Theory of Stochastic Processes
Vol. 17 (33), no. 2, 2011, pp. 35–54

V. KNOPOVA

ASYMPTOTIC BEHAVIOUR OF THE DISTRIBUTION DENSITY OF SOME LÉVY FUNCTIONALS IN $\mathbb{R}^n$

The paper is devoted to the asymptotic behaviour of the distribution density of some Lévy functionals in $\mathbb{R}^n$. We generalize the results obtained in [18] for the case when $\theta(t)+\|x\|\to\infty$, where $\theta(t)$ is some "scaling" function, and (t,x) belong to a suitable domain of $\mathbb{R}_+\times\mathbb{R}^n$.

1. Introduction

The objective of this paper is to find the exact asymptotic behaviour of certain Lévy functionals in $\mathbb{R}^n$. The one-dimensional situation is studied in detail in [18] and (in the case of fractional Lévy motion with $0<H<\frac{1}{2}$) in [19], see also [20] for the upper estimate for the transition probability density of Lévy and affine processes. The approach developed in this paper relies on the n-dimensional version of the saddle point method. We start with some preliminary notions.

Let $(X_t)_{t\geq 0}$ be a real-valued Lévy process on a probability space $(\Omega,\mathcal{F},P)$ with the state space $\mathbb{R}^n$. Its characteristic function can be written as

$$Ee^{izX_t}=e^{t\psi(z)},\quad t>0, \tag{1.1}$$

where the *characteristic exponent* $\psi:\mathbb{R}^n\to\mathbb{C}$ admits the Lévy-Khinchin representation

$$\psi(z)=ia\cdot z-\frac{1}{2}z\cdot Qz+\int_{\mathbb{R}^n}\left(e^{iu\cdot z}-1-iz\cdot u\,1_{\|u\|\leq 1}\right)\mu(du), \tag{1.2}$$

where $a\in\mathbb{R}^n$, $Q\in\mathbb{R}^{n\times n}$ is a positive semi-definite matrix, and $\mu(dy)$ is a Lévy measure, i.e. a measure on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n}(1\wedge\|y\|^2)\,\mu(dy)<\infty$. In what follows we assume that μ satisfies the exponential integrability condition:

$$\int_{\|y\|\geq 1}e^{\alpha\cdot y}\,\mu(dy)<\infty\quad\text{for all }\alpha\in\mathbb{R}^n. \tag{1.3}$$

Finally, we assume that $Q\equiv 0$ and that μ is centered, i.e. that ψ can be written as

$$\psi(z)=\int_{\mathbb{R}^n}\left(e^{iu\cdot z}-1-iz\cdot u\right)\mu(du).$$

Let $\mathbb{T}$, $I\subset\mathbb{R}$, $(t,s)\in\mathbb{T}\times I$; let $\mathcal{F}(t,s)=(\mathcal{F}_{ij}(t,s))_{i,j=1}^n$ be an $n\times n$ matrix-valued function with real-valued elements, bounded in s for fixed t, such that

$$\int_I\|\mathcal{F}(t,s)\|^2ds<\infty\quad\text{for all }t\in\mathbb{T}. \tag{1.4}$$

Under (1.4) and (1.3) the Lévy driven stochastic integral

$$Y_t:=\int_I\mathcal{F}(t,s)dX_s,\quad t\in\mathbb{T}, \tag{1.5}$$

2000 *Mathematics Subject Classification.* Primary 60G51; Secondary 60J35; 60G22.

Key words and phrases. Lévy process, Lévy functionals, distribution density, saddle point method, Laplace method.

The scholarship of NAS of Ukraine during 2010-2011 is gratefully acknowledged.

is well-defined as a limit in L_2 of the respective integral sums, see [15], p.152-158. Our goal is to find the conditions under which the distribution density of Y_t exists, and to investigate its asymptotic behaviour.

Similarly to the Lévy case, the characteristic function of Y_t can be written explicitly:
(1.6)

$$Ee^{izY_t} = \exp\left[\int_I \int_{\mathbb{R}^n} \left(e^{iz\cdot\mathcal{F}(t,s)u} - 1 - iz\cdot\mathcal{F}(t,s)u\right)\mu(du)ds\right], \quad z\in\mathbb{R}^n, \quad t\in\mathbb{T}.$$

For $n = 1$ the representation (1.6) was obtained in [27], Theorem 2.7; in the general case (1.6) can be obtained in the same way. We denote by $\Phi(t, z)$ the characteristic exponent of Y_t.

Under certain condition (see (2.3) or (2.4) below) the function $e^{\Phi(t,\cdot)}$ is integrable, and hence the distribution density $p_t(x)$ of the process Y_t exists and admits the integral representation as the inverse Fourier transform of the characteristic function (1.6):

$$p_t(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-iz\cdot x+\Phi(t,z)}\,dz. \tag{1.7}$$

We investigate the integral (1.7) by developing the multi-dimensional version of the saddle point method, see [14], also [13] for the one-dimensional case. First, applying the Cauchy-Poincaré theorem (see [30]) we change the integration domain in (1.7):

$$p_t(x) = \frac{1}{(2\pi)^n}\int_{i\xi(t,x)+\mathbb{R}^n} e^{-iz\cdot x+\Phi(t,z)}\,dz, \tag{1.8}$$

where $\xi(t, x) \in \mathbb{R}^n$ will be specified below. Then we use a version of the saddle point method to investigate the asymptotic behaviour of the integral (1.8), see [18] for the result in the one dimensional case.

Estimates for the transition probability density of Lévy and, more generally, Markov processes, received a lot of attention during the last years, see [7], [1], [2], [11], [12], [8], [9], [10], [20], [16]. Although the classes of processes which can be investigated by our method, and those, treated in [8], [10] intersect, they are substantially different. For example, our approach does not apply to many symmetric Markov processes treated in [8] and [10], but can be applied for non-symmetric Markov processes, such as the Lévy driven Ornstein-Uhlenbeck process, as well as for non-Markov processes such as the fractional Lévy motion.

