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ON PASTING TOGETHER TWO INHOMOGENEOUS DIFFUSION PROCESSES ON A LINE WITH THE GENERAL FELLER–WENTZELL CONJUGATION CONDITION

By the method of classical potential theory, we obtain an integral representation for the two-parameter semigroup of operators that describes the inhomogeneous Feller process on a line being a result of pasting together two diffusion processes with the general Feller–Wentzell conjugation condition.

1. Introduction

Let $D_i = \{x \in \mathbb{R} : \ (-1)^i x > 0\}$, $i = 1, 2$, be the two domains on the line $\mathbb{R}$ with the common boundary $S = \{0\}$ and the closures $\overline{D}_i = D_i \bigcup \{0\}$, and let $T > 0$ be fixed. If Γ is $\overline{D}_i$ or $\mathbb{R}$, then we denote by $C_b(\Gamma)$ the Banach space of all real-valued bounded and continuous on Γ functions φ with the norm

$$\|\varphi\| = \sup_{x\in\Gamma} |\varphi(x)|,$$

and by $C_2(\Gamma)$ the set of all functions φ that are bounded and uniformly continuous on Γ together with their first- and second-order derivatives. Let φ_i be the restriction of any function $\varphi \in C_b(\mathbb{R})$ to D_i.

Assume that an inhomogeneous diffusion process is given in D_i, $i = 1, 2$, and it is generated by a second-order differential operator $A_s^{(i)}$, $s \in [0, T]$, that acts on $C_2(\overline{D}_i)$:

$$A_s^{(i)}\varphi_i(x) = \frac{1}{2} b_i(s, x)\frac{d^2\varphi_i(x)}{dx^2} + a_i(s, x)\frac{d\varphi_i(x)}{dx}, \quad i = 1, 2, \tag{1}$$

where $b_i(s, x)$ and $a_i(s, x)$ are real-valued continuous bounded functions in the domain $(s, x) \in [0, T] \times \overline{D_i}$, and $b_i(s, x) \geq 0$.

We denote by $C_{2,0}(\mathbb{R})$ the subset of $C_b(\mathbb{R})$ consisting of all functions $\varphi(x)$ such that $\varphi_i \in C_2(\overline{D}_i)$ for $i = 1, 2$, $A_s^{(1)}\varphi_1(0) = A_s^{(2)}\varphi_2(0)$, and define the operator A_s, acting on $C_{2,0}(\mathbb{R})$ as follows:

$$A_s\varphi(x) = A_s^{(i)}\varphi_i(x), \ x \in \overline{D}_i, \ i = 1, 2. \tag{2}$$

Assume also that the conjugation operator of Feller-Wentzell's type is given which acts on the function $\varphi \in C_{2,0}(\mathbb{R})$ by the formula

$$\begin{aligned} L_s\varphi(0) = r(s)A_s\varphi(0) + q_1(s)\varphi'(0-) - q_2(s)\varphi'(0+) + \gamma(s)\varphi(0) + \\ + \int_{D_1\cup D_2} [\varphi(0) - \varphi(y)]\mu(s, dy) = 0, \quad s \in [0, T], \end{aligned} \tag{3}$$

where the coefficients $r, \ q_1, \ q_2, \ \gamma$ and the measure μ satisfy the following conditions:

1.1. The functions $r(s), \ q_1(s), \ q_2(s), \ \gamma(s)$ are nonnegative and continuous on $[0, T]$.

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1.2. For a fixed s, $\mu(s,\cdot)$ is a nonnegative measure on $D_1 \cup D_2$ such that for any $\delta > 0$ and for all functions $f \in C_b(\mathbb{R})$ the integrals

$$F_f^{(i)}(s) = \int_{D_{i,\delta}} y f(y)\mu(s,dy), \quad G_f^{(i)}(s) = \int_{D_i \setminus D_{i,\delta}} f(y)\mu(s,dy)$$

are continuous on $[0,T]$ as functions of s, where $D_{i,\delta} = \{x \in D_i : |x| < \delta\}$, $i = 1,2$.

1.3. $r(s) + q_1(s) + q_2(s) + \gamma(s) + \mu(s, D_1 \cup D_2) > 0$ for all $s \in [0,T]$.

It is known (see [1]) that the operator L_s determines the most general condition of conjugation, which is of the form

$$L_s\varphi(0) = 0, \quad s \in [0,T], \tag{4}$$

and which restricts the differential operator A_s in (2) to an infinitesimal generator of a Feller semigroup in the space of bounded continuous functions. Thus, the condition (4) is the general Feller-Wentzell conjugation condition by which one can describe all the possible types of behavior of a diffusing particle at the time when it reaches the point zero. These types of behavior are as follows: the sticking (viscosity) ($r > 0,\ q_1 \equiv q_2 \equiv \gamma \equiv \mu \equiv 0$), the partial reflection ($q_1 + q_2 > 0,\ r \equiv \gamma \equiv \mu \equiv 0$), the absorption ($\gamma > 0,\ r \equiv q_1 \equiv q_2 \equiv \mu \equiv 0$), the jump ($\mu > 0,\ r \equiv q_1 \equiv q_2 \equiv \gamma \equiv 0$) as well as their combinations (the linear combinations of the corresponding boundary conditions).

Note that the result obtained in the paper [1] concerns the special case, where the diffusion processes given in the domains $D_i,\ i = 1,2$, are homogeneous, i.e., the diffusion coefficients $b_i(s,x)$ and $a_i(s,x),\ i = 1,2$, in (1) as well as the coefficients $r,\ q_1,\ q_2,\ \gamma$ and the measure μ in (3) do not depend on the variable s. However, the scheme of establishing of the conjugation condition of the form (4) introduced there, extends with obvious changes to an inhomogeneous case. Recall that the question about the most general boundary condition restricting an elliptic second-order differential operator with the coefficients defined in a bounded domain in $\mathbb{R}^n$ to an infinitesimal generator of a one-parameter Feller semigroup, has been studied for the first time in one-dimensional case and it has been completely solved in works of W. Feller ([2]) and A.D. Wentzell ([3]). Besides, in [3], by the methods of functional analysis the assertion on the existence of the operator semigroup corresponding to the given boundary conditions has been proved. Such assertions will be established in the present paper.

Thus, our problem is to clarify the question about the existence of a two-parameter semigroup of operators $T_{st},\ 0 \le s < t \le T$, describing the sufficiently general classes of the inhomogeneous Feller processes in $\mathbb{R}$ such that in the domains D_1 and D_2 they coincide with the given diffusion processes generated by the operators $A_s^{(1)}$ and $A_s^{(2)}$, respectively, and their behavior at the point zero is determined by the corresponding to these classes versions of the general Feller-Wentzell conjugation condition (4). This problem is also often called a problem of pasting together two diffusion processes on a line or a problem of constructing of the mathematical model for the diffusion phenomenon on a line with a membrane located at a fixed point that separates different (by their diffusion characteristics) mediums (see [4], [5]).

