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POINCARÉ INEQUALITY AND EXPONENTIAL INTEGRABILITY OF THE HITTING TIMES OF A MARKOV PROCESS

Extending the approach of the paper [Mathieu, P. (1997) Hitting times and spectral gap inequalities, Ann. Inst. Henri Poincaré 33, 4, 437–465], we prove that the Poincaré inequality for a (possibly non-symmetric) Markov process yields the exponential integrability of the hitting times of this process. For symmetric elliptic diffusions, this provides a criterion for the Poincaré inequality in the terms of hitting times.

1. Introduction

In this paper, we investigate the relations between the following two topics:

- the *Poincaré inequality* for the Dirichlet form associated to a Markov process;
- the exponential integrability for *hitting times* of the process.

It is well known that when the state space of a (symmetric) Markov process is finite, the topics listed above are, in fact, equivalent; see the detailed exposition in [1], Chapters 2 – 4. When the state space is infinite, the relations between these topics are more complicated, see the detailed discussion in [8]. The purpose of this paper is two-fold. First, we extend the approach of the paper [11], where it was proved that hitting times of a Markov process are integrable assuming some weak version of the Poincaré inequality holds true. In this paper, we prove that the Poincaré inequality itself provides the *exponential integrability* for hitting times. Next, we show that, on the contrary, under some additional assumptions the exponential integrability for hitting times yields the Poincaré inequality. According to the recent paper [8], for a given Markov process X the *spectral gap* property can be verified in the following way. First, one proves some of the *local non-degeneracy* conditions on the transition probabilities of the Markov process X (minorization condition, Doeblin condition, Dobrushin condition). Second, one finds *some* Lyapunov-type function ϕ such that the recurrence conditions 1) – 3) of Theorem 2.2 in [8] hold true. Then X admits an *exponential ϕ-coupling* (see Definition 2.2 in [8]) according to Theorem 2.2 in [8]. When X is time-reversible (i.e. symmetric), this provides the spectral gap property, which in this case is equivalent to the the Poincaré inequality. The case where X is time-irreversible is more intrinsic; because we do not address this case in the current paper we skip the discussion of this case and refer the interested reader to the paper [8]. Note that in the strategy outlined above there is a lot of freedom in the choice of the Lyapunov-type function ϕ. Proposition 2.4 in [8] reduces such a choice to the class of the functions of the form

$$\phi(x) = E_x e^{\alpha \tau_K}, \quad \text{where} \quad \tau_K = \inf\{t : X_t \in K\} \tag{1}$$

is the hitting time of a compact set K. In Section 3 below we demonstrate that, for particularly important class of symmetric diffusions, this reduction corresponds to the matter of the problem precisely, and deduce the Poincaré inequality under the exponential integrability assumption for the hitting times. Together with the result of Section 2 this provides an equivalence for symmetric diffusions of the Poincaré inequality on one

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hand, and the the exponential integrability of the hitting times on the other hand. Such an equivalence for linear diffusions was established in the recent preprint [9]; note that the case of the symmetric elliptic diffusion on a multidimensional manifold, considered in the current paper, is more complicated because one can not apply here such particularly useful characteristics of the linear diffusion as the scale function and the speed measure. It should be mentioned that in the recent years the relations between the functional inequalities related to a Markov process (like the Poincaré inequality) and the ergodic properties of this process (like the Lyapunov-type condition) have been studied extensively, e.g. [2], [4], [5]. In this paper, we show, in particular, that for a symmetric elliptic diffusion this relation is one-to-one, and the Poincaré inequality is *equivalent* to the Lyapunov-type condition with the Lyapunov-type function of the form (1) (this equivalence was investigated, as well, in the recent preprint [5]).

2. Exponential moments for hitting times under Poincaré inequality

We consider a time homogeneous Markov process $X = \{X_t, t \in \mathbb{R}^+\}$ with a locally compact metric space $(\mathbb{X}, \rho)$ as the state space. The process X is supposed to be strong Markov and to have cádlág trajectories. The transition function for the process X is denoted by $P_t(x, dy), t \in \mathbb{R}^+, x \in \mathbb{X}$. We use standard notation P_x for the distribution of the process X conditioned that $X_0 = x, x \in \mathbb{X}$, and E_x for the expectation w.r.t. P_x. All the functions on $\mathbb{X}$ considered in the paper are assumed to be measurable w.r.t. the Borel σ-algebra $\mathcal{B}(\mathbb{X})$. The set of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is denoted by $\mathcal{P}(\mathbb{X})$. For a given $\mu \in \mathcal{P}(\mathbb{X})$ and $t \in \mathbb{R}^+$, we denote $\mu_t(dy) \stackrel{df}{=} \int_{\mathbb{X}} P_t(x, dy)\, \mu(dx)$. The probability measure μ is called an invariant measure for X if $\mu_t = \mu, t \in \mathbb{R}^+$.

