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ASYMPTOTIC EXPANSIONS FOR DISTRIBUTIONS OF THE SURPLUS PRIOR AND AT THE TIME OF RUIN

Asymptotic expansions for the distribution of the surplus prior to and at the time of a ruin are given for nonlinearly perturbed risk processes.

1. Introduction

Let $X_u^{(\varepsilon)}(t), t \geq 0$, be a standard risk process defined for every $\varepsilon \geq 0$ (perturbation parameter) in the following way:

$$X_u^{(\varepsilon)}(t) = u + c^{(\varepsilon)}t - \sum_{k=1}^{N^{(\varepsilon)}(t)} Z_k^{(\varepsilon)}, \ t \geq 0. \tag{1}$$

The process $X_u^{(\varepsilon)}(t)$ is a classical model used to describe functioning of an insurance company. Here, **(a)** u is a nonnegative constant that denotes an initial capital of a company; **(b)** a positive constant $c^{(\varepsilon)}$ denotes the gross premium rate; **(c)** $N^{(\varepsilon)}(t), t \geq 0$ is a Poisson process (with a parameter $\lambda^{(\varepsilon)}$) that counts the number of claims to the company in the time-intervals $[0, t], t \geq 0$; **(d)** $Z_k^{(\varepsilon)}, k = 1, 2, \ldots$, is a sequence of nonnegative i.i.d. random variables ($Z_k^{(\varepsilon)}$ is the amount of the k^{th} claim) with a distribution function $G^{(\varepsilon)}(u)$ which have a finite positive mean $\mu^{(\varepsilon)} = \int_0^\infty uG^{(\varepsilon)}(du) \in (0, \infty)$; **(e)** the sequence of the random variables $Z_k^{(\varepsilon)}, k = 1, 2, \ldots$, and the process $N^{(\varepsilon)}(t), t \geq 0$, are independent.

We assume to hold the following condition, which let us consider the risk process $X_u^{(\varepsilon)}(t)$, for $\varepsilon > 0$, as a perturbed version of the risk process $X_u^{(0)}(t)$:

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A: **(a)** $c^{(\varepsilon)} \to c^{(0)}$ as $\varepsilon \to 0$; **(b)** $\lambda^{(\varepsilon)} \to \lambda^{(0)}$ as $\varepsilon \to 0$; **(c)** $G^{(\varepsilon)}(\cdot) \Rightarrow G^{(0)}(\cdot)$ as $\varepsilon \to 0$.

Let us introduce, for $u \geq 0$, a random variable, which is the time of a ruin,

$$\tau_u^{(\varepsilon)} = \inf(t \geq 0 : \ X_u^{(\varepsilon)}(t) < 0).$$

This random variable takes values in the interval $(0, \infty]$ with probability 1. By the definition, the probability of ruin can be expressed as,

$$\mathsf{P}^{(\varepsilon)}(u) = \mathsf{P}\{\tau_u^{(\varepsilon)} < \infty\}, \ u \geq 0.$$

Let us introduce parameter $\alpha^{(\varepsilon)} = \frac{\lambda^{(\varepsilon)}\mu^{(\varepsilon)}}{c^{(\varepsilon)}}$ referred usually as a safety loading coefficient. If $\alpha^{(\varepsilon)} \geq 1$ then the ruin probability $\mathsf{P}^{(\varepsilon)}(u) = 1$. That is why, we also assume the following condition:

B: $\alpha^{(\varepsilon)} \in (0, 1]$ for every $\varepsilon \geq 0$.

Let us now introduce, for $u, x, y \geq 0$, random variables $X_u^{(\varepsilon)}(\tau_u^{(\varepsilon)} - 0)$ and $-X_u^{(\varepsilon)}(\tau_u^{(\varepsilon)})$ that are, respectively, the surplus (of capital) prior to and at the time of the ruin. These random variables are well defined if $\tau_u^{(\varepsilon)} < \infty$. Let us also assign them value $+\infty$ if $\tau_u^{(\varepsilon)} = \infty$.

