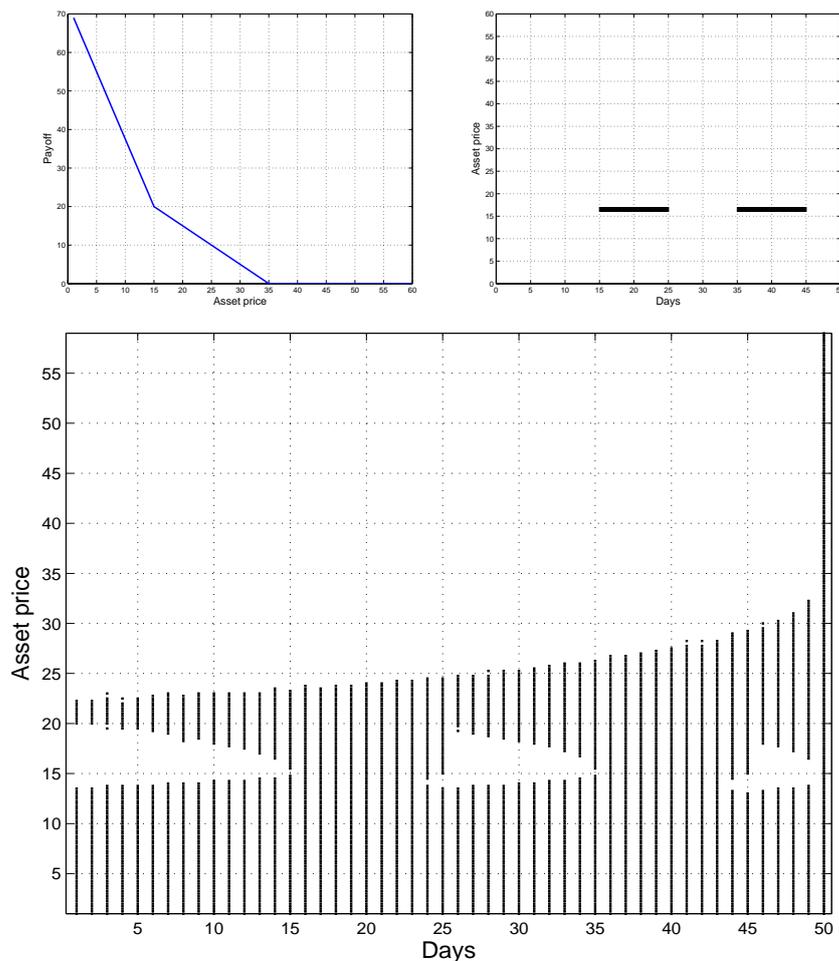


Figure 7: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

In the next example we consider the piecewise linear payoff function together with a small knock out domain in shape of a box. The knock out box is placed underneath the *harbor* in Figure 4. The box is described by equation (16) having parameters $c = 15$, $d = 25$, $\alpha = 16$ and $\beta = 17$ and the result is presented in Figure 6. Again we compare with the result in Figure 4 and first thing we note is that the *harbor* in Figure 4 is extended from day 17 to day 0. Secondly, it is seen that this small knock out domain generate an *inner lake* as in Figure 5.

But a difference from Figure 5 is that a contract starting at S_0 has an opportunity to get the higher payoff underneath the *harbor*.

In the last example of this section we consider the piecewise payoff function together with two knock out boxes. The setting is similar with Figure 6 but we now have two knock out boxes underneath the *harbor*. The boxes are placed after each other with some space in between the boxes.

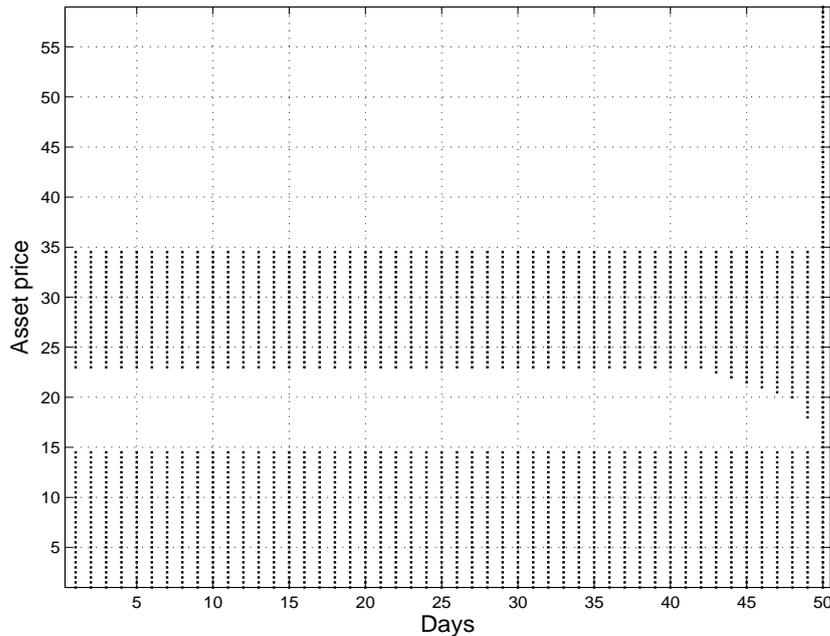


Figure 8: The stopping domain generated by a stepwise payoff function having parameters $K_1 = 40$, $K_2 = 25$ and $B_1 = 10$, $B_2 = 20$

The knockout domains are defined by equation (16) and having parameters $c_1 = 15$, $c_2 = 35$, $d_1 = 25$, $d_2 = 45$, $\alpha = 16$ and $\beta = 17$. We see that introducing one more box, it will generate one more *inner lake* in the stopping domain. The result is presented in Figure 7.