The paper is organized as follows. The main result is contained in Section 2, Theorem 2.1. It states that under certain assumptions on the Lévy measure and the kernel $\mathcal{F}$ the distribution density $p_t(x)$ satisfies

$$p_t(x) \sim \frac{1}{\sqrt{(2\pi)^n\mathcal{K}(t,x)}}e^{\mathcal{D}(t,x)}, \quad \theta(t)+\|x\|\to\infty, \quad (t,x)\in\mathcal{A}\subset\mathbb{T}\times\mathbb{R}^n, \tag{1.9}$$

where the functions θ, $\mathcal{D}$ and $\mathcal{K}$ are explicitly described. In Section 3 we give some examples under which the assumptions of Theorem 2.1 are satisfied. In Section 3.1 we study the fixed time case; in Section 3.2 we investigate the situation when the kernel $\mathcal{F}$ satisfies some self-similarity assumption, which makes it possible to write the asymptotic representation (1.9) in a more explicit form reflecting the structure of $\mathcal{F}$. In Section 4 we prove the ratio limit theorem for the distribution density $p(x)$ of X_1 as $\|x\| \to \infty$.

2. Main result

2.1. **Settings.** Let $\|x\| := \sqrt{x_1^2 + ... + x_n^2}$ for $x \in \mathbb{R}^n$; $\|A\| := \sup_{v\neq 0}\frac{\|Av\|}{\|v\|}$ for $n\times n$ matrix A; $\mathbb{S}^n$ denotes a sphere in $\mathbb{R}^n$, ℓ is unit vector; $x\cdot y$ is the scalar product in $\mathbb{R}^n$. We write $f \asymp g$, if for some positive constants c_1, c_2 we have $c_1 f \le g \le c_2 f$; $f \sim g$ (resp., $f \ll g$, $f \gg g$) as $x \to \infty$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$ (resp, $= 0$, $= \infty$).

Denote by $\mu_{t,\ell}$ the image measure of $ds\mu(du)$ under the mapping $I\times\mathbb{R}^n\ni(s,u)\mapsto \ell\cdot\mathcal{F}(s,t)u\in\mathbb{R}$. In what follows we assume that

$$\inf_{\ell\in\mathbb{S}^n}\mu_{t,\ell}(\mathbb{R}_+)>0. \tag{2.1}$$

For $t\in\mathbb{T}$, $z\in\mathbb{R}^n$, define

$$\Lambda(t,z):=-\mathrm{Re}\,\Phi(t,z)\equiv\iint_{(s,u)\in I\times\mathbb{R}^n}(1-\cos(z\cdot\mathcal{F}(t,s)u))\mu(du)ds. \tag{2.2}$$

If for a given $t\in\mathbb{T}$ and $\|z\|\geq R$, where R is large enough, we have for some $\delta>0$

$$\Lambda(t,z)\geq(k+n+\delta)\ln\|z\|,\quad k\geq 0, \tag{2.3}$$

then Y_t has a distribution density p_t, which belongs to the class $C_b^k(\mathbb{R}^n)$ of k times differentiable functions, whose derivatives are bounded. Indeed, under (2.3) we have $|e^{\Phi(t,z)}|\leq\|z\|^{-(n+k+\delta)}$ for $\|z\|\geq R$, which implies the statement. In particular, if for a given $t\in\mathbb{T}$

$$\Lambda(t,z)\gg\ln\|z\|\quad\text{as}\quad\|z\|\to\infty, \tag{2.4}$$

then Y_t has a distribution density $p_t\in C_b^\infty$. Conditions (2.3) and (2.4) are modifications of the Hartman-Wintner condition [17], see also [21] for the equivalent conditions in the case of a Lévy process.

Let

$$\mathcal{M}_0(t,\xi):=\int_I\int_{\mathbb{R}^n}\big(e^{\xi\cdot\mathcal{F}(t,s)u}-1-\xi\cdot\mathcal{F}(t,s)u\big)\mu(du)ds, \tag{2.5}$$

$$\mathcal{M}_i(t,\xi):=\int_I\int_{\mathbb{R}^n}(\mathcal{F}(t,s)u)_i(e^{\xi\cdot\mathcal{F}(t,s)u}-1)\mu(du)ds,\quad i=1,..,n, \tag{2.6}$$

$$\mathcal{M}_{i_1,..,i_k}(t,\xi):=\int_I\int_{\mathbb{R}^n}\prod_{l=1}^k(\mathcal{F}(t,s)u)_{i_l}e^{\xi\cdot\mathcal{F}(t,s)u}\mu(du)ds,\quad k\geq 2, \tag{2.7}$$

and put

$$\mathbb{M}:=\mathbb{M}(t,\xi)=(\mathcal{M}_{ij}(t,\xi))_{i,j=1}^n\,. \tag{2.8}$$

The matrix $\mathbb{M}$ is positive semi-definite: for any $z\in\mathbb{C}^n$

$$\begin{aligned}(\mathbb{M}z,z)&=\sum_{i,j=1}^n\int_I\int_{\mathbb{R}^n}(\mathcal{F}(t,s)u)_i\cdot z_i(\mathcal{F}(t,s)u)_j\cdot\overline{z}_je^{\xi\cdot\mathcal{F}(t,s)u}\mu(du)ds\\&=\int_I\int_{\mathbb{R}^n}\Big|\sum_{i=1}^n(\mathcal{F}(t,s)u)_i\cdot z_i\Big|^2e^{\xi\cdot\mathcal{F}(t,s)u}\mu(du)ds\geq 0.\end{aligned} \tag{2.9}$$

In the sequel we assume that

(A0) for all $(t,\xi)\in\mathbb{T}\times\mathbb{R}^n$ the matrix $\mathbb{M}$ is non-degenerate.

Denote by $\lambda_i(t,\xi)$, $i=1,..,n$, the eigenvalues of $\mathbb{M}$. By non-degeneracy of $\mathbb{M}$ we have $\lambda_i(t,\xi)>0$, $i=1,..,n$. We denote by $\lambda_{max}(t,\xi)$ and $\lambda_{min}(t,\xi)$, respectively, the maximal and the minimal eigenvalues of $\mathbb{M}$. Recall that

$$\|\mathbb{M}\|=\lambda_{\max}(t,\xi),\quad\|\mathbb{M}^{-1}\|=\max_{i=1,..,n}\lambda_i^{-1}(t,\xi)=\lambda_{min}^{-1}(t,\xi), \tag{2.10}$$

and that the eigenvalues of $\mathbb{M}^2$ and $\mathbb{M}^{1/2}$ are, respectively, $\lambda_i^2(t,\xi)$ and $\lambda_i^{1/2}(t,\xi)$, $i=1,...n$.