In what follows, we suppose an invariant measure π for the process X to be fixed. The process X generates the semigroup $\{T_t\}$ in $L_2 = L_2(\mathbb{X}, \pi)$:

$$T_t f(x) = \int_{\mathbb{X}} f(y) P_t(x, dy), \quad f \in L_2, \quad t \in \mathbb{R}^+.$$

Let A be the generator of this semigroup, and $\mathcal{E}$ be the associated Dirichlet form; which is defined as the completion of the bilinear form

$$Dom(A) \times Dom(A) \ni (f, g) \mapsto -(Af, g)_{L_2}$$

with respect to the norm $\| \cdot \|_{\mathcal{E},1} \stackrel{df}{=} \Big[\| \cdot \|_2^2 - (A \cdot, \cdot)_{L_2} \Big]^{1/2}$ (e.g. [10], Chapter 2). Within this paper, we are mainly interested in the following *Poincaré inequality*:

$$\mathrm{Var}_\pi(f) \stackrel{df}{=} \int_{\mathbb{X}} f^2 d\pi - \left(\int_{\mathbb{X}} f d\pi \right)^2 \le c\, \mathcal{E}(f, f), \quad f \in Dom(\mathcal{E}). \tag{2}$$

In what follows, the form $\mathcal{E}$ is supposed to be regular; that is, the set $Dom(\mathcal{E}) \cap C_0(\mathbb{X})$ is claimed to be dense both in $Dom(\mathcal{E})$ w.r.t. the norm $\| \cdot \|_{\mathcal{E},1}$ and in $C_0(\mathbb{X})$ w.r.t. uniform convergence on a compacts ($C_0(\mathbb{X})$ is the set of continuous functions with compact supports). We also assume that the following *sector condition* holds true:

$$\exists D \in \mathbb{R}^+ : \quad |\mathcal{E}(f, g)| \le D \|f\|_{\mathcal{E},1} \|g\|_{\mathcal{E},1}, \quad f, g \in Dom(\mathcal{E}).$$

It is well-known (see the discussion in Introduction to [11] and references therein) that the hitting times τ_K have natural application in the probabilistic representation for the family of *α-potentials* for the Dirichlet form $\mathcal{E}$. The α-potential, for given $\alpha > 0$ and closed $K \subset \mathbb{X}$, is defined as the function $h_\alpha^K \in Dom(\mathcal{E})$ such that $h_\alpha^K = 1$ quasi-everywhere on K, and $\mathcal{E}(h_\alpha^K, u) = -\alpha(h_\alpha^K, u)$ for every quasi-continuous function $u \in Dom(\mathcal{E})$ such that $u = 0$ quasi-everywhere on K. On the other hand,

$$h_\alpha^K(x) = E_x e^{-\alpha \tau_K}, \quad x \in \mathbb{X}.$$

It is a straightforward corollary of the part (i) of the main theorem from [11] that, if X possesses (2) with some $c > 0$, then $E_\pi \tau_K < +\infty$ for every K with $\pi(K) > 0$ (here and below, $E_\pi \stackrel{df}{=} \int_{\mathbb{X}} E_x \, \pi(dx)$). We will prove the following stronger version of this statement.

Theorem 2.1. *Assume X possess (2) with some $c > 0$. Then for every closed set $K \subset \mathbb{X}$ with $\pi(K) > 0$*

$$E_\pi e^{\alpha\tau_K} < +\infty, \quad \alpha < \frac{\pi(K)}{c}.$$

Moreover, the function $h^K_{-\alpha}(x) \stackrel{df}{=} E_x e^{\alpha\tau_K}, x \in \mathbb{X}$ possesses the following properties:

a) $h^K_{-\alpha} \in Dom(\mathcal{E})$ and $h^K_{-\alpha} = 1$ on K;

b) $\mathcal{E}(h^K_{-\alpha}, u) = \alpha(h^K_{-\alpha}, u)$ for every quasi-continuous function $u \in Dom(\mathcal{E})$ such that $u = 0$ quasi-everywhere on K.

We assume K to be fixed and omit the respective index in the notation, e.g. write τ for τ^K and h_α for h^K_α. For $z \in \mathbb{C}$ with $\mathrm{Re}\, z > 0$, define respective z-potential:

$$h_z(x) = E_x e^{-z\tau}, \quad x \in \mathbb{X}.$$

Denote by $H_{\mathcal{E}}$ the $Dom(\mathcal{E})$ considered as a Hilbert space with the scalar product

$$(f, g)_{\mathcal{E},1} \stackrel{df}{=} (f, g)_{L_2} + \mathcal{E}(f, g).$$

The following lemma shows that $\{h_z, \mathrm{Re}\, z > 0\}$ can be considered as an analytical extension of the family of α-potentials $\{h_\alpha, \alpha > 0\} \subset H_{\mathcal{E}}$ that, in addition, keeps the properties of this family.

Lemma 2.1. *1) The function $z \mapsto h_z$ is analytic as a function taking values in the Hilbert space $H_{\mathcal{E}}$.*

2) For every z with $\mathrm{Re}\, z > 0$, the following properties hold:

(i) $h_z = 1$ quasi-everywhere on K;

(ii) $\mathcal{E}(h_z, u) = -z(h_z, u)$ for every quasi-continuous function $u \in Dom(\mathcal{E})$ such that $u = 0$ quasi-everywhere on K.