So to summarize this section: By introducing one more box on top of the stopping domain as in Figure 3 it will generate one *wave* shaped stopping domain on top. By introducing more boxes on top close to the boundary, the stopping domain will have more *waves*. By moving the boxes further away from the boundary of the stopping domain small islands will appear as in Figure 15, that will prevent the price process to enter the knock out domain. Finally, by introducing more boxes underneath the *harbor*, more *inner lakes* will appear.

Domains generated from stepwise payoff function. Now we consider stopping domains generated from the two-step stepwise payoff function defined in equation (11). The considered payoff function have parameters as $B_1 = 10$, $B_2 = 20$, $K_1 = 40$, $K_2 = 25$. This option contract corresponds to a portfolio of ten digital options with strike price 40, and ten digital options with strike price 25, all having the same maturity fifty days into the future.

The first considered case is the stepwise payoff function without any knock out domain at all as a benchmark. The result is presented in Figure 8. When looking at the figure it is firstly seen that the stopping domain again form a *harbor* shaped stopping domain at the upper strike price, but now it starts at day 0 and continues all the time up to maturity. The *harbor*

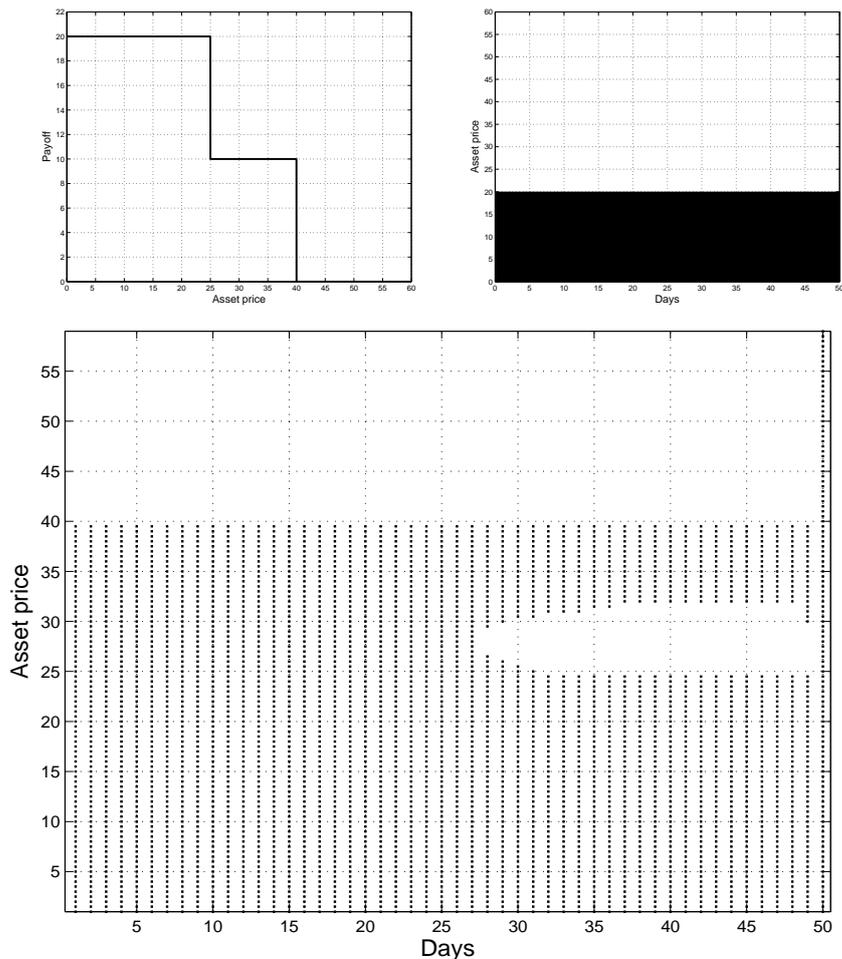


Figure 9: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

shaped stopping domain starts at strike price K_1 and continues until the probability of crossing second strike price K_2 is large enough. Secondly it is seen that because of the constant reward you receive for this contract you should exercise the option as soon as you cross a strike price K_1 or K_2 . If however S_0 of this option contract is below K_1 but above K_2 the strategy can be to continue to hold on to the option because of the high probability of crossing also K_2 and receive a greater reward. If more steps is introduced in the payoff function, then for each step, a new *harbor* will appear in the stopping domain, see Jönsson (2001) for further studies of stopping domains for this type of payoff function.