Let us now introduce the following surplus probabilities,

$$\mathsf{P}_{x,y}^{(\varepsilon)}(u) = \mathsf{P}\{\tau_u^{(\varepsilon)} < \infty, -X_u^{(\varepsilon)}(\tau_u^{(\varepsilon)}) > x, X_u^{(\varepsilon)}(\tau_u^{(\varepsilon)} - 0) > y\}, \ u, x, y \geq 0.$$

Note that, by the definition, both random variables, the surplus at the time of the ruin, $-X_u^{(\varepsilon)}(\tau_u^{(\varepsilon)})$, and the surplus prior to the ruin, $X_u^{(\varepsilon)}(\tau_u^{(\varepsilon)} - 0)$, are positive random variables with probability 1. This implies that $\mathsf{P}^{(\varepsilon)}(u) = \mathsf{P}_{0,0}^{(\varepsilon)}(u), u \geq 0$, i.e., the ruin probability $\mathsf{P}^{(\varepsilon)}(u)$ is a particular case of the surplus probability $\mathsf{P}_{x,y}^{(\varepsilon)}(u)$.

The surplus prior and at the time of ruin and its asymptotics have been studied in Gerber, Goovarerts, and Kaas (1987), Dufresne and Gerber (1988), Dickson (1992), Dickson, Dos Reis, and Waters (1995), Willmot and Lin (1998), Rolski, Schmidli, Schmidt, and Teugels (1999), Schmidli (1999), and Badescu, Breuer, Drekic, Latouche, and Stanford (2005).

In this paper, we give asymptotic exponential expansions for the distribution of the surplus prior to and at the time of a ruin. These expansions generalise to the case of surplus probabilities results on Cramér-Lundberg and diffusion approximations of ruin probabilities for nonlinearly perturbed risk processes obtained in Gyllenberg and Silvestrov (1999, 2000a).

2. Main results

A starting point in our asymptotic analysis is the following renewal equation for the surplus probability $\mathsf{P}^{(\varepsilon)}_{x,y}(u)$, as function in $u \geq 0$, given in Schmidli (1999):

$$\mathsf{P}^{(\varepsilon)}_{x,y}(u) = \alpha^{(\varepsilon)}(1-\bar{G}^{(\varepsilon)}(u \vee y + x)) + \alpha^{(\varepsilon)} \int_0^u \mathsf{P}^{(\varepsilon)}_{x,y}(u-s)\bar{G}^{(\varepsilon)}(ds), \ u \geq 0, \quad (2)$$

where the distribution function $\bar{G}^{(\varepsilon)}(s)$, referred usually as a steady claim distribution, is defined by the following formula,

$$\bar{G}^{(\varepsilon)}(s) = \frac{1}{\mu^{(\varepsilon)}} \int_0^s (1 - G^{(\varepsilon)}(v))dv, \ s \geq 0.$$

This renewal equation (2) reduces to the well known renewal equation satisfied by the ruin probabilities $\mathsf{P}^{(\varepsilon)}(u)$ in the case where $x, y = 0$.

We apply to this equation asymptotic results concerned perturbed renewal equation obtained in Silvestrov (1978, 1979, 1995) and Gyllenberg and Silvestrov (2000a, 2000b).

Consider the following moment generating function:

$$\varphi^{(\varepsilon)}(\rho) = \int_0^\infty e^{\rho s}(1 - G^{(\varepsilon)}(s))ds = \mu^{(\varepsilon)} \int_0^\infty e^{\rho s}\bar{G}^{(\varepsilon)}(ds), \ \rho \in R_1.$$

The following condition is the Cramér type condition for the claim distribution:

C: There exists $\delta > 0$ such that

- **(a)** $\overline{\lim}_{0 \leq \varepsilon \to 0} \varphi^{(\varepsilon)}(\delta) < \infty$;
- **(b)** $\frac{\lambda^{(0)}}{c^{(0)}}\varphi^{(0)}(\delta) = \alpha^{(0)} \int_0^\infty e^{\delta s}\bar{G}^{(0)}(ds) \in (1, \infty)$.

Let us also consider the following characteristic equation

$$\alpha^{(\varepsilon)} \int_0^\infty e^{\rho s}(1 - \bar{G}^{(\varepsilon)}(s))ds = 1. \quad (3)$$

Conditions **A**, **B**, and **C** guarantee that **(a)** there exist $\varepsilon_1 = \varepsilon_1(\delta) > 0$ such that the characteristic equation (3) has, for every $0 \leq \varepsilon \leq \varepsilon_1$, a unique non-negative root $\rho^{(\varepsilon)} < \delta$, and **(b)** $\rho^{(\varepsilon)} \to \rho^{(0)}$ as $\varepsilon \to 0$.