Proof. Denote $h^m_z(x) = (-1)^m E_x \tau^m e^{-z\tau}, x \in \mathbb{X}, m \geq 1$. One can verify easily that, for every $m \in \mathbb{N}$,

$$\frac{d^m}{dz^m} h_z = h^m_z \tag{3}$$

on the set $\{z : \mathrm{Re}\, z > 0\}$, with the function $z \mapsto h_z$ is considered as a function taking values in L_2. In addition,

$$\|h^m_z\|_2^2 \leq E_\pi \left|\tau^m e^{-z\tau}\right|^2 = E_\pi \tau^{2m} e^{-2\tau \mathrm{Re}\, z} \leq \frac{(2m)!}{(2\mathrm{Re}\, z)^{2m}},$$

since $\frac{(2\tau \mathrm{Re}\, z)^{2m}}{(2m)!} \leq e^{2\tau \mathrm{Re}\, z}$. Therefore,

$$\frac{\|h^m_z\|_2}{m!} \leq \sqrt{\frac{C^m_{2m}}{2^{2m}}} (\mathrm{Re}\, z)^{-m} < (\mathrm{Re}\, z)^{-m}, \quad m \in \mathbb{N}, \tag{4}$$

and hence the function

$$\{z : \mathrm{Re}\, z > 0\} \ni z \mapsto h_z \in L_2$$

is analytic.

For every $\alpha, \alpha' > 0$ we have $h_\alpha - h_{\alpha'} = 0$ quasi-everywhere on K. Hence

$$\begin{aligned} \mathcal{E}(h_\alpha - h_{\alpha'}, h_\alpha - h_{\alpha'}) &= \mathcal{E}(h_\alpha, h_\alpha - h_{\alpha'}) - \mathcal{E}(h_{\alpha'}, h_\alpha - h_{\alpha'}) \\ &= -\alpha(h_\alpha, h_\alpha - h_{\alpha'}) + \alpha'(h_{\alpha'}, h_\alpha - h_{\alpha'}) \\ &= (\alpha' - \alpha)(h_{\alpha'}, h_\alpha - h_{\alpha'}) + \alpha(h_{\alpha'} - h_\alpha, h_\alpha - h_{\alpha'}). \end{aligned} \tag{5}$$

For a given $\alpha > 0$ and $\alpha' \to \alpha$, the family $\{\frac{h_{\alpha'}-h_\alpha}{\alpha'-\alpha}\}$ converges to h^1_α in L_2, see (3). Then (5) yields that this family is bounded in $H_\mathcal{E}$, and thus is weakly compact in $H_\mathcal{E}$. Combined with the fact that this family is converges in L_2, this yields that the function $(0, +\infty) \in \alpha \mapsto h_\alpha \in H_\mathcal{E}$ is differentiable in a weak sense, and h^1_α equals its (weak) derivative at the point α.

We have $h^1_\alpha = 0$ quasi-everywhere on K, since

$$h_\alpha(x) = 1 \Leftrightarrow e^{-\alpha\tau} = 1\ P_x - \text{a.s.} \Leftrightarrow \tau = 0\ P_x - \text{a.s.} \Leftrightarrow h^1_\alpha(x) = 0.$$

In addition, since h^1_α is a weak derivative of h_α, we have $\mathcal{E}(h^1_\alpha, u) = -(h_\alpha, u) - \alpha(h^1_\alpha, u)$ for every quasi-continuous function $u \in Dom(\mathcal{E})$ such that $u = 0$ quasi-everywhere on K. Now, repeating the same arguments, we get by induction that, for every $m \geq 1$, the function $(0, +\infty) \in \alpha \mapsto h_\alpha \in H_\mathcal{E}$ is m times weakly differentiable, h^m_α is the corresponding weak derivative of the m-th order, and the following properties hold:

(i^m) $h^m_\alpha = 0$ quasi-everywhere on K;

(ii^m) $\mathcal{E}(h^m_\alpha, u) = -(h^{m-1}_\alpha, u) - \alpha(h^m_\alpha, u)$ for every quasi-continuous function $u \in Dom(\mathcal{E})$ such that $u = 0$ quasi-everywhere on K.

Property (ii) with $u = h^m_\alpha$ and estimate (4) yield that, for a given α, series

$$H_z \stackrel{df}{=} h_\alpha + \sum_{m=1}^\infty \frac{z^m}{m!} h^m_\alpha \in H_\mathcal{E}$$

converge in the circle $\{|z - \alpha| < \alpha\}$. The sum is a weakly analytic $H_\mathcal{E}$-valued function, and hence is analytic ([12], Theorem 3.31). On the other hand, the same series converge in L_2 to h_z. This yields that $h_z = H_z$ in the circle $\{|z - \alpha| < \alpha\}$. By taking various $\alpha \in (0, +\infty)$, we get that the function $z \mapsto h_z$ is an $H_\mathcal{E}$-valued analytic function inside the angle $\mathcal{D}_1 \stackrel{df}{=} \{z : \operatorname{Re} z > |\operatorname{Im} z|\}$. In addition, properties (i^m), (ii^m) of the m-th coefficients of the series ($m \geq 1$) provide that h_z satisfy (i),(ii) inside the angle.