First considered combination of the stepwise payoff function with a knock out domain, is when the knock out domain corresponds to a barrier option of down and out type with a barrier at 20. This is given by equation (15) having parameter $\alpha = 20$. The result are presented in Figure

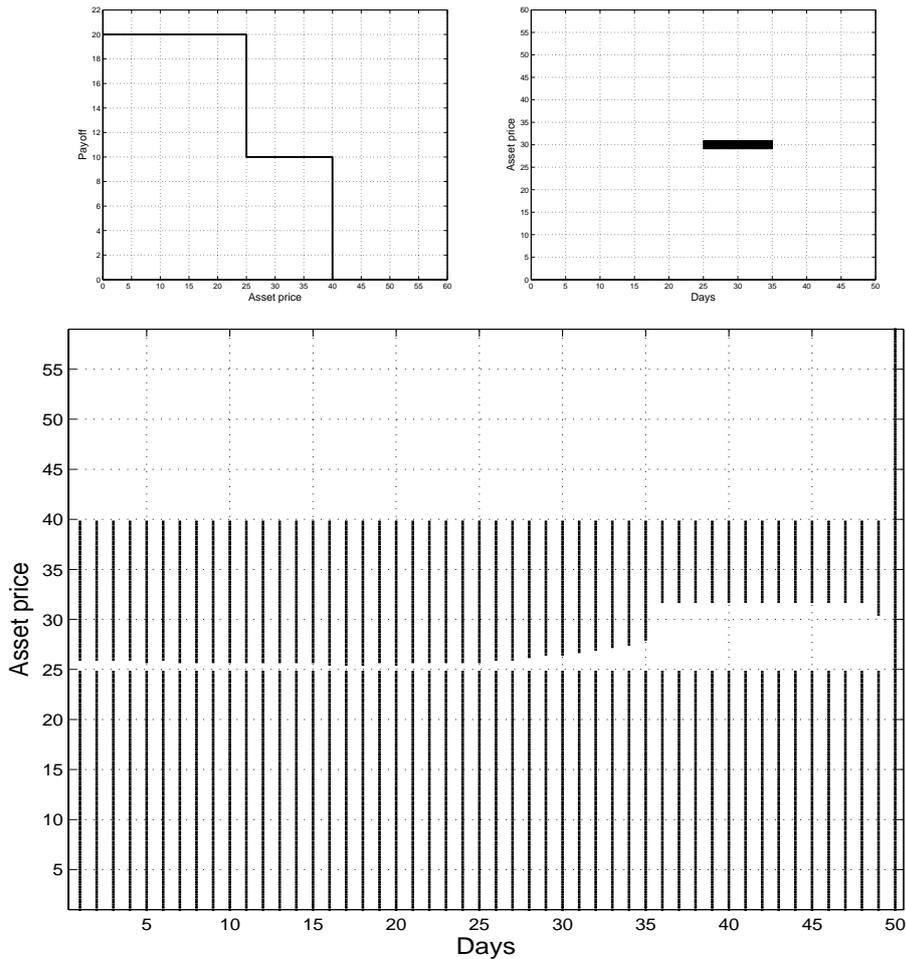


Figure 10: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

9 and it is seen that due to the large probability of hitting the knock out domain, starting from an early moment, the *harbor* in Figure 8 have grown together with the lower part of the stopping domain and, as for the piecewise payoff function, formed an *inner lake*. Again it is worth noting that if the contract starts at day 30 the stopping domain that are giving the *inner lake* structure, is then in the past for the process, and instead we again have a *harbor* structure. as in Figure 8.

Second considered combination is the stepwise payoff function together with a knock out domain having the shape of a small box given by equation (16), having parameters $c = 25$, $d = 35$, $\alpha = 31$ and $\beta = 29$. The result is presented in Figure 10. It is seen that the stopping domain generated by this combination almost create an *inner lake* similar with Figure 9 or Figure 6 for example.