The investigation of the problem formulated above is performed by the analytical methods. Such an approach (see [4]-[8]) allows to determine the required operator family by means of the solution of the corresponding problem of conjugation for a linear parabolic equation of the second order with variable coefficients, discontinuous at the zero point. This problem is to find out a function $u(s,x,t) = T_{st}\varphi(x)$ satisfying the following conditions:

$$\frac{\partial u(s,x,t)}{\partial s} + A_s^{(i)} u(s,x,t) = 0, \quad 0 \le s < t \le T,\ x \in D_i,\ i = 1,2, \tag{5}$$

$$\lim_{s \uparrow t} u(s,x,t) = \varphi(x), \quad x \in \mathbb{R}, \tag{6}$$

$$u(s,0-,t) = u(s,0+,t), \quad 0 \le s < t \le T, \tag{7}$$

$$L_s u(s,0,t) = 0, \quad 0 \le s < t \le T, \tag{8}$$

where $\varphi \in C_b(\mathbb{R})$ is a given function. As we see, the condition (7) in the problem (5)-(8) is the consequence of the Feller property of the required semigroup T_{st}, and the equality (8) corresponds to the general Feller-Wentzell conjugation condition (3), (4). In comparison with classical cases, the peculiarity of the conjugation condition (8) is that it is nonlocal, furthermore, the measure μ in the integral term of (3) can be infinite. From this point of view the parabolic problem of conjugation (5)-(8) formulated in such a manner is considered presumably for the first time. In addition, as far as the condition (8) is concerned, the only two cases are studied. In the first case the coefficients of the operator L_s are assumed to satisfy the conditions 1.1, 1.2 and 3.1, and in the second one they are assumed to satisfy the conditions 1.1, 1.2 and 4.1, 4.2. One more case, when the condition 1.3 holds while $r(s) = q_1(s) = q_2(s) \equiv 0$, $s \in [0,T]$, will be considered separately and published in another paper.

A classical solvability of the problem (5)-(8) is established by the boundary integral equations method with the use of the ordinary parabolic simple-layer potentials that are constructed using the fundamental solutions of the uniformly parabolic operators. Application of this method permits us not only to prove the existence of the solution of the problem (5)-(8), but also to obtain its integral representation. The integral representation of the semigroup T_{st}, will be used in the present paper to construct the required processes and to establish some of their important additional properties. It is necessary to observe that we derived a nontrivial generalization of the corresponding results obtained earlier in [7], [8], where a similar problem was analyzed for the case of homogeneous diffusion processes. Furthermore, the condition of conjugation (4) considered there, had no term corresponding to the termination of the process at zero. We should also mention the works [9]-[11], where the problem of constructing of mathematical models for diffusion processes in mediums with membranes was studied by the methods of stochastic analysis.

2. Auxiliary propositions

Consider the Kolmogorov backward equations (5) ($i = 1,2$). Assume that their coefficients $a_i(s,x)$ and $b_i(s,x)$ are defined on $[0,T] \times \mathbb{R}$ and satisfy the following conditions:

2.1. There exist the constants b and B such that $0 < b \le b_i(s,x) \le B$ for all $(s,x) \in [0,T] \times \mathbb{R}$.

2.2. The functions $a_i(s,x)$ are bounded on $[0,T] \times \mathbb{R}$.

2.3. For all $s, s' \in [0,T]$, $x, x' \in \mathbb{R}$ the next inequalities hold:

$$|b_i(s,x) - b_i(s',x')| \le c\left(|s-s'|^{\frac{\alpha}{2}} + |x-x'|^{\alpha}\right),$$

$$|a_i(s,x) - a_i(s',x')| \le c\left(|s-s'|^{\frac{\alpha}{2}} + |x-x'|^{\alpha}\right),$$

where c and α are the positive constants, $0 < \alpha < 1$.

From the conditions 2.1-2.3 it follows the existence of the fundamental solutions of equations (5) in the domain $[0,T] \times \mathbb{R}$, i.e., the existence of the functions $G_i(s,x,t,y)$ defined for $0 \le s < t \le T$, $x, y \in \mathbb{R}$ such that:

a) they are continuous in the aggregate of the variables;

b) for fixed $t \in (0,T]$, $y \in \mathbb{R}$ they satisfy equations (5);

c) for any function $\varphi(x)$, bounded continuous on $\mathbb{R}$ and for any $t \in (0,T],\ x \in \mathbb{R}$

$$\lim_{s\uparrow t} \int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy = \varphi(x).$$

Furthermore, in the domain $0 \le s < t \le T,\ x, y \in \mathbb{R}$ the following estimations for permissible derivatives of the functions $G_i(s,x,t,y)$ hold:

$$|D_s^r D_x^p G_i(s,x,t,y)| \le c(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-h\frac{(y-x)^2}{t-s}\right\}, \tag{9}$$

where r and p are the nonnegative integers such that $2r + p \le 2$; D_s^r is the partial derivative with respect to s of order r; D_x^p is the partial derivative with respect to x of order p; c, h are positive constants[1]. Recall also that $G_i(s,x,t,y),\ i = 1,2$, are represented as

$$G_i(s,x,t,y) = Z_{i0}(s,y-x,t,y) + Z_{i1}(s,x,t,y), \tag{10}$$

where

$$Z_{i0}(s,x,t,y) = [2\pi b_i(t,y)(t-s)]^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{2b_i(t,y)(t-s)}\right\}, \tag{11}$$

and the functions $Z_{i1}(s,x,t,y)$ satisfy the inequalities

$$|D_s^r D_x^p Z_{i1}(s,x,t,y)| \le c(t-s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-h\frac{(y-x)^2}{t-s}\right\}, \tag{12}$$

where $0 \le s < t \le T,\ x, y \in \mathbb{R},\ 2r + p \le 2$, α is the constant from 2.3.

We mention the following relations ([4, p. 53]), valid for $0 \le s < t \le T,\ \ i = 1,2$:

$$\int_{\mathbb{R}} (y-x)G_i(s,x,t,y)dy = \int_s^t d\tau \int_{\mathbb{R}} G_i(s,x,\tau,z)a_i(\tau,z)dz, \tag{13}$$

$$\begin{aligned}\int_{\mathbb{R}} (y-x)^2 G_i(s,x,t,y)dy = &\int_s^t d\tau \int_{\mathbb{R}} G_i(s,x,\tau,z)b_i(\tau,z)dz + \\ &+ 2\int_s^t d\tau \int_{\mathbb{R}} G_i(s,x,\tau,z)a_i(\tau,z)(z-x)dz.\end{aligned} \tag{14}$$

Given the fundamental solution G_i, we can determine a parabolic potentials that will be used to solve the problem (5)-(8): the Poisson potential

$$u_{i0}(s,x,t) = \int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy, \quad 0 \le s < t \le T,\ x \in \mathbb{R},$$

and the simple-layer potential

$$u_{i1}(s,x,t) = \int_s^t G_i(s,x,\tau,0)V_i(\tau,t,\varphi)d\tau, \quad 0 \le s < t \le T,\ x \in \mathbb{R}, \tag{15}$$

where φ is the function from (6), and $V_i(s,t,\varphi),\ i = 1,2$, are some functions, continuous in $0 \le s \le t \le T$. We mention the following properties of potentials u_{i0} and u_{i1} (see, e.g., [4], [6], [12]):

a) the functions $u_{i0},\ u_{i1},\ i = 1,2$, are continuous in $0 \le s < t \le T,\ x \in \mathbb{R}$, bounded with respect to x, satisfy the equations (5) in the domains $(s,x) \in [0,t) \times \mathbb{R},\ (s,x) \in [0,t) \times (D_1 \cup D_2)$ respectively, and the initial conditions

$$\lim_{s\uparrow t} u_{i0}(s,x,t) = \varphi(x), \quad \lim_{s\uparrow t} u_{i1}(s,x,t) = 0, \quad x \in \mathbb{R};$$

[1] We will subsequently denote various positive constants by the same symbol c (or h).

b) for the potential $u_{i0},\ i=1,2$, the following estimations are valid:

$$|D_s^r D_x^p u_{i0}(s,x,t)| \le c\|\varphi\|(t-s)^{-\frac{2r+p}{2}}, \tag{16}$$

where $0 \le s < t \le T,\ x,y \in \mathbb{R},\ 2r+p \le 2$;

c) the potential $u_{i1},\ i=1,2$, satisfies the relation

$$\frac{\partial u_{i1}(s,0\mp,t)}{\partial x} = \pm\frac{V_i(s,t,\varphi)}{b_i(s,0)} + \int_s^t \frac{\partial Z_{i1}(s,0,\tau,0)}{\partial x} V_i(\tau,t,\varphi)d\tau, \tag{17}$$

that is often called the formula on the jump of the conormal derivative of a simple-layer potential.