Let

$$\Psi(t,z):=\Phi(t,-z)=\int_I\int_{\mathbb{R}^n}\Big(e^{-iz\cdot\mathcal{F}(t,s)u}-1+iz\cdot\mathcal{F}(t,s)u\Big)\,\mu(du)ds,\quad t\in\mathbb{T},\quad z\in\mathbb{C}^n.$$

Since for fixed t the elements of $\mathcal{F}$ are bounded in s, the exponential integrability assumption (1.3) implies that for $t \in \mathbb{T}$ the function $\Psi(t,z)$ is well defined and analytic in $\mathbb{C}^n$ with respect to z. Making the change of variables $z \mapsto -z$ we can rewrite (1.7) as

$$p_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{H(t,x,z)}\, dz, \quad x \in \mathbb{R}^n, \tag{2.11}$$

where

$$H(t,x,z) = ix \cdot z + \Psi(t,z). \tag{2.12}$$

Observe that $\big(\frac{\partial^2 H(t,x,i\xi)}{\partial \xi_k \partial \xi_l}\big)_{k,l=1}^n = \mathbb{M}(t,\xi)$, $\xi \in \mathbb{R}^n$. Since $\mathbb{M}$ is positive definite, the function $H(t,x,\cdot)$ is convex on $i\mathbb{R}^n$. Hence there exists at most one solution to the equation $grad_\xi H(t,x,i\xi) = 0$, or, equivalently, the solution to

$$x = \int_I \int_{\mathbb{R}^n} \mathcal{F}(t,s)u(e^{\xi\cdot \mathcal{F}(t,s)u} - 1)\mu(du)ds. \tag{2.13}$$

By (2.1), there exists $U \subset \mathbb{R}_+$ such that $\inf_{u\in\mathbb{S}^n} \mu_{t,\ell}(U) > 0$. Since

$$\Psi(t,iz) = \int_{\mathbb{R}} \Big(e^{\|z\|v} - 1 - \|z\|v\Big)\mu_{\ell_z}(dv) \geq \inf_{\ell\in\mathbb{S}^n} \int_U \Big(e^{\|z\|v} - 1 - \|z\|v\Big)\mu_{t,\ell}(dv),$$

where $\ell_z := \frac{z}{\|z\|}$, the function $\Psi(t,i\cdot)$ is coercive, i.e.

$$\liminf_{\|\xi\|\to\infty} \frac{\Psi(t,i\xi)}{\|\xi\|} = \infty, \tag{2.14}$$

which implies the existence of the solution $\xi \equiv \xi(t,x)$ to (2.13) (see also Example 11.9 from [28]). Moreover, by (2.13) we have $x\cdot\xi > 0$, and $\|\xi(t,x)\| \to \infty$ as $\|x\| \to \infty$.

Define for $A \subset \mathbb{R}$

$$\Theta(t,r,A) := \inf_{\ell\in\mathbb{S}^n} \iint_{\substack{\ell\cdot\mathcal{F}(t,s)u\in A\\ (s,u)\in I\times\mathbb{R}^n}} (1 - \cos(r\ell\cdot\mathcal{F}(t,s)u))\mu(du)ds, \tag{2.15}$$

and

$$\mathcal{D}(t,x) := H(t,x,i\xi(t,x)), \quad \mathcal{K}(t,x) := \det \mathbb{M}(t,\xi(t,x)) = \prod_{i=1}^n \lambda_i(t,\xi(t,x)).$$

When it does not lead to misunderstanding, we write ξ instead of $\xi(t,x)$.

Let $\mathcal{A} \subset \mathbb{T}\times\mathbb{R}^n$,

$$\mathcal{T} := \{t : \exists x \in \mathbb{R}^n, (t,x) \in \mathcal{A}\}, \quad \mathcal{B} := \{(t,\xi(t,x)) : (t,x) \in \mathcal{A}\}.$$

Finally, let θ and χ be two functions, such that $\theta : \mathcal{T} \to (0,+\infty)$ is bounded away from zero on $\mathcal{T}$, and $\chi : \mathcal{T} \to (0,+\infty)$ is bounded away from zero on every set $\{t : \theta(t) \leq c\}$, $c > 0$. As we will see below, these functions reflect the structure of the kernel $\mathcal{F}$.

2.2. **Formulation and the proof.**

Theorem 2.1. *Assume that the Assumptions (A0) and (A1) – (A4) below are satisfied:*

(A1) $\max_{ijkl} |\mathcal{M}_{ijkl}(t,\xi)| \ll \lambda^3_{min}(t,\xi)\lambda^{-1}_{max}(t,\xi)$, *as* $\theta(t) + \|\xi\| \to \infty$, $(t,\xi) \in \mathcal{B}$;

(A2)

$$\ln\Big(\Big(\chi^{-2}(t)\frac{\max_{ijkl}|\mathcal{M}_{ijkl}(t,\xi)|}{\lambda_{min}(t,\xi)}\Big) \vee 1\Big) + \ln\big(\ln[(1\vee\chi^{-1}(t))\lambda_{max}(t,\xi)] \vee 1\big) \ll \ln\theta(t) + \chi(t)\|\xi\|,$$

as $\theta(t) + \|\xi\| \to \infty$, $(t,\xi) \in \mathcal{B}$;

(A3) There exists $R > 0$ *and* $\delta > 0$ *such that*

$$\Theta(t, r, \mathbb{R}_+) \geq (n + \delta) \ln(\chi(t)r), \quad t \in \mathcal{T}, \quad r > R;$$

(A4) There exists $r > 0$ *such that for every* $\varepsilon > 0$

$$\inf_{h \geq \varepsilon} \Theta(t, h, [\chi(t)r, \infty)) \geq \theta(t) \left((\varepsilon\chi(t))^2 \wedge 1 \right).$$

Then for every $t \in \mathcal{T}$ *the law of* Y_t *has a continuous bounded distribution density* $p_t(x)$*, and*

$$p_t(x) \sim \frac{1}{\sqrt{(2\pi)^n \mathcal{K}(t,x)}} e^{\mathcal{D}(t,x)}, \quad \theta(t) + \|x\| \to \infty, \quad (t,x) \in \mathcal{A}. \tag{2.16}$$

Proof. Step 1: changing the integration contour. We prove that

$$p_t(x) = \frac{1}{(2\pi)^n} \int_{i\xi(t,x)+\mathbb{R}^n} e^{H(t,x,z)}\, dz = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{H(t,x,\eta+i\xi(t,x))}\, d\eta. \tag{2.17}$$

For this we apply the Cauchy-Poincaré theorem, see [20] for similar argument in the case of a Lévy process. Consider the domain

$$G := \left\{ z \in \mathbb{C}^n \ : \ \operatorname{Im} z = v\xi(t,x), 0 \leq v \leq 1, \ \operatorname{Re} z \in \prod_{j=1}^n [-M_j, M_j], M_j > 0, \ 1 \leq j \leq n \right\}.$$