The structure in Figure 10 is due to the fact that the small knock out

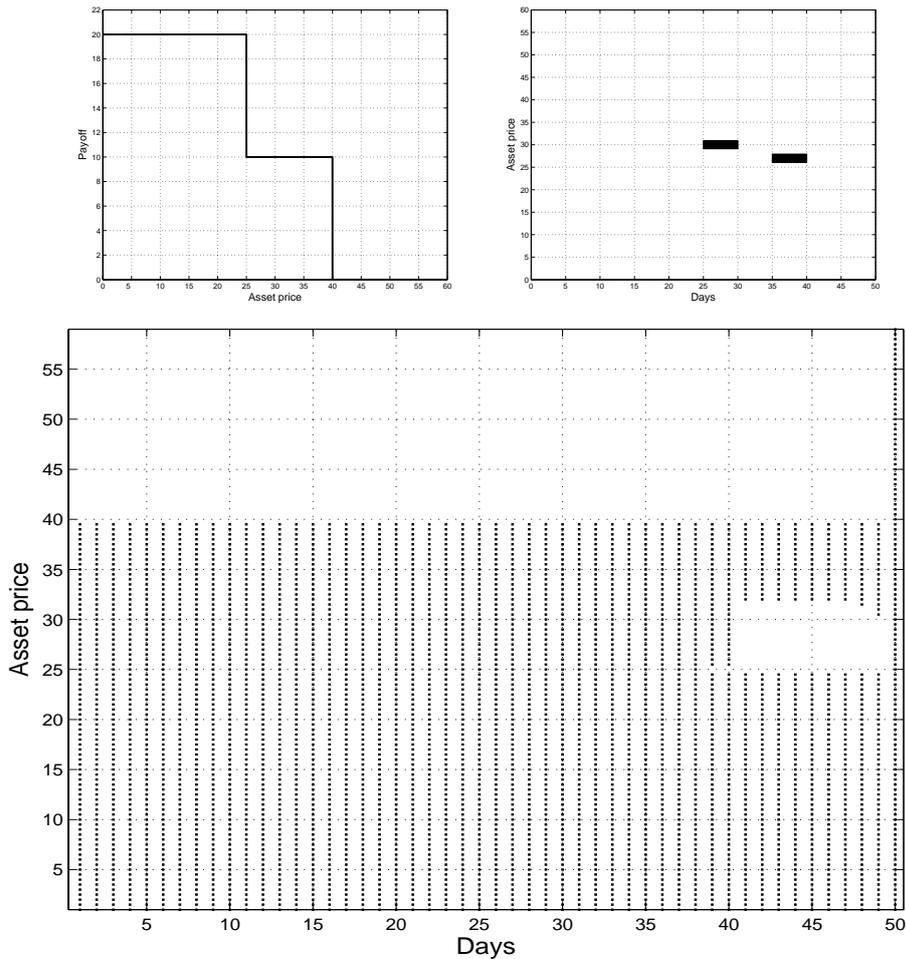


Figure 11: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

box are placed in a small continuation region, that's why the possibility for the price process to get past is almost 0. As in previous sections if a contract between day 35 and day 50 is constructed, then the starting point for this contract will be after the knock out domain in time, and the stopping domain for this contract will be as in Figure 10 from day 35 and forward.

Next combination is the stepwise payoff function together with two knock out boxes.

Both knock out boxes are placed in the small continuation region underneath the *harbor* giving the price process a small possibility to get past them without either hit the knock out domain or the stopping domain. The boxes are described by equation (16) and having parameters $c_1 = 25$, $c_2 = 35$, $d_1 = 30$, $d_2 = 40$, $\alpha_1 = 29$, $\alpha_2 = 26$ and $\beta_1 = 31$, $\beta_2 = 28$. The result is presented in Figure 11. Due to the different levels in prices of the two boxes, which gives the price process a small probability of passing

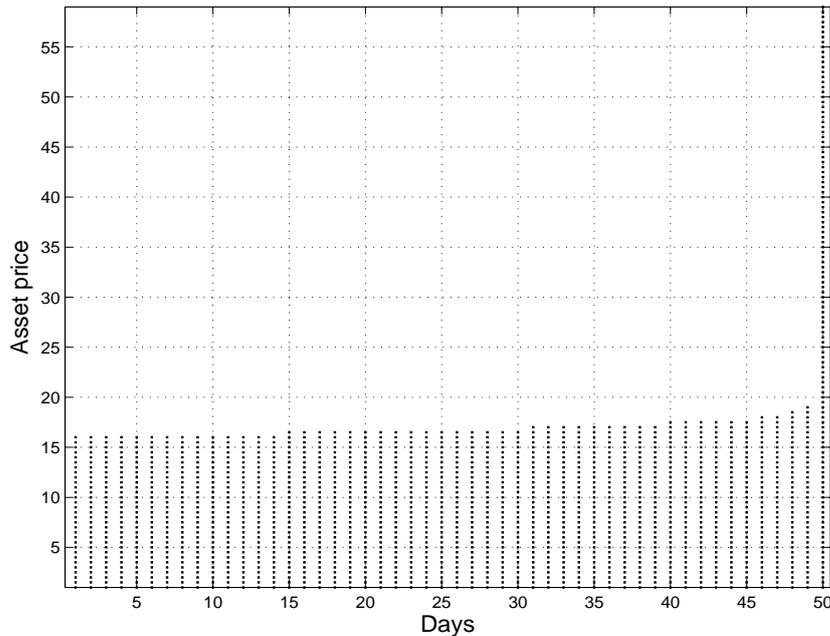


Figure 12: The stopping domain generated by a quadratic put payoff function having parameters $K = 35$ and $a = 1.0$

the boxes, an *inner lake* are again generated. If however the distance between the *harbor* and the stopping domain should be greater there would be a small opening to get pass the knock out domains and to enter the *inner lake* structure in Figure 11.

The same reasoning as earlier can be applied here too. That if the process starts at moment 41 the stopping domain creating the *inner lake* are in the past and in this case the stopping domain would be similar with the stopping domain in Figure 8 from moment 41 and forward. By adding more small boxes as in Figure 11 small openings or *inner lakes* can be constructed. There should however be noted that for this contract the possibility of changing the structure is limited to the small continuation regions under the *harbors*.

Domains generated from quadratic payoff function. In this section several combinations of knock out domains together with the quadratic payoff function are considered. The quadratic payoff function is given by equation (12), and we will consider following parameters, $K = 35$ $a = 1.0$.

Again the first considered case is the quadratic payoff function with no knock out domain at all, as a benchmark. The result are presented in Figure 12.