Note that the aforementioned properties of the simple-layer potential will be used, obviously, under somewhat more general assumptions on the functions $V_i,\ i=1,2$, in (15).

We will also use the next lemma.

Lemma 1. *Let $Q_f(s),\ s\in[0,T]$ be a family of linear functionals defined on $C_b(\mathbb{R})$ such that for all $f\in C_b(\mathbb{R})$ the functions $Q_f(s)$ are Hölder continuous with the same exponent $\beta\in(0,1)$ on a closed interval $[0,T]$. Then for every $M>0$ there exist a common constant $c>0$ such that for all the functions $f\in C_b(\mathbb{R})$, bounded by M and for all $s,\ s'\in[0,T]$ the inequality*

$$|Q_f(s)-Q_f(s')| \le c|s-s'|^\beta$$

holds.

Proof. For every $f\in C_b(\mathbb{R})$ we consider the Hölder coefficient c_f of the function $Q_f(s), s\in[0,T]$. Taking into account the linearity of the functionals $Q_f(s)$ it is easy to establish that c_f, as a functional on $C_b(\Gamma)$, satisfies the following conditions:

a) $c_{f_1+f_2}\le c_{f_1}+c_{f_2}$, for all $f_1,\ f_2\in C_b(\Gamma)$;

b) $c_{\lambda f}=|\lambda|\cdot c_f$, for an arbitrary $\lambda\in\mathbb{R}$.

Hence the functional c_f is a seminorm. But this implies the assertion of the lemma. □

3. Processes with reflections, absorption and jumps

In this section we consider the problem (5)-(8) in the case of the coefficients of $L_s,\ s\in[0,T]$, satisfying the condition

3.1. $r(s)=0,\quad q_1(s)+q_2(s)>0\quad$ for all $s\in[0,T]$.

Theorem 1. *Assume that the coefficients of the operators $A_s^{(i)}$, i=1,2, as well as the functions $r,\ q_1,\ q_2,\ \gamma$ and the measure μ satisfy conditions 2.1-2.3 and 1.1, 1.2, 3.1. Then for every function $\varphi\in C_b(\mathbb{R})$ the problem (5)-(8) has a unique solution*

$$u(s,x,t)\in C^{1,2}([0,t)\times D_1\cup D_2)\cap C([0,t]\times\mathbb{R}). \tag{18}$$

Furthermore,

$$|u(s,x,t)|\le c\|\varphi\|,\quad 0\le s<t\le T, \tag{19}$$

and this solution is represented as follows

$$u(s,x,t)=u_{i0}(s,x,t)+u_{i1}(s,x,t),\quad x\in\overline{D}_i,\quad 0\le s<t\le T, \tag{20}$$

where a pair of functions $(V_1,\ V_2)$ is a solution of some system of Volterra integral equations of the second kind.

Proof. We find a solution of the problem (5)-(8) of the form (20) with the unknown functions $V_i,\ i=1,2$, that will be determined from the conjugation conditions (7), (8). If we substitute the expression (20) for $u(s,x,t)$ into (7), (8) and use therewith the formula on the jump for a simple-layer potential (17), we obtain the following system of integral equations for V_i

$$\sum_{i=1}^{2}\int_s^t(-1)^iG_i(s,0,\tau,0)V_i(\tau,t,\varphi)d\tau=\Upsilon(s,t,\varphi),$$
$$\sum_{i=1}^{2}\left(\frac{q_i(s)}{b_i(s,0)}V_i(s,t,\varphi)-\int_s^t\overline{K}_i(s,\tau)V_i(\tau,t,\varphi)d\tau\right)=\Psi(s,t,\varphi), \tag{21}$$

where

$$\Upsilon(s,t,\varphi)=u_{10}(s,0,t)-u_{20}(s,0,t)$$
$$\Psi(s,t,\varphi)=\sum_{i=1}^{2}\Bigg((-1)^iq_i(s)\frac{\partial u_{i0}(s,0,t)}{\partial x}-\frac{\gamma(s)}{2}u_{i0}(s,0,t)-$$
$$-\int_{D_i}\left[u_{i0}(s,0,t)-u_{i0}(s,y,t)\right]\mu(s,dy)\Bigg),$$
$$\overline{K}_i(s,\tau)=(-1)^iq_i(s)\frac{\partial Z_{i1}(s,0,\tau,0)}{\partial x}-\frac{\gamma(s)}{2}G_i(s,0,\tau,0)-$$
$$-\int_{D_i}\left[G_i(s,0,\tau,0)-G_i(s,y,\tau,0)\right]\mu(s,dy).$$

The two equations in (21) are the Volterra integral equations of the first and second kinds, respectively. By the Holmgren's method, we reduce the first one to an equivalent Volterra integral equation of the second kind. To this end, we define the operator

$$\mathcal{E}(s,t)\Upsilon=\sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_s^t(\rho-s)^{-\frac{1}{2}}\Upsilon(\rho,t,\varphi)d\rho,\quad 0\le s<t\le T, \tag{22}$$

and apply it to the both sides of this equation. After some straightforward simplifications, we obtain

$$\sum_{i=1}^{2}\int_s^t(-1)^i\widetilde{K}_i(s,\tau)V_i(\tau,t,\varphi)d\tau+(-1)^{i-1}\frac{V_i(s,t,\varphi)}{\sqrt{b_i(s,0)}}=\Phi(s,t,\varphi), \tag{23}$$

where

$$\widetilde{K}_i(s,\tau)=\sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_s^\tau(\rho-s)^{-\frac{1}{2}}Z_{i1}(\rho,0,\tau,0)d\rho=$$
$$=\frac{1}{\sqrt{2\pi}}\int_s^\tau(\rho-s)^{-\frac{3}{2}}\left(Z_{i1}(\rho,0,\tau,0)-Z_{i1}(s,0,\tau,0)\right)d\rho-\sqrt{\frac{2}{\pi}}Z_{i1}(s,0,\tau,0)(\tau-s)^{-\frac{1}{2}},$$

$$\Phi(s,t,\varphi)=\frac{1}{\sqrt{2\pi}}\int_s^t(\rho-s)^{-\frac{3}{2}}\left(\Upsilon(\rho,t,\varphi)-\Upsilon(s,t,\varphi)\right)d\rho-\sqrt{\frac{2}{\pi}}\Upsilon(s,t,\varphi)(t-s)^{-\frac{1}{2}}.$$