This is an $n+1$-dimensional cube with base

$$\left\{ z \in \mathbb{C}^n \ : \ \operatorname{Re} z \in \prod_{j=1}^n [-M_j, M_j], \ \operatorname{Im} z = 0 \right\}$$

and lid

$$\left\{ z \in \mathbb{C}^n \ : \ \operatorname{Re} z \in \prod_{j=1}^n [-M_j, M_j], \ \operatorname{Im} z = \xi(t,x) \right\}.$$

Since the number of sides of G is even, we can fix some orientation on ∂G such that base and lid have opposite orientation. By the Cauchy-Poincaré theorem

$$\int_{\partial G} e^{H(t,x,z)}\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n = 0. \tag{2.18}$$

Consider the integrals over the sides (except the base and the lid)

$$\int_0^1 e^{H(t,x,M \pm iv\xi(t,x))} dv, \quad \text{where} \quad M = (\pm M_1, \ldots, \pm M_n). \tag{2.19}$$

By definition,

(2.20)
$$\begin{aligned} \operatorname{Re} H(t,x,\eta + i\xi) &= -x \cdot \xi - \int_I \int_{\mathbb{R}^n} \Big(1 - e^{\xi \cdot \mathcal{F}(t,s)u} \cos(\eta \cdot \mathcal{F}(t,s)u) + \xi \cdot \mathcal{F}(t,s)u \Big) \mu(du) ds \\ &= H(t,x,i\xi) - \int_I \int_{\mathbb{R}^n} e^{\xi \cdot \mathcal{F}(t,s)u} \Big(1 - \cos(\eta \cdot \mathcal{F}(t,s)u) \Big) \mu(du) ds, \quad \xi, \eta \in \mathbb{R}^n. \end{aligned}$$

As we have shown above, the function $\xi \mapsto H(t,x,i\xi)$ is real-valued, convex, and attains its minimal value at the point $\xi(t,x)$. Then $H(t,x,iv\xi) \leq H(t,x,0) = 0$ for $0 \leq v \leq 1$.

On the other hand,

$$\int_I \int_{\mathbb{R}^n} e^{\xi\cdot\mathcal{F}(t,s)u}(1-\cos(\eta\cdot\mathcal{F}(t,s)u))\mu(du)ds$$
$$\geq \iint_{\substack{\ell_\xi\cdot\mathcal{F}(t,s)u>0\\ (s,u)\in I\times\mathbb{R}^n,}} (1-\cos(\eta\cdot\mathcal{F}(t,s)u))\mu(du)ds$$
$$\geq \inf_{\ell\in\mathbb{S}^n} \iint_{\substack{\ell\cdot\mathcal{F}(t,s)u>0\\ (s,u)\in I\times\mathbb{R}^n,}} (1-\cos(\|\eta\|\ell\cdot\mathcal{F}(t,s)u))\mu(du)ds$$
$$= \Theta(t,\|\eta\|,\mathbb{R}_+).$$

Therefore

$$\operatorname{Re} H(t,x,\pm M + iv\xi(t,x)) \leq -\Theta(t,\|M\|,\mathbb{R}_+), \quad v\in[0,1].$$

Thus, condition (A3) implies that the integrals in (2.19) tend to 0 as $\|M\|\to+\infty$, which gives (2.17). Since $p_t(x)$ is real-valued, we derive from (2.17)

$$p_t(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{R(t,x,\eta)}\cos(I(t,x,\eta))d\eta, \tag{2.21}$$

where

$$R(t,x,\eta) := \operatorname{Re} H(t,x,\eta+i\xi(t,x)), \quad I(t,x,\eta) := \operatorname{Im} H(t,x,\eta+i\xi(t,x)), \tag{2.22}$$

and

$$\operatorname{Im} H(t,x,\eta+i\xi) = x\cdot\eta - \int_I\int_{\mathbb{R}^n}\Big(e^{\xi\cdot\mathcal{F}(t,s)u}\sin(\eta\cdot u) - \eta\cdot\mathcal{F}(t,s)u\Big)\mu(du)ds. \tag{2.23}$$

Step 2: choosing α,β. Split the integral (2.21) into the sum

$$\frac{1}{(2\pi)^n}\Big[\int_{\|\eta\|\leq\alpha} + \int_{\|\eta\|\in(\alpha,\beta]} + \int_{\|\eta\|>\beta}\Big]\Big(e^{R(t,x,\eta)}\cos I(t,x,\eta)\,d\eta\Big) = J_1(t,x)+J_2(t,x)+J_3(t,x), \tag{2.24}$$

where

$$\beta\equiv\beta(t,x) := \sqrt{\frac{\lambda_{min}(t,\xi(t,x))}{n^2\max\limits_{ijkl}|\mathcal{M}_{ijkl}(t,\xi(t,x))|}}, \tag{2.25}$$

and $\alpha\equiv\alpha(t,x)$ is chosen such that

$$\frac{1}{\lambda_{min}(t,\xi(t,x))} \ll \alpha^2(t,x) \ll \frac{\lambda_{min}(t,\xi(t,x))}{\max\limits_{ijkl}|\mathcal{M}_{ijkl}(t,\xi(t,x))|}, \tag{2.26}$$

and

$$\alpha^3(t,x) \ll \frac{1}{\max\limits_{ijk}|\mathcal{M}_{ijk}(t,\xi(t,x))|}, \quad \theta(t)+\|x\|\to\infty, \quad (t,x)\in\mathcal{A}. \tag{2.27}$$

Let us show that such $\alpha(t,x)$ exists. By the Cauchy inequality and (A1) we have

$$|\mathcal{M}_{ijk}(t,\xi)|^2 \leq \max_{ij}|\mathcal{M}_{ij}(t,\xi)|\max_{ijkl}|\mathcal{M}_{ijkl}(t,\xi)| \ll \lambda^3_{min}(t,\xi), \quad \theta(t)+\|\xi\|\to\infty, \quad (t,\xi)\in\mathcal{B}. \tag{2.28}$$