The structure generated by the quadratic payoff function is monotonic and looks similar with the structure generated by the standard put option given in Figure 1, the difference is that the quadratic put payoff makes the continuation strategy more competitive because if the underlying make a

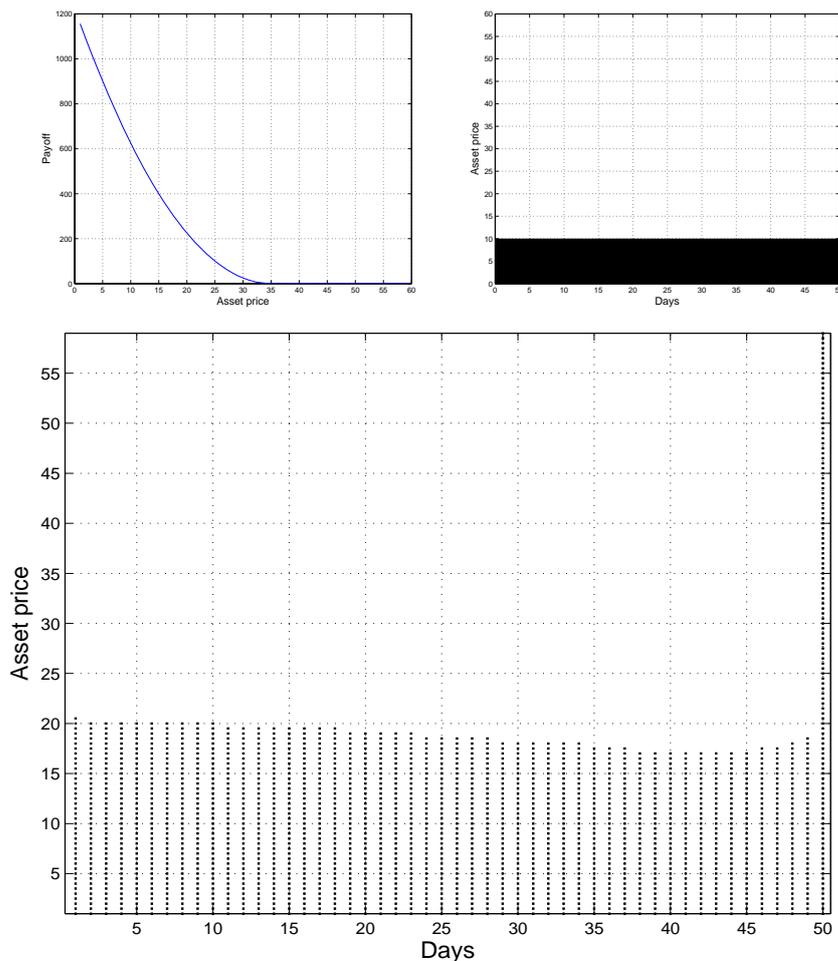


Figure 13: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

small change in price down, the payoff will have a big change.

Now we consider the quadratic payoff function together with a knock out domain. The knock out domain corresponds to a straight barrier and it is given by equation (15) with parameter $\alpha = 10$. This corresponds to a barrier put option of down and out type having a barrier at 10, with a quadratic payoff function. The result is given in Figure 13 and it is seen that the stopping domain is no longer monotonic. That because of the probability of entering the knock out domain is greater when t is far from T . This will give the effect of the boundary going down and giving the opportunity to stop at a larger payoff at a later date. But at day 45 due to so short time left to maturity the stopping strategy will become more profitable. These two factors will give the structure of the stopping domain. It is also noted that this stopping domain has similar structure as the standard put option with a straight barrier given in Figure 2. But with the quadratic put the

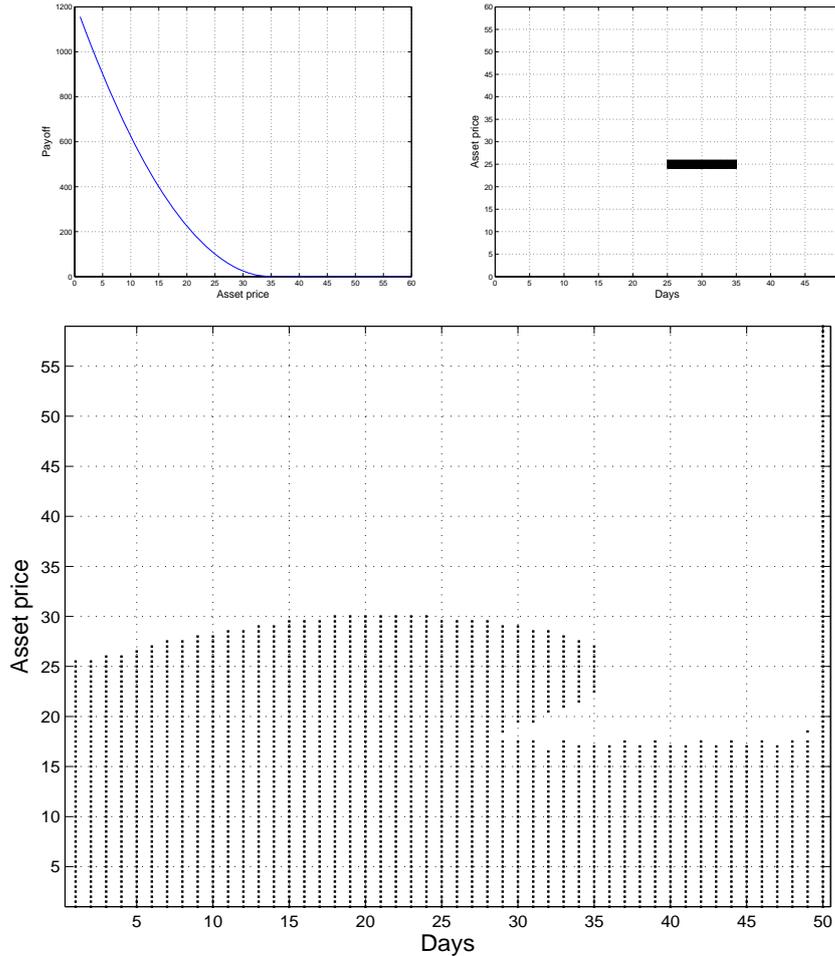


Figure 14: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

continuation strategy is more competitive which will make the continuation region larger than for the standard put payoff.