Note also that the root $\rho^{(0)} > 0$ if the limit value of the safety loading coefficient $\alpha^{(0)} < 1$, and $\rho^{(0)} = 0$ if $\alpha^{(0)} = 1$. These cases correspond, respectively, to the models of Cramér-Lundberg and diffusion approximations.

Let us introduce, for $n = 0, 1, \ldots$, the mixed power-exponential moment generating functions

$$\varphi^{(\varepsilon)}[\rho, n] = \int_0^\infty s^n e^{\rho s}(1 - G^{(\varepsilon)}(s))ds = \mu^{(\varepsilon)} \int_0^\infty s^n e^{\rho s}\bar{G}^{(\varepsilon)}(ds), \ \rho \in R_1.$$

By the definition, $\varphi^{(\varepsilon)}[\rho, 0] = \varphi^{(\varepsilon)}(\rho)$.

Let us choose an arbitrary $\rho^{(0)} < \beta < \delta$. Condition **C** implies that **(c)** there exists $\varepsilon_2 = \varepsilon_2(\beta) > 0$ such that, for $\varepsilon \leq \varepsilon_2$ and $n = 0, 1, \dots$,

$$\varphi^{(\varepsilon)}[\beta, n] \leq c_n \int_0^\infty e^{\delta s}(1 - G^{(\varepsilon)}(s))ds < \infty, \tag{4}$$

where $c_n = c_n(\delta, \beta) = \sup_{s \geq 0} s^n e^{-(\delta-\beta)s} < \infty$.

Note also that $\varphi^{(\varepsilon)}[\rho, n]$, for $\rho \leq \beta$, is the derivative of order n of the function $\varphi^{(\varepsilon)}(\rho)$.

Denote

$$\begin{aligned} \pi^{(\varepsilon)}_{x,y}(\rho^{(\varepsilon)}) &= \frac{\int_0^\infty e^{\rho^{(\varepsilon)}s}(1 - \bar{G}^{(\varepsilon)}(s))ds}{\int_0^\infty s e^{\rho^{(\varepsilon)}s}\bar{G}^{(\varepsilon)}(ds)} \qquad (5) \\ &= \frac{\int_0^\infty e^{\rho^{(\varepsilon)}s}(\int_s^\infty (1 - G^{(\varepsilon)}(u))du)ds}{\int_0^\infty s e^{\rho^{(\varepsilon)}s}(1 - G^{(\varepsilon)}(s)ds}. \end{aligned}$$

Relation **(b)** implies that **(d)** there exists $\varepsilon_3 = \varepsilon_3(\beta) > 0$ such that $\rho^{(\varepsilon)} < \beta$.

Define $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Conditions **A**, **B**, and **C** imply, due to relations **(b)** and **(d)**, that **(e)** $\int_0^\infty e^{\rho^{(\varepsilon)}s}(1 - \bar{G}^{(\varepsilon)}(s))ds < \infty$ and **(f)** $\int_0^\infty s e^{\rho^{(\varepsilon)}s}\bar{G}^{(\varepsilon)}(ds) < \infty$ for $\varepsilon \leq \varepsilon_3$. Therefore, the quantity $\pi^{(\varepsilon)}(\rho^{(\varepsilon)})$ is well defined for all $\varepsilon \leq \varepsilon_0$.

The following theorem describe asymptotic behaviour of the surplus probabilities.