Therefore, by (A1) and (2.28) there exists a function $k(t,\xi)$ such that

$$
\begin{aligned}
&1 \ll k(t,\xi) \ll \frac{\lambda_{min}(t,\xi)}{\max\limits_{ijkl}|\mathcal{M}_{ijkl}(t,\xi)|^{\frac{1}{2}}},\\
&k(t,\xi) \ll \frac{\lambda_{min}^{\frac{1}{2}}(t,\xi)}{\max\limits_{ijk}|\mathcal{M}_{ijk}(t,\xi)|^{\frac{1}{3}}}, \quad \theta(t)+\|\xi\|\to\infty, \quad (t,\xi)\in\mathcal{B}.
\end{aligned} \tag{2.29}
$$

Chose

$$
\alpha(t,x) = ck(t,\xi(t,x))\lambda_{min}^{-\frac{1}{2}}(t,\xi(t,x)), \tag{2.30}
$$

where $c>0$ is some constant. Then α satisfies (2.26) and (2.27). Since $k(t,\xi)$ is locally bounded, the constant c can be chosen such that

$$
0<\alpha(t,x)\le\beta(t,x), \quad \theta(t)+\|x\|\to\infty, \quad (t,x)\in\mathcal{A}.
$$

Step 3: estimating $J_1(t,x)$ *in* (2.24). We have

$$
\frac{\partial}{\partial\eta_i}R(t,x,\eta) = -\int_I\int_{\mathbb{R}^n} e^{\xi\cdot\mathcal{F}(t,s)u}(\mathcal{F}(t,s)u)_i\sin(\eta\cdot\mathcal{F}(t,s)u)\mu(du)ds,
$$

$$
\frac{\partial}{\partial\eta_i}R(t,x,\eta)\Big|_{\eta=0}=0, \quad \frac{\partial^3}{\partial\eta_i\partial\eta_j\partial\eta_k}R(t,x,\eta)\Big|_{\eta=0}=0, \quad i,j,k\in\{1,..,n\}, \tag{2.31}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial\eta_i\partial\eta_j}R(t,x,\eta)\Big|_{\eta=0} &= -\int_I\int_{\mathbb{R}^n}(\mathcal{F}(t,s)u)_i(\mathcal{F}(t,s)u)_j e^{\xi\cdot\mathcal{F}(t,s)u}\cos(\eta\cdot\mathcal{F}(t,s)u)\mu(du)ds\Big|_{\eta=0}\\
&= -\mathcal{M}_{ij}(t,\xi(t,x)),
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{\partial^4}{\partial\eta_i\partial\eta_j\partial\eta_k\partial\eta_l}R(t,x,\eta)\right| &= \left|\int_I\int_{\mathbb{R}^n}\prod_{\iota=i,j,k,l}(\mathcal{F}(t,s)u)_\iota\cos(\eta\cdot\mathcal{F}(t,s)u)e^{\xi\cdot\mathcal{F}(t,s)u}\mu(du)ds\right|\\
&\le|\mathcal{M}_{ijkl}(t,\xi)|, \quad i,j,k,l\in\{1,..,n\}.
\end{aligned}
$$

Therefore decomposing $\cos(\eta\cdot\mathcal{F}(t,s)u)$ in the representation of $\frac{\partial^2}{\partial\eta_i\partial\eta_j}R(t,x,\eta)$ we get for some η^* from the segment joining 0 and η

$$
\begin{aligned}
(\nabla^2R\eta,\eta) :&= \sum_{i,j=1}^n\frac{\partial^2}{\partial\eta_i\partial\eta_j}R(t,x,\eta)\eta_i\eta_j\\
&= -\sum_{i,j=1}^n\mathcal{M}_{ij}(t,\xi)\eta_i\eta_j+\frac{1}{2}\sum_{i,j,k,l=1}^n\frac{\partial^4}{\partial\eta_i\partial\eta_j\partial\eta_k\partial\eta_l}R(t,x,\eta^*)\eta_i\eta_j\eta_k\eta_l.
\end{aligned}
$$

For all $\eta\in\mathbb{R}^n$

$$
\begin{aligned}
\Big|\sum_{ijkl=1}^n\frac{\partial^4}{\partial\eta_i\partial\eta_j\partial\eta_k\partial\eta_l}R(t,x,\eta)\eta_i\eta_j\eta_k\eta_l\Big| &\le \max_{ijkl}|\mathcal{M}_{ijkl}(t,\xi)|n^2\|\eta\|^4\\
&\le n^2\frac{\max\limits_{ijkl}|\mathcal{M}_{ijkl}(t,\xi)|}{\inf_{\|v\|\ne0}\frac{(\mathbb{M}v,v)}{\|v\|^2}}(\mathbb{M}\eta,\eta)\|\eta\|^2\\
&= n^2\frac{\max\limits_{ijkl}|\mathcal{M}_{ijkl}(t,\xi)|}{\lambda_{min}(t,\xi)}(\mathbb{M}\eta,\eta)\|\eta\|^2,
\end{aligned} \tag{2.32}
$$

where we used

$$\inf_{\|v\|\neq 0} \frac{(\mathbb{M}v,v)}{\|v\|^2} = \inf_{\|v\|\neq 0} \frac{\|\mathbb{M}^{\frac12}v\|^2}{\|v\|^2} = \left(\sup_{\|v\|\neq 0} \frac{\|\mathbb{M}^{-\frac12}v\|^2}{\|v\|^2}\right)^{-1} = \left(\max_{i=1,..,n} \frac{1}{\lambda_i}\right)^{-1} = \min_{i=1,..,n} \lambda_i = \lambda_{min}.$$

We have for $\|\eta\| \le \alpha$, where α is defined in (2.30),

$$-(\mathbb{M}\eta,\eta)\Big(1+n^2\alpha^2 \frac{\max_{ijkl}|\mathcal{M}_{ijkl}|}{\lambda_{min}}\Big) \le (\nabla^2 R\eta,\eta) \le -(\mathbb{M}\eta,\eta)\Big(1-n^2\alpha^2 \frac{\max_{ijkl}|\mathcal{M}_{ijkl}|}{\lambda_{min}}\Big).$$

By the right-hand side estimate on α in (2.26)

$$\inf_{\|\eta\|\le\alpha} (\nabla^2 R\eta,\eta) \sim \sup_{\|\eta\|\le\alpha} (\nabla^2 R\eta,\eta) \sim -(\mathbb{M}\eta,\eta), \quad \theta(t)+\|x\|\to\infty, \quad (t,x)\in\mathcal{A}. \tag{2.33}$$