Now instead of considering a knock out domain that is homogeneous in time, we consider a knock out domain that has the shape of a small box together with the quadratic payoff function. The box is given by equation (16) with parameters $c = 25$, $d = 35$, $\alpha = 26$ and $\beta = 24$. The result is given in Figure 14. Since the box lies close enough to the stopping domain generated by the quadratic payoff function, the stopping domain generated by the small knock out domain has grown together with the stopping domain given by the quadratic put payoff function, and generated one joint set of the stopping domain. The result in Figure 14 is similar with the result for the standard put payoff with similar knock out domain given in Figure 3, but with the difference coming from the quadratic payoff function.

In the next case all the settings are the same apart from that the knock

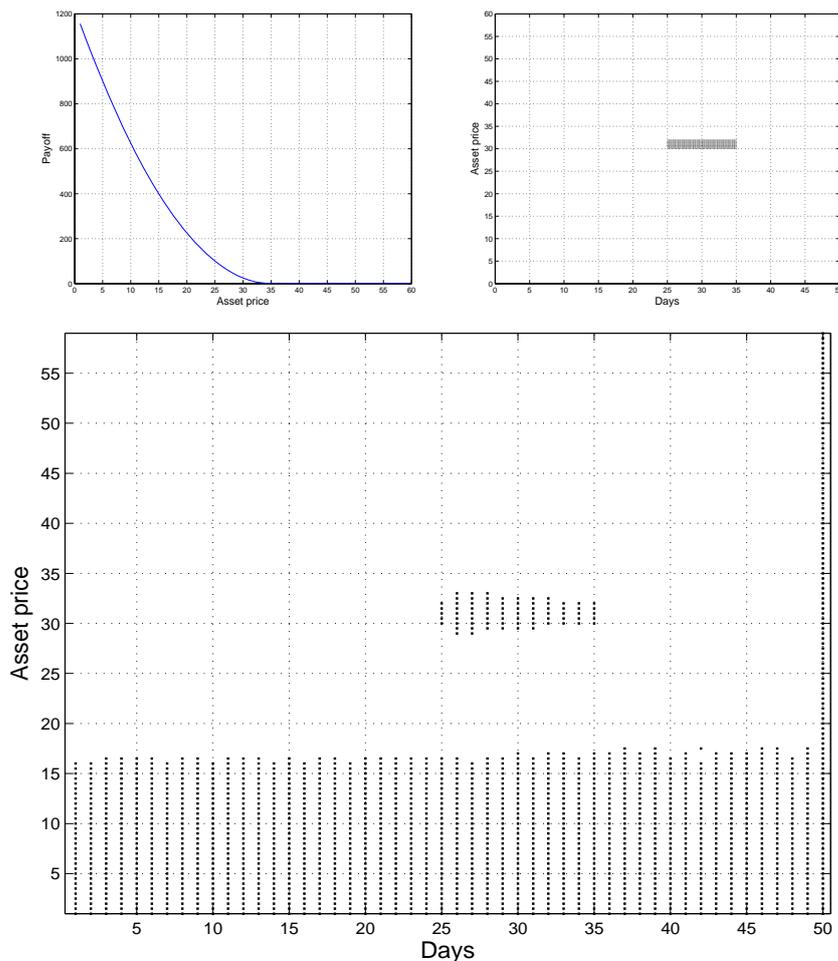


Figure 15: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

out domain has been moved further away from the stopping domain given in Figure 12 and the box is now given by equation (16) but with parameters $c = 25$, $d = 35$, $\alpha = 31$ and $\beta = 29$. The result is given in Figure 15. Now when the distance is greater the stopping domain is now two disjoint sets one similar with what the quadratic put payoff would generate. Compare with Figure 12, and we see that we first get a stopping domain similar with Figure 12 that corresponds to the stopping domain given by the payoff function. We also have one set that is covering the knock out domain and some small area around the knock out domain to make sure that the option will be exercised before entering the knock out domain. The effect would be similar if a knock out domain is introduced far from the stopping domain generated by other payoff functions too.

Domains generated from logarithmic payoff functions. In this section several combinations of knock out domains together with the loga-