Theorem 1. *Let conditions* **A**, **B**, *and* **C** *hold. Then, for any* $u^{(\varepsilon)} \to \infty$ *as* $\varepsilon \to 0$, *the following asymptotic relation holds for every* $x, y \geq 0$,

$$\frac{\mathsf{P}^{(\varepsilon)}_{x,y}(u^{(\varepsilon)})}{\exp\{-\rho^{(\varepsilon)}u^{(\varepsilon)}\}} \to \pi^{(0)}_{x,y}(\rho^{(0)}) \ as\ \varepsilon \to 0. \tag{6}$$

Let now assume that the following nonlinear perturbation conditions hold for some integer $k \geq 1$:

$\mathbf{D_1^{(k)}}$: $c^{(\varepsilon)} = c^{(0)} + c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k)$, where $|c_l| < \infty, l = 1, \dots, k$;

$\mathbf{D_2^{(k)}}$: $\lambda^{(\varepsilon)} = \lambda^{(0)} + d_1\varepsilon + \cdots + d_k\varepsilon^k + o(\varepsilon^k)$, where $|d_l| < \infty, l = 1, \dots, k$;

$\mathbf{D_3^{(k)}}$: $\varphi^{(\varepsilon)}[\rho^{(0)}, n] = \varphi^{(0)}[\rho^{(0)}, n] + v_1[\rho^{(0)}, n]\varepsilon + \cdots + v_{k-n}[\rho^{(0)}, n]\varepsilon^{k-n} + o(\varepsilon^{k-n})$, where $|v_l[\rho^{(0)}, n]| < \infty, l = 1, \dots, k-n, n = 0, \dots, k$.

Conditions $\mathbf{D_1^{(k)}}$, $\mathbf{D_2^{(k)}}$, and $\mathbf{D_3^{(k)}}$ mean that the characteristic quantities of the perturbed risk processes penetrating the above perturbation conditions are nonlinear functions of ε. These conditions correspond to the model of smooth perturbation where these functions have k derivatives at $\varepsilon = 0$, i.e., they can be expanded in a power series with respect to ε up to and including the order k.

The relationship between the rates at which the perturbation parameter ε tends to zero and the initial capital u tends to infinity has an influence upon the obtained results. Without loss of generality it can be assumed that $u = u^{(\varepsilon)}$ is a function of the parameter ε. The relationship between the rate of perturbation and the rate of growth of the initial capital is characterized by the following balancing condition that is assumed to hold for some integer $1 \leq r \leq k$:

$\mathbf{E^{(r)}}$: $u^{(\varepsilon)} \to \infty$ as $\varepsilon \to 0$ such that $\varepsilon^r u^{(\varepsilon)} \to \varrho_r$ as $\varepsilon \to 0$, where $\varrho_r \in [0, \infty)$.

The following theorem presents the asymptotic exponential expansions for surplus probabilities $\mathsf{P}^{(\varepsilon)}_{x,y}(u)$ for nonlinearly perturbed risk processes.

Theorem 2. *Let conditions* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D_1^{(k)}}$, $\mathbf{D_2^{(k)}}$, $\mathbf{D_3^{(k)}}$, *and* $\mathbf{E^{(r)}}$ *hold. Then, the following asymptotic relation holds for every* $x, y \geq 0$,

$$\frac{\mathsf{P}^{(\varepsilon)}_{x,y}(u^{(\varepsilon)})}{\exp\{-(\rho^{(0)} + a_1\varepsilon + \cdots + a_{r-1}\varepsilon^{r-1})u^{(\varepsilon)}\}} \to e^{-\varrho_r a_r}\pi^{(0)}_{x,y}(\rho^{(0)}) \text{ as } \varepsilon \to 0, \quad (7)$$

where the coefficients $a_1, \ldots a_k$ *are given by explicit recurrence formulas as functions of coefficients in the expansions penetrating the perturbation conditions* $\mathbf{D_1^{(k)}}$, $\mathbf{D_2^{(k)}}$, *and* $\mathbf{D_3^{(k)}}$.

3. Conclusion

In conclusion, I would like to note that this paper presents the result of the author included in a new book [9] written in cooperation with Professor Mats Gyllenberg. The algorithm for calculation of the coefficients in the asymptotic expansions (7) and proofs of Theorems 1 and 2 are given in this book.

The book mentioned above is devoted to studies of quasi-stationary phenomena in nonlinearly perturbed stochastic systems. The methods based on exponential asymptotics for nonlinearly perturbed renewal equation are used. Mixed ergodic and large deviation theorems are presented for nonlinearly perturbed regenerative processes, semi-Markov processes and Markov chains. Applications to nonlinearly perturbed population dynamics and epidemic models, queueing systems and risk processes are considered. The book also includes an extended bibliography of works in the area.

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