Similarly to (2.31), for all $i,\ j,\ k \in \{1,..,n\}$

$$I(t,x,\eta)\Big|_{\eta=0} = \frac{\partial}{\partial\eta_i} I(t,x,\eta)\Big|_{\eta=0} = \frac{\partial}{\partial\eta_i\partial\eta_j} I(t,x,\eta)\Big|_{\eta=0} = 0,$$

$$\Big|\frac{\partial}{\partial\eta_i\partial\eta_j\partial\eta_k} I(t,x,\eta)\Big| \le |\mathcal{M}_{ijk}(t,\xi)|, \quad \text{for all } \eta\in\mathbb{R}^n.$$

(the equality for $\frac{\partial}{\partial\eta_i}I$ is due to the fact that $i\xi(t,x)$ is a critical point of $H(t,x,\cdot)$). By the estimate (2.27) on α we get decomposing $\sin(\eta\cdot\mathcal{F}(t,s)u)$ in the representation for $I(t,x,\eta)$ and using the inequality

$$\left(\sum_{i=1}^n x_i\right)^3 \le n^{3/2}\|x\|^3,$$

and

$$\sup_{\|\eta\|\le\alpha} |I(t,x,\eta)| \le \frac{n^{\frac32}}{3!} \max_{ijk} |\mathcal{M}_{ijk}(t,\xi)|\alpha^3 \to 0, \quad \theta(t)+\|x\|\to\infty, \quad (t,x)\in\mathcal{A}. \tag{2.34}$$

Recall our notation $\mathcal{K}(t,x) = \det \mathbb{M}(t,\xi(t,x))$ and

$$\mathcal{D}(t,x) \equiv H(t,x,i\xi(t,x)) = R(t,x,0).$$

From (2.33) and (2.34) we get

$$\begin{aligned} \int_{\|\eta\|\le\alpha} e^{R(t,x,\eta)} \cos I(t,x,\eta)d\eta &\sim e^{R(t,x,0)} \int_{\|\eta\|\le\alpha} e^{-\frac{(\mathbb{M}\eta,\eta)}{2}} d\eta \\ &= \sqrt{\frac{(2\pi)^n}{\mathcal{K}(t,x)}} e^{\mathcal{D}(t,x)} \int_{\|\mathbb{M}^{-\frac12}v\|\le\alpha} \frac{e^{-\frac{\|v\|^2}{2}}}{(2\pi)^{\frac n2}} dv. \end{aligned} \tag{2.35}$$

The integral on the right-hand side can be estimated as

$$\int_{\|v\|\le\alpha\lambda_{min}^{\frac12}} \frac{e^{-\frac{\|v\|^2}{2}}}{(2\pi)^{\frac n2}} dv \le \int_{\|\mathbb{M}^{-\frac12}v\|\le\alpha} \frac{e^{-\frac{\|v\|^2}{2}}}{(2\pi)^{\frac n2}} dv \le 1.$$

By (2.26), the left-hand side tends to 1 as $\|x\|\to\infty$. Therefore

$$J_1(t,x) \sim \frac{1}{\sqrt{(2\pi)^n \mathcal{K}(t,x)}} e^{\mathcal{D}(t,x)}, \quad \theta(t)+\|x\|\to\infty, \quad (t,x)\in\mathcal{A}. \tag{2.36}$$

Step 4: proving that $J_2(t,x)$ is negligible. Decompose $R(t,x,\eta)$ in Taylor series:

$$R(t,x,\eta) = R(t,x,0) - \frac{1}{2!}\sum_{i,j=1}^{n} \mathcal{M}_{ij}(t,\xi(t,x))\eta_i\eta_j + \frac{1}{4!}\sum_{i,j,k,l=1}^{n} \frac{\partial^4}{\partial\eta_i\partial\eta_j\partial\eta_k\partial\eta_l} R(t,x,\eta^*)\eta_i\eta_j\eta_k\eta_l,$$

where we used that

$$\frac{\partial^2}{\partial\eta_i\partial\eta_j} R_{ij}(t,x,\eta)\big|_{\eta=0} = -\mathcal{M}_{ij}(t,\xi(t,x)), \qquad i,j=1,\dots,n,$$

and η^* is some point on the segment joining 0 and η. From (2.25) and (2.32) we have for $\|\eta\| \le \beta$

$$\Big|\sum_{i,j,k,l=1}^{n} \frac{\partial^4}{\partial\eta_i\partial\eta_j\partial\eta_k\partial\eta_l} R(t,x,\eta^*)\eta_i\eta_j\eta_k\eta_l\Big| \le (\mathbb{M}\eta,\eta).$$

Thus for $\|\eta\| \le \beta$

$$R(t,x,\eta) \le R(t,x,0) - \frac{11}{24}(\mathbb{M}\eta,\eta)$$

which, together with the lower estimate for α in (2.26), gives

(2.37)
$$|J_2(t,x)| \le \int_{\|\eta\|\in(\alpha,\beta]} e^{R(t,x,\eta)}d\eta \le e^{R(t,x,0)}\int_{\|\eta\|>\alpha} e^{-\frac{11}{24}(\mathbb{M}\eta,\eta)}d\eta$$
$$= \frac{e^{\mathcal{D}(t,x)}}{\sqrt{\mathcal{K}(t,x)}}\int_{\|\mathbb{M}^{-\frac{1}{2}}\eta\|>\alpha} e^{-\frac{11}{24}\|\eta\|^2}\,d\eta \ll J_1(t,x), \quad \theta(t)+\|x\|\to\infty, \quad (t,x)\in\mathcal{A}.$$

Step 5: proving that $J_3(t,x)$ in (2.24) is negligible. By (2.20),

$$|J_3(t,x)| \le \int_{\|\eta\|>\beta} e^{R(t,x,\eta)}d\eta$$
$$\le e^{\mathcal{D}(t,x)}\int_{\|\eta\|>\beta} \exp\left\{-\int_I\int_{\mathbb{R}^n} e^{\xi\cdot\mathcal{F}(t,s)u}(1-\cos(\eta\cdot\mathcal{F}(t,s)u))\mu(du)ds\right\}\,d\eta.$$

Therefore, by (2.36), to prove

$$J_3(t,x) \ll J_1(t,x)$$

we need to check that

$$\int_{\|\eta\|>\beta} e^{-\Delta(t,x,\eta)}d\eta \ll \mathcal{K}(t,x)^{-1/2}, \quad \theta(t)+\|x\|\to\infty, \quad (t,x)\in\mathcal{A}, \tag{2.38}$$

where

$$\Delta(t,x,\eta) = \int_I\int_{\mathbb{R}^n} e^{\xi(t,x)\cdot\mathcal{F}(t,s)u}(1-\cos(\eta\cdot\mathcal{F}(t,s)u))\mu(du)ds.$$