Next, writing the equation (23) instead of the first equation in (21) we find that the system (21) can be replaced by an equivalent system of Volterra integral equations of the second kind:

$$V_i(s,t,\varphi)=\sum_{j=1}^{2}\int_s^tK_{ij}(s,\tau)V_j(\tau,t,\varphi)d\tau+\Psi_i(s,t,\varphi),\quad i=1,2, \tag{24}$$

where

$$K_{ij}(s,\tau) = d_i(s)\left(\overline{K}_j(s,\tau) + (-1)^{i+j}\frac{q_{3-i}(s)}{\sqrt{b_{3-i}(s,0)}}\widetilde{K}_j(s,\tau)\right),$$

$$\Psi_i(s,t,\varphi) = d_i(s)\left(\Psi(s,t,\varphi) + (-1)^{i-1}\frac{q_{3-i}(s)}{\sqrt{b_{3-i}(s,0)}}\Phi(s,t,\varphi)\right),$$

$$d_i(s) = \frac{b_i(s,0)\sqrt{b_{3-i}(s,0)}}{q_1\sqrt{b_2(s,0)} + q_2\sqrt{b_1(s,0)}}.$$

Let us show that there exists a solution of the system of equations (24) and it can be obtained by the method of successive approximations

$$V_i(s,t,\varphi) = \sum_{k=0}^{\infty} V_i^{(k)}(s,t,\varphi), \quad 0 \le s < t \le T, \quad i = 1,2, \tag{25}$$

where

$$V_i^{(0)}(s,t,\varphi) = \Psi_i(s,t,\varphi),$$

$$V_i^{(k)}(s,t,\varphi) = \sum_{j=1}^{2}\int_s^t K_{ij}(s,\tau)V_j^{(k-1)}(\tau,t,\varphi)d\tau, \quad k = 1,2,\ldots.$$

For this purpose, we have first to estimate the functions Ψ_i and the kernels K_{ij} in (24).

The estimation for Ψ_i can be easily established by using the inequalities (16). Consider, for example, the integral (this is the last term of the formula for the function Ψ):

$$I_i(s,t,\varphi) = \int_{D_i}\left[u_{i0}(s,0,t) - u_{i0}(s,y,t)\right]\mu(s,dy) =$$

(26)

$$= \int_{D_{i,1}}\left[u_{i0}(s,0,t) - u_{i0}(s,y,t)\right]\mu(s,dy) + \int_{D_i\setminus D_{i,1}}\left[u_{i0}(s,0,t) - u_{i0}(s,y,t)\right]\mu(s,dy).$$

We denote by I_{i1} and I_{i2} the first and second terms in the expression (26), respectively. In order to estimate I_{i1}, first, we apply the Lagrange formula to the integrand $u_{i0}(s,0,t) - u_{i0}(s,y,t)$

$$u_{i0}(s,0,t) - u_{i0}(s,y,t) = \left.\frac{\partial u_{i0}(s,x,t)}{\partial x}\right|_{x=\theta y}\cdot y,$$

where θ is some real number from the interval $(0,1)$. Then, using the inequality (16) when $r = 0$ and $p = 1$, we find that

$$|I_{i1}(s,t,\varphi)| \le c\|\varphi\|(t-s)^{-\frac{1}{2}}.$$

It is clear that the same estimation is also valid for the integral I_{i2}. Estimating all the rest terms in the expression for Ψ_i, we conclude that

$$|\Psi_i(s,t,\varphi)| \le c\|\varphi\|(t-s)^{-\frac{1}{2}}. \tag{27}$$

We proceed to estimate the kernels $K_{ij}(s,\tau)$, $i = 1,2$, $j = 1,2$, in (24). For this purpose, in the expression for $\overline{K}_j$ we take the component

$$N_{j,\delta}(s,\tau) = \int_{D_{j,\delta}}\big[Z_{j0}(s,0,\tau,0) - Z_{j0}(s,y,\tau,0)\big]\mu(s,dy) \tag{28}$$

(here δ is an arbitrary positive number) and write K_{ij} in the form

$$K_{ij}(s,\tau) = K_{ij}^{(1)}(s,\tau) + K_{ij}^{(2)}(s,\tau), \quad 0 \le s < \tau < t \le T, \tag{29}$$

where

$$K_{ij}^{(1)}(s,\tau)=d_i(s)N_{j,\delta}(s,\tau),$$

$$K_{ij}^{(2)}(s,\tau)=d_i(s)\Bigg((-1)^j q_j(s)\frac{\partial Z_{j1}(s,0,\tau,0)}{\partial x}-\frac{\gamma(s)}{2}G_j(s,0,\tau,0)-\int_{D_{j,\delta}}\big[Z_{j1}(s,0,\tau,0)- - Z_{j1}(s,y,\tau,0)\big]\mu(s,dy)-\int_{D_j\setminus D_{j,\delta}}\big[G_j(s,0,\tau,0)-G_j(s,y,\tau,0)\big]\mu(s,dy)+ +(-1)^{i+j}\frac{q_{3-i}(s)}{\sqrt{b_{3-i}(s,0)}}\widetilde{K}_j(s,\tau)\Bigg).$$

By the inequalities (9), (12), we estimate each of the five terms on the right hand side of the expression for $K_{ij}^{(2)}$. Especially, for the difference $Z_{j1}(s,0,\tau,0)-Z_{j1}(s,y,\tau,0)$ we additionally use the finite-increments formula with respect to the variable y. We obtain

$$\left|K_{ij}^{(2)}(s,\tau)\right|\le c(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}, \tag{30}$$

where $c(\delta)$ is some positive constant, depending on δ.

Proceeding by the same considerations, using the inequalities (9) and the Lagrange formula for the difference $Z_{j0}(s,0,\tau,0)-Z_{j0}(s,y,\tau,0)$, we can also estimate the function $K_{ij}^{(1)}(s,\tau)$. However, on the right hand side of the estimation for $|K_{ij}(s,\tau)|$, in contrast to (30), the factor $(\tau-s)^{-1}$ appears. This means that the function $K_{ij}^{(1)}(s,\tau)$, and thus, the function $K_{ij}(s,\tau)$ has non-integrable singularity. Nevertheless, we show that to the system of integral equations (24) the method of successive approximations can be applied. Indeed, using the representation (29) as well as the inequalities (27) and (30) by mathematical induction method, we prove that for the terms of series (25) the following inequality is valid ($0\le s<t\le T$):

$$\left|V_i^{(k)}(s,t,\varphi)\right|\le c\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{n=0}^{k}C_k^n\cdot a^{(k-n)}M(\delta)^n,\quad k=0,1,2, \tag{31}$$

where

$$a^{(n)}=\frac{\big(2c(\delta)T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\big)^n\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+n\alpha}{2}\right)},\quad n=0,1,2,\dots,k,$$

$$M(\delta)=\frac{B}{b}\max_{s\in[0,T]}\mu\left(s,D_{1,\delta}\cup D_{2,\delta}\right)\downarrow 0,\ \text{as}\ \delta\downarrow 0.$$

Let us fix $\delta=\delta_0$ such that $M(\delta_0)<1$. Then, in view of (31), we have ($i=1,2$)

$$\sum_{k=0}^{\infty}\left|V_i^{(k)}(s,t,\varphi)\right|\le c\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}C_k^n a^{(k-n)}M(\delta_0)^n= =c\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}a^{(k)}\sum_{n=0}^{\infty}C_{k+n}^n M(\delta_0)^n=c\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{a^{(k)}}{(1-M(\delta_0))^{k+1}}= =c\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\left(\frac{c(\delta_0)}{1-M(\delta_0)}T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\right)^k}{\Gamma\left(\frac{1+k\alpha}{2}\right)}\cdot\frac{\Gamma(\frac{1}{2})}{1-M(\delta_0)}\ . \tag{32}$$

The estimation (32) ensures the absolute and uniform convergence of series (25) in $0\le s<t\le T$. This means that the functions $V_i(s,t,\varphi),\ i=1,2$, do exist. Furthermore,

they are continuous in $s \in [0,t)$ and satisfy the inequality

$$|V_i(s,t,\varphi)| \le c\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad 0 \le s < t \le T. \tag{33}$$

Inequalities (9) and (33) imply that the function $u(s,x,t)$, defined by the formula (20) is a solution of the conjugation problem (5)-(8) and it satisfies the estimation (19). Using the relations (9)-(12) and the estimation (33), we prove also that the constructed solution belongs to (18).