Now, we complete the proof using the following iterative procedure. Assume that the function $z \mapsto h_z \in H_\mathcal{E}$ is analytic in some domain $\mathcal{D} \subset \{z : \operatorname{Re} z > 0\}$ and satisfy (i),(ii) in this domain. Then the same arguments with those used above show that, for every $z_0 \in \mathcal{D}$, the domain $\mathcal{D}$ can be extended to $\mathcal{D}' \stackrel{df}{=} \mathcal{D} \cup \{z : |z - z_0| < \operatorname{Re} z_0\}$ with the function $z \mapsto h_z$ still being analytic in $\mathcal{D}'$ and satisfying (i),(ii) in the extended domain. Therefore, we prove iteratively that the required statement holds true in every angle $\mathcal{D}_k \stackrel{df}{=} \{z : \operatorname{Re} z > \frac{1}{k}|\operatorname{Im} z|\}$. Since $\cup_k \mathcal{D}_k = \{z : \operatorname{Re} z > 0\}$, this completes the proof. □

Next, we consider "ψ-potentials" that correspond to functions $\psi : \mathbb{R}^+ \to \mathbb{R}$. Denote

$$h_\psi(x) = E_x\psi(\tau), \quad x \in \mathbb{X}.$$

The following statement is an appropriate modification of the inversion formula for the Laplace transform.

Lemma 2.2. *Let $\psi \in C^2(\mathbb{R})$ have a compact support and* $\operatorname{supp}\psi \subset [0, +\infty)$. *Denote $\Psi(z) = \int_\mathbb{R} e^{zt}\psi(t)\,dt$, $z \in \mathbb{C}$.*

The function h_ψ belongs to $H_\mathcal{E}$ and admits integral representation

$$h_\psi = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \Psi(z) h_z\, dz, \tag{6}$$

where $\sigma > 0$ is arbitrary, and the integral is well defined as an improper Bochner integral of an $H_\mathcal{E}$-valued function.

Proof. First, let us show that the integral in the right hand side of (6) is well defined. We have by condition (ii) of Lemma 2.1 that

$$\mathcal{E}(h_z, h_z) = \mathcal{E}(h_z, h_z - 1) = -z(h_z, h_z - 1).$$

For any z with $\operatorname{Re} z > 0$, we have $|h_z(x)| \le E_x e^{-\tau \operatorname{Re} z} \le 1$, and thus $|h_z(x) - 1| \le 2$. Hence,

$$\|h_z\|_{H_\mathcal{E}} = \sqrt{\|h_z\|_2^2 + \mathcal{E}(h_z, h_z)} \le \sqrt{1 + 2|z|}.$$

On the other hand, for ψ satisfying conditions of the lemma,

$$z^2\Psi(z) = \int_0^\infty e^{zt}\psi''(t)\,dt, \quad |z^2\Psi(z)| \le \int_0^\infty e^{t\operatorname{Re} z}|\psi''(t)|\,dt.$$

Thus, on the line $\sigma + i\mathbb{R} \stackrel{df}{=} \{z : \operatorname{Re} z = \sigma\}$, the function $z \mapsto \Psi(z)h_z \in H_\mathcal{E}$ admits the following estimate:

$$\|\Psi(z)h_z\|_{H_\mathcal{E}} \le C|z|^{-\frac{3}{2}},$$

and therefore it is integrable on $\sigma + i\mathbb{R}$. Denote by $g_\psi \in H_\mathcal{E}$ corresponding integral. In order to prove that $h_\psi = g_\psi$, it is sufficient to prove that h_ψ and g_ψ coincide as elements of L_2. Hence, we have reduced the proof of the lemma to verification of the following "weak L_2-version" of (6):

$$\int_\mathbb{X} h_\psi v\,d\pi = \frac{1}{2\pi i}\int_\mathbb{X}\int_{\sigma - i\infty}^{\sigma + i\infty} \Psi(z)h_z(x)v(x)\,dz\pi(dx), \quad v \in L_2. \tag{7}$$

Recall that $h_z(x) = E_x e^{-z\tau}$, and hence the right hand side of (7) can be rewritten to the form

$$\frac{1}{2\pi i}\int_\mathbb{X}\int_{\sigma - i\infty}^{\sigma + i\infty} E_x\Psi(z)e^{-z\tau}v(x)\,dz\pi(dx) = \frac{1}{2\pi i}\int_\mathbb{X} E_x\int_{\sigma - i\infty}^{\sigma + i\infty} \Psi(z)e^{-z\tau}v(x)\,dz\pi(dx).$$

Here, we have changed the order of integration using Fubini's theorem. This can be done, because $|\Psi(z)| \le C|z|^{-2}$, and therefore

$$E_x\int_{\sigma - i\infty}^{\sigma + i\infty} |\Psi(z)e^{-z\tau}|\,dz = h_\sigma(x)\int_{\sigma - i\infty}^{\sigma + i\infty} |\Psi(z)|\,dz \le Ch_\sigma(x).$$