We have for any $\sigma \in (0,1)$ and some $r > 0$

$$
\begin{aligned}
\Delta(t,x,\eta) &\geq \iint_{\substack{\ell_\xi\cdot\mathcal{F}(t,s)u>0\\(s,u)\in I\times\mathbb{R}^n}} e^{\|\xi\|\ell_\xi\cdot\mathcal{F}(t,s)u}(1-\cos(\eta\cdot\mathcal{F}(t,s)u))\mu(du)ds\\
&\geq (1-\sigma)\iint_{\substack{\ell_\xi\cdot\mathcal{F}(t,s)u>0\\(s,u)\in I\times\mathbb{R}^n}}(1-\cos(\eta\cdot\mathcal{F}(t,s)u))\mu(du)ds+\\
&\quad+\sigma e^{r\chi(t)\|\xi\|}\iint_{\substack{\ell_\xi\cdot\mathcal{F}(t,s)u>\chi(t)r\\(s,u)\in I\times\mathbb{R}^n}}(1-\cos(\eta\cdot\mathcal{F}(t,s)u))\mu(du)ds\\
&\geq (1-\sigma)\inf_{\ell\in\mathbb{S}^n}\iint_{\substack{\ell\cdot\mathcal{F}(t,s)u>0\\(s,u)\in I\times\mathbb{R}^n}}(1-\cos(\|\eta\|\ell\cdot\mathcal{F}(t,s)u))\mu(du)ds\\
&\quad+\sigma e^{r\chi(t)\|\xi\|}\inf_{\ell\in\mathbb{S}^n}\iint_{\substack{\ell\cdot\mathcal{F}(t,s)u>\chi(t)r\\(s,u)\in I\times\mathbb{R}^n}}(1-\cos(\|\eta\|\ell\cdot\mathcal{F}(t,s)u))\mu(du)ds\\
&\geq (1-\sigma)\Theta(t,\|\eta\|,\mathbb{R}_+)+\sigma e^{r\chi(t)\|\xi\|}\Theta(t,\|\eta\|,[\chi(t)r,\infty)),
\end{aligned}
\tag{2.39}
$$

where Θ is defined in (2.15). Choosing σ such that $(1-\sigma)(n+\delta) > n$ we have by (A3)

$$
\int_{\mathbb{R}^n} e^{-(1-\sigma)\Theta(t,\|\eta\|,\mathbb{R}_+)}d\eta \leq c_1 + \int_{\|z\|\geq R} e^{-(1-\sigma)(n+\delta)\ln(\chi(t)\|z\|)}dz \leq c_2\Big(1\vee\chi^{-n}(t)\Big),
\tag{2.40}
$$

where $c_1,\ c_2 > 0$ are independent of t, and R is given by (A3). Therefore, in view of (2.40) and (A4), to show (2.38) it is enough to prove for every $\sigma > 0$

$$
(1\vee\chi^{-n}(t))\exp\Big[-\sigma e^{r\chi(t)\|\xi\|}\theta(t)\Big((\beta(t,x)\chi(t))^2\wedge 1\Big)\Big] \ll \mathcal{K}(t,x)^{-1/2},
\tag{2.41}
$$

as $\theta(t)+\|x\|\to\infty$, $(t,x)\in\mathcal{A}$. By the definition (2.25) of $\beta(t,x)$, we have

$$
\Big((\beta(t,x)\chi(t))^2\wedge 1\Big) = \left(\Big(\chi^{-2}(t)\frac{n^2\max_{ijkl}|\mathcal{M}_{ijkl}(t,\xi(t,x))|}{\lambda_{min}(t,\xi(t,x))}\Big)\vee 1\right)^{-1}.
\tag{2.42}
$$

Observe that (A2) implies

$$
\left(\Big(\chi^{-2}(t)\frac{\max_{ijkl}|\mathcal{M}_{ijkl}(t,\xi)|}{\lambda_{min}(t,\xi)}\Big)\vee 1\right)\ln\big((1\vee\chi^{-n}(t))\lambda_{max}(t,\xi)\big) \ll \sigma\theta(t)e^{r\chi(t)\|\xi\|},
$$

as $\theta(t)+\|\xi\|\to\infty$, $(t,\xi)\in\mathcal{B}$. By the definition of the set $\mathcal{B}$ this relation, combined with (2.42) and the estimate $\mathcal{K}(t,x)\leq\lambda_{max}^n(t,\xi(t,x))$, yields

$$
\ln\Big((1\vee\chi^{-n}(t))\mathcal{K}(t,x)\Big) \ll \sigma\theta(t)e^{r\chi(t)\|\xi\|}\Big((\beta(t,x)\chi(t))^2\wedge 1\Big),
\qquad \theta(t)+\|x\|\to\infty,\quad (t,x)\in\mathcal{A},
$$

which in turn implies (2.41). Thus,

$$
J_3(t,x) \ll J_1(t,x)\quad\text{as } \theta(t)+\|\xi\|\to\infty,\quad (t,\xi)\in\mathcal{A}.
$$

Combining the results obtained on the steps 3, 4 and 5, we arrive at the statement of the theorem. □

3. Explicit conditions

In this section we give some explicit conditions under which the assumptions of Theorem 2.1 are easy to verify. To simplify the formulation we assume that the matrix $\mathcal{F}$ is of the form $\mathcal{F}(t,s) = f(t,s)\mathbb{I}$, where $\mathbb{I}$ is the identity matrix, and $f(t,\cdot)$ is a bounded function, satisfying

$$\int_I f^2(t,s)ds < \infty \tag{3.1}$$

and

$$\int_I (f(t,s) \vee 0)^2\, ds > 0, \quad t \in \mathbb{T}. \tag{3.2}$$

For such a kernel we can use a simplified version of condition (2.1). Let $f(s) \equiv f(1,s)$, and let $\mu_\ell(\cdot)$ be the image measure of $ds\,\mu(du)$ under the mapping

$$I \times \mathbb{R}^n \ni (s,u) \mapsto f(s)u \cdot \ell \in \mathbb{R}.$$

We assume that $\mu_\ell(\cdot)$ satisfies

$$\inf_{\ell \in \mathbb{S}^n} \mu_\ell(\mathbb{R}_+) > 0. \tag{3.3}$$

First we consider the fixed time setting; then under the assumption that f satisfies some self-similarity assumption we show that conditions (A1)–(A4) hold for $t \in \mathcal{T}$ provided that they hold for $t = 1$.