Thus, in order to complete the proof of the theorem it remains to establish the uniqueness of the solution of the conjugation problem (5)-(8). For this purpose, it suffices to note that the constructed function $u(s,x,t)$ in each of two domains $0 \le s < t \le T,\ x \in \overline{D}_1$ and $0 \le s < t \le T,\ x \in \overline{D}_2$ can be treated as a unique solution of the following second boundary-value parabolic problem:

$$\frac{\partial \omega(s,x,t)}{\partial s} + A_s^{(i)}\omega(s,x,t) = 0, \quad 0 \le s < t \le T,\ x \in D_i,\ i = 1,2, \tag{34}$$

$$\lim_{s \uparrow t} \omega(s,x,t) = \varphi(x), \quad x \in \overline{D}_i,\ i = 1,2, \tag{35}$$

$$\frac{\partial \omega}{\partial x}(s,0,t) = \upsilon_i(s,t), \quad 0 \le s < t \le T,\ i = 1,2, \tag{36}$$

where

$$\begin{aligned} \upsilon_1(s,t) = \frac{-2}{q_1(s)+q_2(s)} \Bigg(& \frac{q_1(s)-q_2(s)}{2} \frac{\partial u}{\partial x}(s,0-,t) - q_2(s)\frac{\partial u}{\partial x}(s,0+,t) + \\ & + \gamma(s)u(s,0,t) + \int_{D_1 \cup D_2} (u(s,0,t) - u(s,y,t))\,\mu(s,dy) \Bigg), \\ \upsilon_2(s,t) = \frac{2}{q_1(s)+q_2(s)} \Bigg(& q_1(s)\frac{\partial u}{\partial x}(s,0-,t) + \frac{q_1(s)-q_2(s)}{2} \frac{\partial u}{\partial x}(s,0+,t) + \\ & + \gamma(s)u(s,0,t) + \int_{D_1 \cup D_2} (u(s,0,t) - u(s,y,t))\,\mu(s,dy) \Bigg). \end{aligned}$$

The proof of Theorem 1 is now complete. □

Consider the two-parameter family of linear operators $T_{st},\ 0 \le s < t \le T$, acting on the function $\varphi \in C_b(\mathbb{R})$ by the formula

$$T_{st}\varphi(x) = \int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy + \int_s^t G_i(s,x,\tau,0)V_i(\tau,t,\varphi)d\tau, \tag{37}$$

where the pair of functions $(V_1,\ V_2)$ is the solution of the system of Volterra integral equations of the second kind (24).

Let us study the properties of the operator family T_{st} under the assumption that the conditions of Theorem 1 are satisfied.

First we note that if the sequence $\varphi_n \in C_b(\mathbb{R})$ is such that $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$ and, in addition, $\sup_n \|\varphi_n\| < \infty$, then $\lim_{n\to\infty} V_i(s,t,\varphi_n) = V_i(s,t,\varphi),\ i = 1,2$, and thus $\lim_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x)$ for all $0 \le s < t \le T,\ x \in \mathbb{R}$. This follows from the fact that the corresponding limit passages can be performed in the series (25) representing the function $V_i(s,t,\varphi),\ i = 1,2$, and under the integral sign on the right hand side of the equality (37) by which the function T_{st} is defined. This property allows us to prove the following properties of the operator family T_{st}, without loss of generality, under the condition that the function φ has a compact support.

Now we prove that the operators $T_{st},\ 0 \le s < t \le T$, remain a cone of nonnegative functions invariant.

Lemma 2. *If $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \ge 0$ for all $x \in \mathbb{R}$, then $T_{st}\varphi(x) \ge 0$ for all $0 \le s < t \le T,\ x \in \mathbb{R}$.*

Proof. Let φ be any nonnegative function in $C_b(\mathbb{R})$ with a compact support. If $\varphi \equiv 0$, then the assertion of the lemma is obvious. Consider now the case where the function φ not everywhere equals zero. Denote by m the minimum of the function $T_{st}\varphi(x)$ in the domain $(s,x) \in [0,t] \times \mathbb{R}$ and assume that $m < 0$. By the minimum principle ([13, Ch. II]), the value m is taken at a point $(s_0, 0), s_0 \in (0,t)$. Then the following inequalities must hold:

$$\gamma(s_0)T_{s_0t}\varphi(0) \le 0, \quad \int_{D_1\cup D_2} (T_{s_0t}\varphi(0) - T_{s_0t}\varphi(y))\,\mu(s_0,dy) \le 0.$$

Furthermore, Theorem 14 in [13, p. 69] implies that

$$\frac{\partial T_{s_0t}\varphi(0-)}{\partial x} < 0, \quad \frac{\partial T_{s_0t}\varphi(0+)}{\partial x} > 0.$$

But since $q_1(s_0) + q_2(s_0) > 0$, it is easily seen that in the case of $s = s_0$, the fulfillment of the conjugation condition (8) is impossible. The contradiction we arrived at indicates that $m \ge 0$. This completes the proof of the lemma. □

By similar considerations to those in proof of Lemma 2, it can be easily established that the operators T_{st} are contractive, i.e.,

$$\|T_{st}\| \le 1, \quad 0 \le s < t \le T.$$

Let us prove that the operator family T_{st} has a semigroup property, i.e., if $0 \le s < \tau < t \le T$, then $T_{st} = T_{s\tau}T_{\tau t}$. This property is a consequence of the assertion of uniqueness of the solution of the problem (5)-(8) which we have already established above. Indeed, to find $u(s,x,t)$ when $\lim_{s\uparrow t} u(s,x,t) = \varphi(x)$, the problem (5)-(8) can be solved first in the time interval $[\tau, t]$, and then with the "initial" function $u(\tau,x,t) = T_{\tau t}\varphi(x)$, we derived, it can be solved in the time interval $[s,\tau]$. In other words, $T_{st}\varphi(x) = T_{s\tau}(T_{\tau t}\varphi)(x),\ \varphi \in C_b(\mathbb{R})$, i.e., $T_{st} = T_{s\tau}T_{\tau t}$.

The properties of the operator family T_{st}, proved above, imply the next theorem (see [6, p.79, Theorem 2.1]).