The function Ψ is the (two-sided) Laplace transform for ψ, up to the change of variables $p \mapsto -z$. We write the inversion formula for the Laplace transform in the terms of Ψ and, after the change of variables, get

$$\psi(t) = \frac{1}{2\pi i}\int_{-\sigma - i\infty}^{-\sigma + i\infty} e^{pt}\Psi(-p)\,dp = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{-zt}\Psi(z)\,dz, \quad t \in \mathbb{R}^+.$$

Hence, the right hand side of (7) is equal

$$\int_\mathbb{X} E_x\psi(\tau)v(x)\pi(dx) = \int_\mathbb{X} h_\psi v\,d\pi,$$

that proves (7). □

Corollary 2.1. *Let $\psi \in C^3(\mathbb{R})$ and $\operatorname{supp}\psi' \subset [0, +\infty)$. Then $h_\psi \in Dom(\mathcal{E})$ and*

$$\mathcal{E}(h_\psi, u) = (h_{\psi'}, u) \tag{8}$$

for every $u \in Dom(\mathcal{E})$ such that $u = 0$ quasi-everywhere on K.

Proof. Assume first that $\int_{\mathbb{R}^+}\psi'(x)\,dx = 0$. Then both ψ and ψ' satisy conditions of Lemma 2.2. We have $\tilde{\Psi}(z) \stackrel{df}{=} \int_\mathbb{R} e^{zt}\psi'(t)\,dt = -z\Psi(z)$. Hence, from the representation (6) for h_ψ and $h_{\psi'}$ and relation $\mathcal{E}(h_z, u) = -z(h_z, u), \operatorname{Re} z > 0$, we get

$$\mathcal{E}(h_\psi, u) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \Psi(z)\mathcal{E}(h_z, u)\,dz = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{\Psi}(z)(h_z, u)\,dz = (h_{\psi'}, u).$$

The general case can be reduced to the one considered above by the following limit procedure. Since $\operatorname{supp}\psi' \subset [0, +\infty)$, there exist $C \in \mathbb{R}$ and $x_* \in \mathbb{R}^+$ such that $\psi(x) =$

$C, x \geq x_*$. Take a function $\vartheta \in C^3(\mathbb{R})$ such that $\vartheta(x) = 0, x \leq 0, \vartheta(x) = C, x \geq 1$, and put

$$\psi_t(x) = \psi(x) - \vartheta(x - t), \quad x \in \mathbb{R}, t > x_*.$$

Then every ψ_t satisfies the additional assumption $\int_{\mathbb{R}^+} [\psi_t]'(x)\,dx = 0$, and thus h_{ψ_t} belongs to $Dom(\mathcal{E})$ and satisfies (8). It can be verified easily that $h_{\psi_t} \to h_\psi, t \to \infty$ in L_2 sense. In addition,

$$\mathcal{E}(h_{\psi_t}, h_{\psi_t}) = (h_{[\psi_t]'}, h_{\psi_t}) \to (h_{[\psi]'}, h_\psi) < +\infty, \quad t \to +\infty$$

(here, we have used (8) with $u = h_{\psi_t}$). This means that the family $\{h_{\psi_t}\}$ is bounded in $H_\mathcal{E}$, and hence is weakly compact in $H_\mathcal{E}$. Therefore, $h_{\psi_t} \to h_\psi, t \to \infty$ weakly in $H_\mathcal{E}$. Since $h_{[\psi_t]'} \to h_{\psi'}, t \to \infty$ in L_2 sense, (8) for ψ follows from (8) for ψ_t. □

Proof of the Theorem 2.1. Now, we are ready to complete the proof of the theorem. Let us fix $\alpha < \frac{\pi(K)}{c}$, and construct the family of the functions $\varrho_t, t \geq 1$ that approximate the function $\varrho : x \mapsto e^{\alpha x} - 1$ appropriately. First, we take function $\chi \in C^3(\mathbb{R})$ such that $\chi \geq 0, \chi' \leq 0, \chi(x) = 1, x \leq 0$, and $\chi(x) = 0, x \geq 1$. We put

$$\rho_t(x) = \int_0^x \alpha e^{\alpha y} \chi(y - t)\,dy, \quad x \geq 0, t \geq 1.$$

By the construction, the derivatives of the functions $\rho_t, t \geq 1$ have the following properties:

a) $[\rho_t]' \geq 0$ and $[\rho_t]'(x) = 0, x \geq t + 1$;

b) $[\rho_s]' \leq [\rho_t]', s \leq t$.