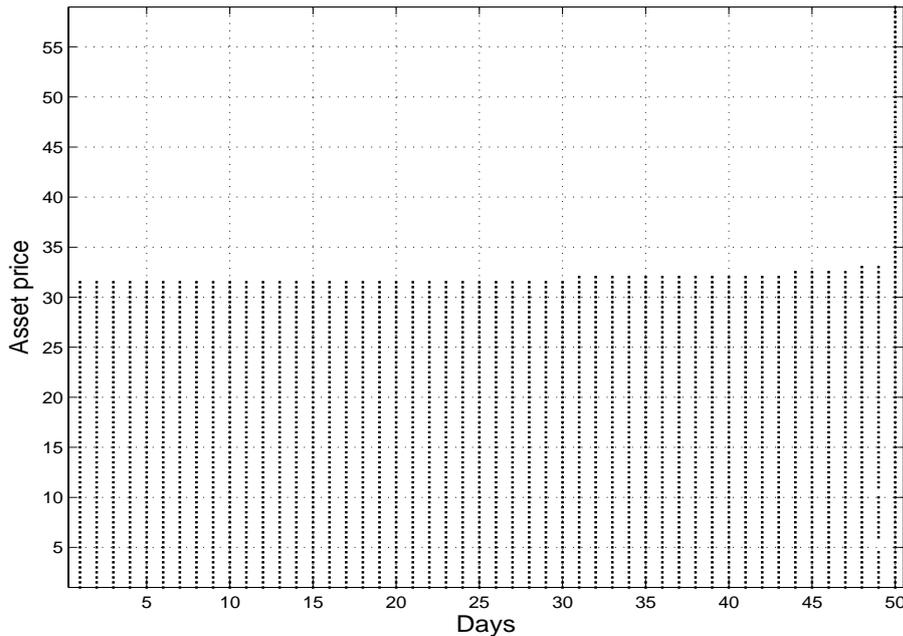


Figure 16: The stopping domain generated by a logarithmic put payoff function having parameters $K = 35$ and $a = 1.0$

rithmic payoff function are considered. The logarithmic payoff function is given by equation (13) with parameters $K = 35$ and $a = 1.0$. The first considered example is again the logarithmic payoff function with no knock out domain at all, as a benchmark, the result is presented in Figure 16. The stopping domain is again similar with Figure 1 but here the stopping region is greater, that due to the logarithmic payoff function, and the fact that a small change in price will give an even smaller change in payoff.

Secondly, we consider a case with the logarithmic payoff function in combination with a constant barrier at 20. The constant barrier are described by equation (15) having parameter $\alpha = 20$. The result here is similar as for the standard put payoff with barrier given in Figure 2, but here the stopping region again is greater.

Now we consider the logarithmic payoff function in combination with a knock out domain having the shape of a small box. The box is given by equation (16) with parameters $c = 25$, $d = 35$, $\alpha = 32$ and $\beta = 30$. The result is presented in Figure 18. We note that the knock out domain give some effects in the stopping domain but not so visible as in previous sections.

More generally, we see that for stopping domains generated by the logarithmic payoff function, the effect coming from the knock out domain is small. That effect can be explained due to the fact that a small change in price of underlying will give an even smaller change in payoff.

Discussion. It is seen that the combination of different payoff func-

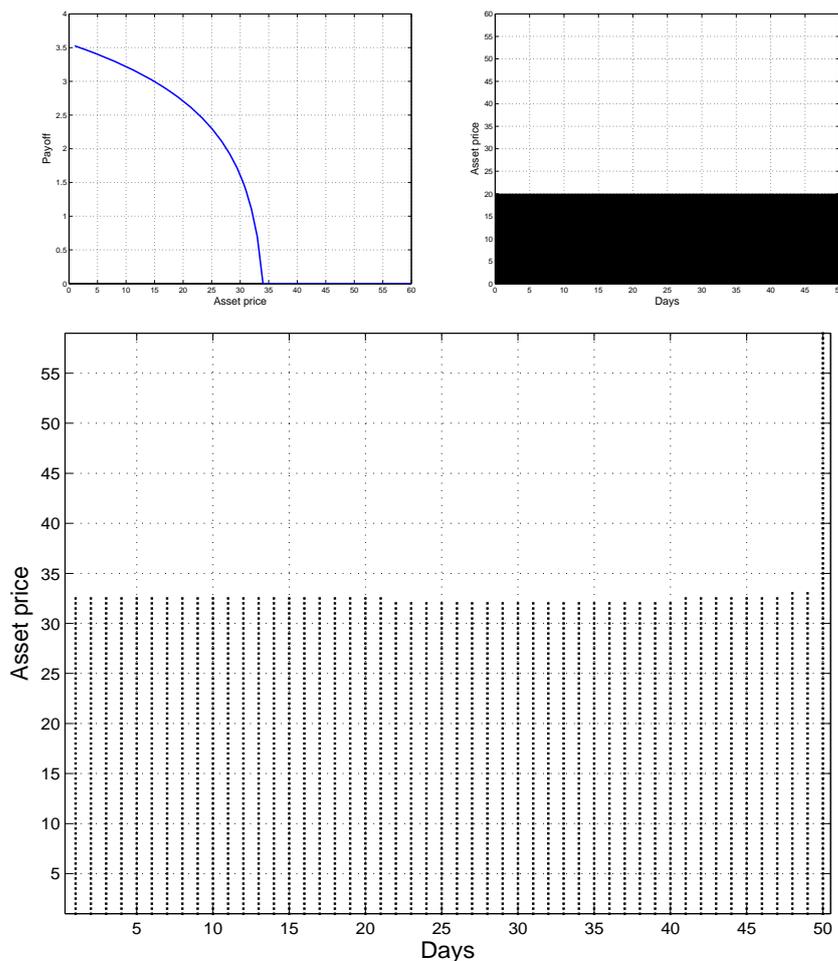


Figure 17: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

tions with different kind of knock out domains can yield complex stopping domains. If the stepwise payoff is excluded similar effects can be observed for the same type of knock out domains. If Figure 2, Figure 13 and Figure 17 are compared many similarities of there structure is found. The payoff function will only determine the size of the stopping domain. For the logarithmic payoff the continuation strategy is less competitive due to the small payoff you gain for the risk you take of keeping the contract. For the quadratic payoff it is the opposite, you will gain a large payoff for keeping the contract which makes the continuation strategy more competitive. Similar effects can also be seen in Figure 5, but it is not as visual since the barrier is far from the upper boundary of the stopping domain. By considering the *inner lake* this effect is also found for the piecewise linear payoff function.