Theorem 2. *Let the conditions of Theorem 1 hold. Then the semigroup of operators $T_{st},\ 0 \le s < t \le T$, defined by formulas (37), (25) describes the inhomogeneous Feller process in $\mathbb{R}$, such that in D_1 and D_2 it coincides with the diffusion processes generated by $A_s^{(1)}$ and $A_s^{(2)}$, respectively, and its behavior on $S = \{0\}$ is determined by the conjugation condition (4). If $P(s,x,t,dy)$ is the transition probability of this process, then, for any function $\varphi \in C_b(\mathbb{R})$*

$$T_{st}\varphi(x) = \int_{\mathbb{R}} P(s,x,t,dy)\varphi(y).$$

The integral representation of the operator family T_{st}, we derived, allows us to calculate for the corresponding Markov process its diffusion characteristics: the diffusion coefficient and the drift coefficient. We establish the existence of these coefficients in the sense of M.I. Portenko ([4]) under the additional assumption that the measure μ in (3) satisfies the condition

$$\int_{D_i} y\mu(s,dy) \in C([0,T]), \qquad \int_{D_i} y^2\mu(s,dy) \in C([0,T]). \tag{38}$$

In order to calculate for the constructed process its diffusion characteristics, we first define the functions $\varphi_1(x) = x,\ \varphi_2(x) = x^2,\ x \in \mathbb{R}$. Substituting them into the expression for $\Psi_i(s,t,\varphi)$, we get

$$\Psi_i(s,t,\varphi_1) = d_i(s)\Bigg(q_2(s) - q_1(s) + \int_{D_1\cup D_2} y\mu(s,dy) + \\ + 2\sqrt{\frac{2}{\pi}}\,[a_2(s,0) - a_1(s,0)]\,\frac{(-1)^{i-1}q_{3-i}(s)}{\sqrt{b_{3-i}(s,0)}}(t-s)^{\frac12}\Bigg) + \overline{\Psi}_i(s,t,\varphi_1), \tag{39}$$

$$\Psi_i(s,t,\varphi_2) = d_i(s)\Bigg(\int_{D_1\cup D_2} y^2\mu(s,dy) + \\ + 2\sqrt{\frac{2}{\pi}}\,[b_2(s,0) - b_1(s,0)]\,\frac{(-1)^{i-1}q_{3-i}(s)}{\sqrt{b_{3-i}(s,0)}}(t-s)^{\frac12}\Bigg) + \widetilde{\Psi}_i(s,t,\varphi_2), \tag{40}$$

where the functions $\overline{\Psi}_i$ and $\widetilde{\Psi}_i$, when $0 \le s < t \le T$, satisfy the inequalities:

$$\left|\overline{\Psi}_i(s,t,\varphi_1)\right| \le c(t-s)^{\frac{1+\alpha}{2}}, \qquad \left|\widetilde{\Psi}_i(s,t,\varphi_2)\right| \le c(t-s)^{\frac{1+\alpha}{2}}.$$

Since the functions $\Psi_i(s,t,\varphi_k),\ i = 1,2,\ k = 1,2$, have a weaker singularity than the functions $\Psi_i(s,t,\varphi)$ when $\varphi \in C_b(\mathbb{R})$ (see inequality (27)), we easily find that the functions $V_i(s,t,\varphi_k)$, defined by formulas (25) are continuous in $s \in [0,t]$ and satisfy the estimation

$$|V_i(s,t,\varphi_k)| \le c, \quad i = 1,2,\ k = 1,2.$$

Further, from the representation (37) and the relations (13), (14) it follows that

$$\int_{\mathbb{R}}(y-x)P(s,x,t,dy) = T_{st}\varphi_1(x) - x + x(1 - T_{st}1) = \int_s^t d\tau \int_{\mathbb{R}} G_i(s,x,\tau,z)a_i(\tau,z)dz + \\ + \int_s^t G_i(s,x,\tau,0)V_i(\tau,t,\varphi_1)d\tau - x\int_s^t G_i(s,x,\tau,0)V_i(\tau,t,1)d\tau, \tag{41}$$

$$\int_{\mathbb{R}}(y-x)^2P(s,x,t,dy) = T_{st}\varphi_2(x) - x^2T_{st}1 - 2x\int_{\mathbb{R}}(y-x)P(s,x,t,dy) = \\ = \int_s^t d\tau \int_{\mathbb{R}} G_i(s,x,\tau,z)b_i(\tau,z)dz + 2\int_s^t d\tau \int_{\mathbb{R}} G_i(s,x,\tau,z)a_i(\tau,z)(z-x)dz + \\ + \int_s^t G_i(s,x,\tau,0)V_i(\tau,t,\varphi_2)d\tau - 2x\int_s^t G_i(s,x,\tau,0)V_i(\tau,t,\varphi_1)d\tau + \\ + x^2\int_s^t G_i(s,x,\tau,0)V_i(\tau,t,1)d\tau. \tag{42}$$

Using the equalities (39)-(42), after the direct calculations we find that for any function $\varphi \in C_b(\mathbb{R})$ with compact support the following relations are fulfilled:

$$\lim_{t\downarrow s}\int_{\mathbb{R}}\varphi(x)\left(\frac{1}{t-s}\int_{\mathbb{R}}(y-x)P(s,x,t,dy)\right)dx = \int_{\mathbb{R}} a(s,x)\varphi(x)dx + a_0(s)\varphi(0), \tag{43}$$

$$\lim_{t\downarrow s}\int_{\mathbb{R}}\varphi(x)\left(\frac{1}{t-s}\int_{\mathbb{R}}(y-x)^2P(s,x,t,dy)\right)dx = \int_{\mathbb{R}} b(s,x)\varphi(x)dx + b_0(s)\varphi(0), \tag{44}$$

where

$$a(s,x) = \begin{cases} a_i(s,x), & s \in [0,T],\ x \in D_i,\ i = 1,2, \\ \sum_{i=1}^2 l_i(s)a_i(s,0), & s \in [0,T],\ x = 0, \end{cases}$$

$$b(s,x) = \begin{cases} b_i(s,x), & s \in [0,T],\ x \in D_i,\ i = 1,2, \\ \sum_{i=1}^2 l_i(s)b_i(s,0), & s \in [0,T],\ x = 0, \end{cases}$$

$$l_i(s) = \frac{q_i(s)\sqrt{b_{3-i}(s,0)}}{q_1\sqrt{b_2(s,0)} + q_2\sqrt{b_1(s,0)}}, \quad i=1,2, \quad l_1(s)+l_2(s)=1,$$

$$a_0(s) = \frac{1}{2}\left(d_1(s)+d_2(s)\right)\left(q_2(s)-q_1(s)+m_1(s)\right), \quad b_0(s) = \frac{1}{2}\left(d_1(s)+d_2(s)\right)m_2(s),$$

$$m_1(s) = \int_{D_1\cup D_2} y\mu(s,dy)), \quad m_2(s) = \int_{D_1\cup D_2} y^2\mu(s,dy)), \quad s\in[0,T].$$

Relations (43), (44) means that for the constructed process with the transition probability $P(s,x,t,dy)$ there exist, in generalized sense, the diffusion coefficient $b(s,x)+b_0(s)\delta(x)$ and the drift coefficient $a(s,x)+a_0(s)\delta(x)$, where $\delta(x)$ is the Dirac δ-function. This completes the proof of the following theorem.

Theorem 3. *Assume that the conditions of Theorem 1 as well as the condition (38) are satisfied. Then the inhomogeneous Feller process generated by the semigroup of operators T_{st}, $0\le s<t\le T$, defined by formulas (37), (25) is a generalized diffusion process with its transition probability satisfying the relations (43), (44).*

4. The general case

The purpose of this section is to investigate the problem (5)-(8) when the conjugation condition of Feller-Wentzell (8) is general, in sense, that it can include all its five terms. Furthermore, the common boundary $S=\{0\}$ of the domains D_1 and D_2 is "sticky". The existence of the required semigroup is established under the following additional assumptions:

4.1. The function $r(s)$ is positive for all $s\in[0,T]$ as well as it is Hölder continuous, with exponent $\frac{\alpha}{2}$ (α is the constant from 2.3), on $[0,T]$.