Since $\rho_t(0) = 0, t \geq 1$, the latter property yields that $\rho_s \leq \rho_t, s \leq t$. In addition,

$$[\rho_t]''(x) = \alpha e^{\alpha x} \chi'(x) + \alpha^2 e^{\alpha x} \chi(x) \leq \alpha^2 e^{\alpha x} \chi(x) = \alpha [\rho_t]'(x),$$

since $\chi' \leq 0$. This and relation $[\rho_t]'(0) = \alpha(\rho_t(0) + 1)$ provide

$$[\rho_t]' \leq \alpha(\rho_t + 1). \tag{9}$$

At last, we take function $\theta \in C^3(\mathbb{R})$ such that $\theta' \geq 0, \theta(x) = 0, x \leq 0$, and $\theta(x) = 1, x \geq 1$. We put

$$\varrho_t(x) = \begin{cases} \theta(xt)\,\rho_t(x), & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad t \geq 1.$$

We have $\varrho_t \uparrow \varrho, t \uparrow \infty$. In addition, by (9),

$$[\varrho_t]'(x) = t\theta'(tx)\rho_t(x) + \theta(tx)[\rho_t]'(x) \leq t \sup_y \theta'(y) \rho_t(t^{-1}) + \alpha(\rho_t(x) + 1) \leq \alpha \varrho_t(x) + C \tag{10}$$

with an appropriate constant C (recall that $t\rho_t(t^{-1}) = t\alpha(e^{\alpha t^{-1}} - 1) \to \alpha^2, t \to \infty$).

Every ϱ_t satisfies conditions of Corollary 2.1, and hence

$$\int_\mathbb{X} h_{\varrho_t}^2\,d\pi - \left(\int_\mathbb{X} h_{\varrho_t}\,d\pi\right)^2 \leq c\mathcal{E}(h_{\varrho_t}, h_{\varrho_t}) = c(h_{[\varrho_t]'}, h_{\varrho_t}) \leq \alpha c(h_{\varrho_t}, h_{\varrho_t}) + C \int_\mathbb{X} h_{\varrho_t}\,d\pi.$$

Here, we have used subsequently property (2), equality (9) with $u = h_{\varrho_t}$, and (10).

We have $h_{\varrho_t} = 0$ on K because $\varrho_t(0) = 0$. Then, by the Cauchy inequality,

$$\begin{aligned} \int_\mathbb{X} h_{\varrho_t}^2\,d\pi - \left(\int_\mathbb{X} h_{\varrho_t}\,d\pi\right)^2 &= \int_\mathbb{X} h_{\varrho_t}^2\,d\pi - \left(\int_{\mathbb{X}\setminus K} h_{\varrho_t}\,d\pi\right)^2 \\ &\geq (1 - \pi(\mathbb{X} \setminus K)) \int_\mathbb{X} h_{\varrho_t}^2\,d\pi = \pi(K)(h_{\varrho_t}, h_{\varrho_t}). \end{aligned}$$

Therefore,

$$(h_{\varrho_t}, h_{\varrho_t}) \leq \frac{\alpha c}{\pi(K)}(h_{\varrho_t}, h_{\varrho_t}) + C \int_\mathbb{X} h_{\varrho_t}\,d\pi,$$

which implies that

$$(h_{\varrho_t}, h_{\varrho_t}) \le \frac{C\pi(K)}{\pi(K)-\alpha c}\int_{\mathbb{X}} h_{\varrho_t}\, d\pi \tag{11}$$

(recall that $\alpha < \frac{\pi(K)}{c}$). One can verify easily that (11) yields that the L_2-norms of the functions h_{ϱ_t} are uniformly bounded. Since $\varrho_t \uparrow \varrho$, this implies that the function

$$h_\varrho(x) \stackrel{df}{=} E_x e^{\alpha\tau} - 1, \quad x \in \mathbb{X}$$

belongs to L_2, and $h_{\varrho_t} \to h_\varrho, t \to \infty$ in L_2. Similarly to the proof of Corollary 2.1, one can verify that $\{h_{\varrho_t}\}$ is a bounded subset in $H_{\mathcal{E}}$, and hence $h_{\varrho_t} \to h_\varrho, t \to \infty$ weakly in $H_{\mathcal{E}}$. This proves statement a) of the theorem. In order to prove statement b), we apply (8) to $\psi = \varrho_t$, and pass to the limit as $t \to +\infty$. The theorem is proved. □

3. Poincaré inequality for symmetric diffusions: criterion in the terms of hitting times

Let $\mathbb{X}$ be a connected locally compact Riemannian manifold of dimension d, and X be a diffusion process on $\mathbb{X}$. Let $\pi \in \mathcal{P}(\mathbb{X})$ be an invariant measure for the process X (we assume invariant measure to exist). We assume that X is symmetric w.r.t. π; that is, $T_t = T_t^*, t \in \mathbb{R}^+$.

On a given local chart of the manifold $\mathbb{X}$, the generator of the process X has the form

$$A = \sum_{j=1}^d a_j \partial_j + \frac{1}{2}\sum_{j,k=1}^d b_{jk}\partial^2_{jk},$$

where $a = \{a_j\}_{j=1}^d$ and $b = \{b_{jk}\}_{j,k=1}^d$ are the drift and diffusion coefficients of the process X on this chart, respectively. We assume the coefficients a, b to be Hölder continuous on every local chart, and the drift b coefficient to satisfy ellipticity condition

$$\sum_{j,k=1}^d b_{jk} v_j v_k \ge \beta \sum_{j=1}^d v_j^2$$

uniformly on every compact. Under these conditions, the transition function of the process X has a positive density w.r.t. Riemannian volume, and this density is a continuous function on $(0, +\infty) \times \mathbb{X} \times \mathbb{X}$. One can easily deduce this from the same statement for diffusions in $\mathbb{R}^d$ (e.g. [6]) and strong Markov property of X. This implies that X satisfies the *extended Doeblin condition* (see Section 2.1 in [8]) on every compact subset of $\mathbb{X}$.