If instead Figure 3, Figure 14 and Figure 18 are compared similarities

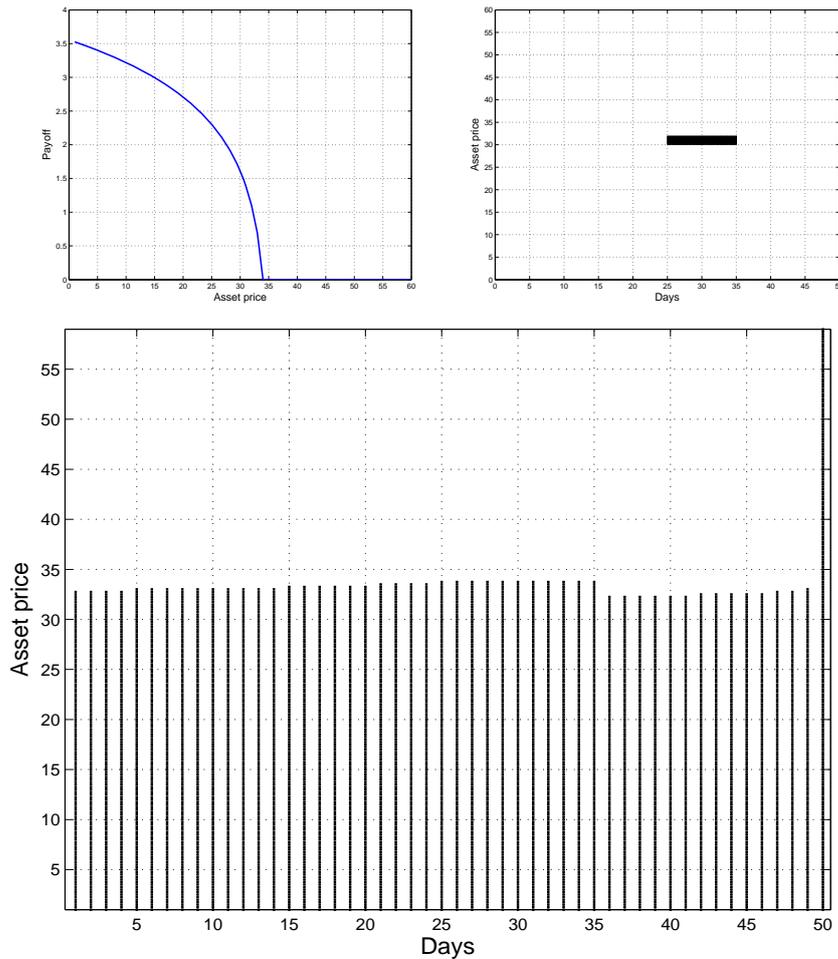


Figure 18: Upper left: The considered payoff function. Upper right: The structure of knock out domain. Down: The corresponding optimal stopping domain.

in their structure will be found for them too, again the payoff function will determine the size of the stopping domain. The effect seen in Figure 15 can be seen for the other payoff functions too, that if a box is placed far away from the stopping domain, but still in-the-money, a small *island* will be created around that box.

By comparing Figure 6, Figure 7, Figure 10 and Figure 11 similar effects again can be found, the difference in the structure depends upon the discrete steps in reward for the stepwise payoff function.

To summarize: A knock out domain of barrier type will due to the probability of entering it give a convex stopping domain for all payoff functions. The stopping domain will also suggest exercise at a lower profit than for an ordinary contract if a barrier type option of down and out is considered. That because the probability of entering the knock out domain will

affect the optimal strategy. A box shaped knock out domain close above the boundary of the stopping domain for a payoff function with no knock out domain, will make the combined stopping domain to cover the knock out domain and form a *wave* shaped stopping domain. If the box is further away the stopping domain will form a small *island* around the box and some small surroundings. If the box is placed in between a *harbor* and the stopping domain the stopping domain can grow together and form an *inner lake*. By adding more boxes more *waves* and *inner lakes* will appear.

7. CONCLUSIONS

This paper presents the results of experimental studying of the structure of optimal stopping domains for American knock out options. The optimal stopping time is given by first hitting moment of the optimal stopping domain (7), the structure of optimal stopping domain is given by the recursion (8).

From Table 1 and Table 3 it is seen that the volatility increases rapidly and the expectation decreases close to the knock out domain. From Table 2 and Table 4, it is seen that the probability of making classification errors has the greatest value near the boundary of the optimal stopping domain and maturity.

Stopping domains can poses complex multi-threshold structure determined by the combinations of different payoff functions and different knock out domains. Several combinations is presented in section 6. By knowing the structure of the stopping domain the interesting points for investigation can be chosen, hence speed of generating the stopping domains can be increased.

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