4.2. For all $f\in C_b(\mathbb{R})$, $\delta>0$ the functions $F_f^{(i)}(s)$ and $G_f^{(i)}(s)$ from 1.2 are Hölder continuous, with exponent $\frac{\alpha}{2}$ (α is the constant from 2.3), on $[0,T]$ ($i=1,2$).

As in previous section we find a solution of the problem (5)-(8) of the form (20) with the unknown functions V_i to be determined. First we note that in view of relations (5)-(7), the condition (8) reduces to

$$u(s,0,t) = \varphi(0) - \int_s^t g(\tau,t)d\tau, \tag{45}$$

where

$$g(\tau,t) = \frac{1}{r(\tau)}\Bigg(q_1(\tau)\frac{\partial u(\tau,0-,t)}{\partial x} - q_2(\tau)\frac{\partial u(\tau,0+,t)}{\partial x} + \gamma(\tau)u(\tau,0,t) + \\ + \int_{D_1\cup D_2}\left[u(\tau,0,t)-u(\tau,y,t)\right]\mu(\tau,dy)\Bigg).$$

Then, substituting instead of function u its expression (20) into both sides of (45), and instead of $\frac{\partial u(s,0\mp,t)}{\partial x}$ the relation (17), after some straightforward simplifications, we obtain the system of Volterra integral equations of the first kind

$$\Lambda_i(s,t,\varphi) = \int_s^t G_i(s,0,\tau,0)V_i(\tau,t,\varphi)d\tau + \sum_{j=1}^{2}\int_s^t P_j(s,\tau)V_j(\tau,t,\varphi)d\tau, \quad i=1,2, \tag{46}$$

where

$$P_j(s,\tau) = \frac{q_j(\tau)}{r(\tau)b_j(\tau,0)} - \int_s^\tau \frac{\overline{K}_j(\rho,\tau)}{r(\rho)}d\rho,$$

$$\Lambda_i(s,t,\varphi) = \varphi(0) - u_{i0}(s,0,t) + \int_s^t \frac{\Psi(\rho,t,\varphi)}{r(\rho)}d\rho,$$

and the functions $\overline{K}_j$ and Ψ are the same as in (21).

Applying the transform (22) to the both sides of each equation of the system (46), we get an equivalent system of Volterra integral equations of the second kind

$$V_i(s,t,\varphi)=\sum_{j=1}^{2}\int_s^t R_{ij}(s,\tau)V_j(\tau,t,\varphi)d\tau+\Delta_i(s,t,\varphi),\quad i=1,2, \tag{47}$$

where

$$R_{ii}(s,\tau)=\sqrt{\frac{2b_i(s,0)}{\pi}}\left[\sqrt{\frac{\pi}{2}}\widetilde{K}_i(s,\tau)-\frac{q_i(\tau)}{r(\tau)b_i(\tau,0)}(\tau-s)^{-\frac{1}{2}}-\int_s^\tau(\rho-s)^{-\frac{1}{2}}\frac{\overline{K}_i(\rho,\tau)}{r(\rho)}d\rho\right]$$
$$R_{ij}(s,\tau)=-\sqrt{\frac{2b_i(s,0)}{\pi}}\left[\frac{q_j(\tau)}{r(\tau)b_j(\tau,0)}(\tau-s)^{-\frac{1}{2}}+\int_s^\tau(\rho-s)^{-\frac{1}{2}}\frac{\overline{K}_j(\rho,\tau)}{r(\rho)}d\rho\right],\quad i\neq j,$$
$$\Delta_i(s,t,\varphi)=\sqrt{\frac{2b_i(s,0)}{\pi}}\left[\frac{1}{2}\int_s^t(\rho-s)^{-\frac{3}{2}}\left(u_{i0}(\rho,0,t)-u_{i0}(s,0,t)\right)d\rho-\right.$$
$$\left.-\left[u_{i0}(s,0,t)-\varphi(0)\right](t-s)^{-\frac{1}{2}}+\int_s^t(\rho-s)^{-\frac{1}{2}}\frac{\Psi(\rho,t,\varphi)}{r(\rho)}d\rho\right],$$

and the functions $\widetilde{K}_i,\ i=1,2$, are the same as in (23).

We now show that the functions Δ_i and the kernels R_{ij} in (47) satisfy the inequalities

$$|\Delta_i(s,t,\varphi)|\le c\|\varphi\|(t-s)^{-\frac{1}{2}}, \tag{48}$$
$$|R_{ij}(s,\tau)|\le c(\tau-s)^{-1+\frac{\alpha}{2}}. \tag{49}$$

To prove (48) it suffices to note that the first term as well as the function Ψ on the right hand side of the expression for Δ_i are the parts of the formula for Ψ_i in (24).

In order to establish the estimation (49), first, we consider the integral term that is a part of the expression for R_{ij}, more precisely, we consider its component

$$L_j(s,\tau)=\int_s^\tau(\rho-s)^{-\frac{1}{2}}\frac{N_{j,\delta}(\rho,\tau)}{r(\rho)}d\rho,$$

that has a stronger singularity than the other components of this integral. Here the function $N_{j,\delta}$ is the part of the formula for $\overline{K}_j$ and it is determined by (28).

We write the integral L_j in the form

$$L_j(s,\tau)=L_j^{(1)}(s,\tau)+L_j^{(2)}(s,\tau), \tag{50}$$

where

$$L_j^{(1)}(s,\tau)=\frac{1}{r(\tau)}\int_s^\tau(\rho-s)^{-\frac{1}{2}}d\rho\int_{D_{j,\delta}}\left[Z_{j0}(\rho,0,\tau,0)-Z_{j0}(\rho,y,\tau,0)\right]\mu(\tau,dy),$$
$$L_j^{(2)}(s,\tau)=\int_s^\tau(\rho-s)^{-\frac{1}{2}}d\rho\int_{D_{j,\delta}}\left[Z_{j0}(\rho,0,\tau,0)-Z_{j0}(\rho,y,\tau,0)\right]\left(\frac{\mu(\rho,dy)}{r(\rho)}-\frac{\mu(\tau,dy)}{r(\tau)}\right).$$

Using the equality

$$Z_{j0}(\rho,0,\tau,0)-Z_{j0}(\rho,y,\tau,0)=\int_0^1\left[2\pi b_j(\tau,0)(\tau-\rho)\right]^{-\frac{1}{2}}\frac{\partial}{\partial\theta}e^{\frac{-y^2\theta}{2b_j(\tau,0)\cdot(\tau-\rho)}}d\theta,$$

we can estimate $L_j^{(1)}(s,\tau)$:

$$\left|L_j^{(1)}(s,\tau)\right| \le \frac{1}{\sqrt{2\pi b}}\left|\int_s^\tau (\rho-s)^{-\frac12}(\tau-\rho)^{-\frac12}d\rho\int_{D_{j,\delta}}\mu(\tau,dy)\int_0^1\frac{\partial}{\partial\theta}e^{\frac{-y^2\theta}{2b\cdot(\tau-\rho)}}d\theta\right| =$$

$$= \frac{1}{2b\sqrt{2\pi b}}\left|\int_s^\tau (\rho-s)^{-\frac12}(\tau-\rho)^{-\frac32}d\rho\int_{D_{j,\delta}}y\mu(\tau,dy)\int_0^1 ye^{\frac{-y^2\theta}{2b\cdot(\tau-\rho)}}d\theta\right| =$$

$$= \frac{1}{2b\sqrt{2\pi b}}\left|\int_{D_{j,\delta}}y\mu(\tau,dy)\int_0^1 ye^{\frac{-y^2\theta}{2b\cdot(\tau-s)}}d\theta\int_s^\tau(\rho-s)^{-\frac12}(\tau-\rho)^{-\frac32}e^{\frac{-y^2\theta}{2b\cdot(\tau-s)}\frac{\rho-s}{\tau-\rho}}d\rho\right| =$$

$$= \frac{1}{2b\sqrt{2\pi b}(\tau-s)}\left|\int_{D_{j,\delta}}y\mu(\tau,dy)\int_0^1 ye^{\frac{-y^2\theta}{2b\cdot(\tau-s)}}d\theta\int_0^\infty z^{-\frac12}e^{\frac{-y^2\theta}{2b\cdot(\tau-s)}\cdot z}dz\right| \le c(\tau-s)^{-\frac12}. \tag{51}$$