Theorem 3.1. *The following statements are equivalent:*

1) the Poincaré inequality (2) holds true with some constant c;

2) the process X admits an exponential ϕ-coupling for some function ϕ, see Definition 2.2 in [8]*;*

3) for every closed subset $K \subset \mathbb{X}$ with $\pi(K) > 0$, there exists $\alpha > 0$ such that

$$E_\pi e^{\alpha\tau_K} < +\infty.$$

In addition, 1) – 3) hold true assuming that

3′) there exists a compact subset $K \subset \mathbb{X}$ and $\alpha > 0$ such that

$$E_x e^{\alpha\tau_K} < +\infty \quad \text{for } \pi\text{-almost all} \quad x \in X.$$

Remark 3.1. Note that the property 2) both provides the uniqueness of the invariant measure and makes it possible to give explicit bounds for the convergence rate of the transition probabilities of the process X to the invariant distribution, see [8]. Hence for symmetric diffusions the above theorem, together with the criterion for the Poincaré inequality in the terms of hitting times, gives a sufficient condition for an (exponential) ergodicity.

Proof. Implication 2) $\Rightarrow$ 1) follows immediately from Theorem 3.4 in [8]. Implication 1) $\Rightarrow$ 3) is provided by Theorem 2.1. Implication 3) $\Rightarrow$ 3′) is trivial. To prove implication 3′) $\Rightarrow$ 2), we will use Proposition 2.4 in [8]. Recall that we have already seen that X satisfies the extended Doeblin condition on K. Hence, in order to apply Proposition 2.4 in [8], it is sufficient to verify for a given $\tilde\alpha\in(0,\alpha)$ the following conditions:

(a) $E_x e^{\tilde\alpha\tau_K}<+\infty, x\in\mathbb{X}$;
(b) there exists $S>0$ such that

$$\sup_{x\in K, t\in[0,S]} E_x e^{\tilde\alpha\tau_K^t}<\infty,\quad \tau_K^t:=\inf\{s\geq 0: X_{t+s}\in K\}.$$

In order to simplify the exposition, we consider the case $\mathbb{X}=\mathbb{R}^d$, only. One can easily extend the proof to the general case by a standard localization procedure.

We put $\phi(x)=E_xe^{\tilde\alpha\tau_K}, \psi(x)=E_xe^{\alpha\tau_K}, x\in\mathbb{X}$. Let us show that ϕ is locally bounded; this would imply the condition (a) above.

Let $x_0\in\mathbb{R}^d$ and $0<r_0<r_1$ be such that $K\subset\{x:\|x-x_0\|<r_0\}$. Denote $D=\{x:\|x-x_0\|<r_1\}\setminus K$, $\theta=\inf\{t: X_t\in\partial D\}$, and $\mu_x(dy)\stackrel{df}{=}P_x(X_\theta\in dy), x\in D$.

Consider auxiliary function

$$h(x)=\int_{\partial D} E_y e^{\alpha\tau_K}\,\mu_x(dy)\in[0,\infty],\quad x\in D.$$

This function can be represented as a monotonous point-wise limit of the functions

$$h_N(x)=\int_{\partial D} g_N(y)\,\mu_x(dy),\quad N\geq 1$$

with bounded and measurable functions g_N. Every function is A-harmonic in D, this can be proved in a standard way using the strong Markov property of X, e.g. Chapter II §5, [3]. Hence every h_N satisfies the Harnack inequality (see [7]). Namely, there exists $C\in\mathbb{R}^+$ independent of N such that

$$h_N(x_1)\leq Ch_N(x_2)$$

for every $y\in D$, and $x_1,x_2\in\{x:\|x-y\|<\frac{1}{2}\mathrm{dist}(y,\partial D)\}$. Then the same relation holds true with h instead of h_N. On the other hand, by the strong Markov property of X, we have

$$E_xe^{\alpha\tau_K}=E_x(e^{\alpha\theta}\psi(X_\theta))\geq E_x\psi(X_\theta)=h(x),\quad x\in D.$$

Hence, under condition 3′), $h(x)<+\infty$ for π-a.a. $x\in D$. In addition, $\mathrm{supp}\,\pi=\mathbb{X}$; one can easily verify this fact using positivity of the transition probability density. Therefore, the function h is bounded on every compact $S\subset D$.

The function h can be written in the form

$$h(x)=E_xe^{\alpha\tau_K^\theta},\quad \tau_K^\theta=\inf\{s\geq 0: X_{s+\theta}\in K\}.$$

For $x\in D$, we have $\tau_K=\theta+\tau_K^\theta$ P_x-a.s., and therefore

$$E_xe^{\tilde\alpha\tau_K}\leq[E_x(e^{\frac{\alpha\tilde\alpha}{\alpha-\tilde\alpha}\theta})]^{\frac{\alpha-\tilde\alpha}{\alpha}}[h(x)]^{\frac{\tilde\alpha}{\alpha}}.$$

Using the Kac formula one can show that, for every $a>0$, the function $x\mapsto E_xe^{a\theta}$ is bounded on D (this fact is quite standard and hence we do not go into details here). Therefore, the function ϕ is bounded on every compact $S\subset D$.