Consider the integral $L_j^{(2)}(s,\tau)$. Applying the Lagrange formula to the difference $Z_{j0}(\rho,0,\tau,0)-Z_{j0}(\rho,y,\tau,0)$, we get

$$L_j^{(2)}(s,\tau) = \int_s^\tau(\rho-s)^{-\frac12}d\rho\int_{D_{j,\delta}}y\frac{\partial Z_{j0}(\rho,x,\tau,0)}{\partial x}\bigg|_{x=\theta y}\left(\frac{\mu(\tau,dy)}{r(\tau)}-\frac{\mu(\rho,dy)}{r(\rho)}\right) =$$

$$= \int_s^\tau(\rho-s)^{-\frac12}(\tau-\rho)^{-1}d\rho\int_{D_{j,\delta}}y(\tau-\rho)\frac{\partial Z_{j0}(\rho,x,\tau,0)}{\partial x}\bigg|_{x=\theta y}\left(\frac{\mu(\tau,dy)}{r(\tau)}-\frac{\mu(\rho,dy)}{r(\rho)}\right),$$

where θ is some positive number from the interval $(0,1)$.

Note that $f_{\tau\rho}^{(j)}(y) = (\tau-\rho)\frac{\partial Z_{j0}(\rho,x,\tau,0)}{\partial x}\big|_{x=\theta y}$, $j=1,2$, as functions of y, belong to $C_b(\mathbb{R})$ for all $0\le s<\rho<\tau<t\le T$, and they are bounded by some common constant (see (11) when $r=0,\ p=1$). Hence, by Lemma 1,

$$\int_{D_{j,\delta}} yf_{\tau\rho}^{(j)}(y)\left(\frac{\mu(\tau,dy)}{r(\tau)}-\frac{\mu(\rho,dy)}{r(\rho)}\right)\le c(\tau-\rho)^{\frac{\alpha}{2}},$$

and hence

$$\left|L_j^{(2)}(s,\tau)\right| \le c(\tau-s)^{-\frac12+\frac{\alpha}{2}},\quad 0\le s<\rho<t\le T. \tag{52}$$

Combining (51) and (52), we find that

$$|L_j(s,\tau)|\le c(\tau-s)^{-\frac12}. \tag{53}$$

It is clear that the inequality (53) is also valid for the integral $\int_s^\tau(\rho-s)^{-\frac12}\frac{\overline{K}_j(\rho,\tau)}{r(\rho)}d\rho$ in the expression for $R_{ij}(s,\tau)$ in (47). Therefore, recalling the estimation

$$\left|\widetilde{K}_j(s,\tau)\right|\le c\|\varphi\|(\tau-s)^{-1+\frac{\alpha}{2}},$$

we conclude that (49) holds.

From (48) and (49) it follows the existence of the solution of the system of integral equations (47) which can be obtained by the method of successive approximations. Besides, the functions $V_i(s,t,\varphi)$, $i=1,2$, are continuous in $s\in[0,t)$ and for them the inequality (33) is valid.

Thus we have shown that if the coefficients of the operators $A_s^{(i)}$ in (1) and L_s in (3) satisfy the conditions 2.1-2.3 and 1.1, 1.2, 4.1, 4.2, respectively, then the solution of the parabolic problem of conjugation (5)-(8) exists, is represented in the form (20) and satisfies the inequality (19). The proof of the uniqueness of the constructed solution $u(s,x,t)$ of the problem (5)-(8) belonging to (18) is a repetition of the proof of the corresponding assertion in section 3 with obvious changes.

Proceeding in the same way as in section 3, we prove that the family of operators $T_{st}\varphi(x) = u(s,x,t)$, $\varphi \in C_b(\mathbb{R})$ is the contractive Feller semigroup, which describes the inhomogeneous Markov process in $\mathbb{R}$, such that in the domains D_1 and D_2 it coincides with the diffusion processes generated by $A_s^{(1)}$ and $A_s^{(2)}$, respectively, and its behavior at zero is determined by the conjugation condition (4) for which the condition 4.1 certainly hold. If we further assume that the measure μ, which corresponds to the jump-like exit from the point zero, satisfies the additional condition (38), then by the direct calculations we find that for the transition probability of the constructed process (we denote it by $P(s,x,t,dy)$) the following relations hold:

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{\mathbb{R}} (y-x) P(s,x,t,dy)dx = a(s,x), \tag{54}$$

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{\mathbb{R}} (y-x)^2 P(s,x,t,dy)dx = b(s,x), \tag{55}$$

where

$$a(s,x) = \begin{cases} a_i(s,x), & s\in[0,T],\ x\in D_i,\ i=1,2, \\ \frac{q_2(s)-q_1(s)+m_1(s)}{r(s)}, & s\in[0,T],\ x=0, \end{cases}$$

$$b(s,x) = \begin{cases} b_i(s,x), & s\in[0,T],\ x\in D_i,\ i=1,2, \\ \frac{m_2(s)}{r(s)}, & s\in[0,T],\ x=0 \end{cases}$$

$$m_1(s) = \int_{D_1\cup D_2} y\mu(s,dy)), \quad m_2(s) = \int_{D_1\cup D_2} y^2\mu(s,dy), \quad s\in[0,T].$$

Relations (54), (55) mean that for the constructed process with the transition probability $P(s,x,t,dy)$ the Kolmogorov local characteristics exist in an ordinary sense. Furthermore, the drift and diffusion coefficients coincide with the functions $a(s,x)$ and $b(s,x)$, respectively.

The result, obtained in the present section can be formulated in the form of the following theorem.

Theorem 4. *Assume that the coefficients of the operators $A_s^{(i)}$, i=1,2 as well as the functions r, q_1, q_2, γ and the measure μ satisfy conditions 2.1-2.3 and 1.1, 1.2, 4.1, 4.2. Then the following two assertions are true:*

(i) *For any function $\varphi \in C_b(\mathbb{R})$ the problem (5)-(8) has the unique solution $u(s,x,t)$ belonging to (18). Furthermore, this solution satisfy the inequality (19) (when $0 \le s < t \le T$, $x \in \mathbb{R}$) and it is represented by (20), where the pair of functions $(V_1,\ V_2)$ is the solution of the system of Volterra integral equations of the second kind (47).*

(ii) *The semigroup of operators $T_{st}\varphi(x) = u(s,x,t)$ describes the inhomogeneous Feller process in $\mathbb{R}$, such that in the domains D_1 and D_2 it coincides with the diffusion processes generated by $A_s^{(1)}$ and $A_s^{(2)}$, respectively, and its behavior on $S = \{0\}$ is determined by the conjugation condition (4). If, in addition, the condition (38) holds, then the transition probability of the constructed process satisfies the relations (54), (54).*

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