Next, consider the closed ball $E=\{x:\|x-x_0\|\leq r_0\}$; note that its boundary $S=\{x:\|x-x_0\|=r_0\}$ is a compact subset of D and therefore the function ϕ is bounded on S. We put $\sigma = \inf\{t : X_t \in S\}$, then by the strong Markov property of X we have for $x \in E$

$$\phi(x) \le E_x(e^{\tilde{\alpha}\sigma}\phi(X_\sigma)) \le (E_x e^{\tilde{\alpha}\sigma}) \sup_{y\in S} \phi(y).$$

The function $x \mapsto E_x e^{\tilde{\alpha}\sigma}$ is bounded on E (again, we do not give a detailed discussion here). Hence ϕ is bounded on E. Since r_0 and r_1 can be taken arbitrarily large, this means that ϕ is locally bounded.

Now, let us verify the condition (b) above. We keep the notation $E = \{x : \|x - x_0\| \le r_0\}, S = \partial E$, and put $\sigma^0 = 0$,

$$\sigma^{2n-1} = \inf\{t \ge \sigma^{2n-2} : X_t \in S\}, \quad \sigma^{2n} = \inf\{t \ge \sigma^{2n-1} : X_t \in K\}, \quad n \ge 1.$$

For any $a > 0$, one has

$$q \overset{df}{=} \max\left[\sup_{x\in K} E_x e^{-a\tau_S} < 1, \sup_{x\in S} E_x e^{-a\tau_K} < 1\right] < 1$$

because dist $(K, S) > 0$ and X is a Feller process with continuous trajectories. Therefore,

$$E\Big[e^{-a(\sigma^{k+1}-\sigma^k)}\Big|\mathcal{F}_{\sigma^k}\Big] \le q \quad \text{a.s.}, \quad k \ge 0. \tag{12}$$

We have

$$E_x e^{\tilde{\alpha}\tau_K^t} = \sum_{k=0}^{\infty} E_x e^{\tilde{\alpha}\tau_K^t} \mathbb{1}_{\sigma^k \le t < \sigma^{k+1}}, \quad x \in K.$$

For k even, $X_t \in E$ a.s. on the set $C_{k,t} \overset{df}{=} \{\sigma^k \le t < \sigma^{k+1}\}$. In addition, $C_{k,t} \in \mathcal{F}_t$. Hence

$$\begin{aligned} E_x e^{\tilde{\alpha}\tau_K^t} \mathbb{1}_{\sigma^k \le t < \sigma^{k+1}} &= E_x \left(\mathbb{1}_{\sigma^k \le t < \sigma^{k+1}} E\Big[e^{\tilde{\alpha}\tau_K^t}\Big|\mathcal{F}_t\Big]\right) = E_x \mathbb{1}_{\sigma^k \le t < \sigma^{k+1}} \phi(X_t) \\ &\le \sup_{y\in E} \phi(y)\, P_x(\sigma^k \le t < \sigma^{k+1}), \quad k = 2n. \end{aligned}$$

For k odd, $\tau_K^t = \sigma^{k+1} - t \le \sigma^{k+1} - \sigma^k$ a.s. on the set $C_{k,t}$. Hence

$$\begin{aligned} E_x e^{\tilde{\alpha}\tau_K^t} \mathbb{1}_{\sigma^k \le t < \sigma^{k+1}} &\le E_x \mathbb{1}_{\sigma^k \ge t} e^{\tilde{\alpha}(\sigma^{k+1}-\sigma^k)} = E_x \left(\mathbb{1}_{\sigma^k \le t} E\Big[e^{\tilde{\alpha}(\sigma^{k+1}-\sigma^k)}\Big|\mathcal{F}_{\sigma^k}\Big]\right) \\ &= E_x \mathbb{1}_{\sigma^k \le t} \phi(X_{\sigma^k}) \le \sup_{y\in E} \phi(y)\, P_x(\sigma^k \le t). \end{aligned}$$

Therefore,

$$E_x e^{\tilde{\alpha}\tau_K^t} \le \sup_{y\in E} \phi(y) \sum_{k=0}^{\infty} P_x(\sigma^k \le t), \quad x \in K.$$

It follows from (12) that $E_x e^{-a\sigma^k} \le q^k, x \in K$. Then

$$P_x(\sigma^k \le t) = P_x(-\sigma^k \ge -t) \le e^{at} q^k, \quad k \ge 0, x \in K,$$

and consequently

$$\sup_{x\in K, t\in[0,S]} E_x e^{\tilde{\alpha}\tau_K^t} \le e^{aS}(1-q)^{-1} \sup_{y\in E} \phi(y) < +\infty.$$

We have verified conditions (a) and (b). Hence the required statement follows from Proposition 2.4 in [8]. □

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