3.1. **Case** $t = 1$**.** To simplify the notation, we drop the index t where appropriate. In particular, we write $\mathcal{M}_{i_1..i_k}(\xi) \equiv \mathcal{M}_{i_1..i_k}(1,\xi)$, and $\Theta(r,A) \equiv \Theta(1,r,A)$, $r \geq 0$.

When $t = 1$ the assumptions (A1) − (A4) reduce to the following:

$(A1')$ $\quad \max_{ijkl} |\mathcal{M}_{ijkl}(\xi)| \ll \lambda^3_{min}(\xi)\lambda^{-1}_{max}(\xi)$ as $\|\xi\| \to \infty$.

$(A2')$

$$\ln\left(\left(\frac{\max_{ijkl}|\mathcal{M}_{ijkl}(\xi)|}{\lambda_{min}(\xi)}\right) \vee 1\right) + \ln\left(\ln \lambda_{max}(\xi) \vee 1\right) \ll \|\xi\|, \quad \text{as} \quad \|\xi\| \to \infty.$$

$(A3')$ There exists $R > 0$ and $\delta > 0$ such that for all $r \geq R$

$$\Theta(r, \mathbb{R}_+) \geq (n+\delta)\ln r; \tag{3.4}$$

$(A4')$ There exists $q > 0$ and $c > 0$ such that for all $\varepsilon > 0$

$$\inf_{h \geq \varepsilon} \Theta(h, [q,\infty)) \geq c(\varepsilon^2 \wedge 1). \tag{3.5}$$

Sometimes it is possible to show the stronger condition than $(A3')$:

$(A3'')$

$$\Theta(r, \mathbb{R}_+) \gg \ln r, \quad r \to +\infty. \tag{3.6}$$

We show that $(A3'')$ holds true under some restrictions on the kernel f and the non-degeneracy condition on μ. We assume that f satisfies one of the assumptions below; these assumptions are taken from [18], where they are discussed in detail.

(F_1) $\int_I (f(s) \vee 0)^2\, ds > 0$.

(F_2) On some interval $[a,b] \subset I$, the function f is positive and has a continuous non-zero derivative.

(F_3) On some interval $(-\infty, b] \subset I$, the function f is positive, convex, and has at most exponential decay at $-\infty$; that is, there exists $\gamma > 0$ such that

$$\lim_{s \to -\infty} e^{-\gamma s} f(s) = +\infty. \tag{3.7}$$

(F_4) On some interval $(-\infty, b) \subset I$, the function f is positive, convex, and has a subexponential decay at $-\infty$; that is, (3.7) holds true for every $\gamma > 0$.

Note that the conditions (F_i) become more strong with increase of i. Let $\mu_\ell(\cdot) := \mu_{1,\ell}(\cdot)$, $\ell \in \mathbb{S}^n$. We assume also that μ satisfies one of the assumptions below:

(N_1')

$$\inf_{\ell_z \in \mathbb{S}^n} \int_{|u\cdot \ell_z| \le \|z\|^{-1}} (u\cdot z)^2 \mu(du) \gg \ln \|z\|, \quad \|z\| \to \infty; \tag{3.8}$$

(N_2')

$$\int_{\mathbb{R}^n} [(u\cdot z)^2 \wedge 1]\mu(du) \gg \ln \|z\|, \quad \|z\| \to \infty; \tag{3.9}$$

(N_3') $\inf_{\ell \in \mathbb{S}^n} \mu_\ell(\mathbb{R}_+) = +\infty$;

(N_4') $\inf_{\ell \in \mathbb{S}^n} \mu_\ell(\mathbb{R}_+) > 0$.

As in the one-dimensional case, the conditions N_i' become more mild when i increases from 1 to 4.

The Lemma below generalizes the one dimensional result proved in [18]. Let

$$F := ess \sup_{s\in I} f(s). \tag{3.10}$$

For $q > 0$, $\ell \in \mathbb{S}^n$, define

$$V_{q,\ell} := \{u \in \mathbb{R}^n : u\cdot \ell > q\}. \tag{3.11}$$

Lemma 3.1. *Assume that for some $i = 1, \dots, 4$ conditions $(N_i') + (F_i)$ hold true. Then (A3") is satisfied.*

Proof. *Case $i = 1$.* By the left-hand side inequality in

$$(1-\cos 1)|x|^2 1_{|x|\le 1} \le 1 - \cos x \le 2(|x|^2 \wedge 1), \quad x \in \mathbb{R}, \tag{3.12}$$

we have

$$\begin{aligned}\Theta(r, \mathbb{R}_+) &\ge \inf_{\ell\in\mathbb{S}^n} \iint_{\substack{(s,u)\in I\times\mathbb{R}^n\\ 0<f(s)u\cdot\ell<r^{-1}}} (1-\cos(rf(s)\ell\cdot u))\mu(du)ds \\ &\ge (1-\cos 1)r^2 \inf_{\ell\in\mathbb{S}^n} \iint_{\substack{(s,u)\in I\times\mathbb{R}^n\\ 0<f(s)u\cdot\ell<r^{-1}}} f_+^2(s)(\ell\cdot u)^2\mu(du)ds \\ &\ge (1-\cos 1)r^2 \int_I f_+^2(s)ds \inf_{\ell\in\mathbb{S}^n} \int_{0<u\cdot\ell\le(Fr)^{-1}} (u\cdot\ell)^2\mu(du).\end{aligned}$$

Thus, for $i = 1$ the statement is implied by (3.8).

Case $i = 2$. The statement follows from (N_2') and the estimate

$$\int_a^b (1-\cos(xf(s)))ds \ge c(x^2\wedge 1), \quad x \in \mathbb{R}, \tag{3.13}$$

see [18] for details.

Case $i = 3$, 4. The inequality

$$\int_{-\infty}^b (1-\cos(xf(s)))\, ds \ge c\ln|x|, \quad x \in \mathbb{R}, \tag{3.14}$$

holds true (i) for some $c > 0$ and $|x|$ large enough provided that f satisfies (F_3); (ii) for every $c > 0$ and $|x|$ large enough provided that f satisfies (F_4), see [18].

In the case $i = 3$, take $c > 0$ and $Q > 0$ such that (3.14) holds true for $|x| \geq Q$. Then for $r \geq q^{-1}Q$

$$\begin{aligned}\Theta(r, \mathbb{R}_+) &\geq \inf_{\ell\in\mathbb{S}^n} \int_{-\infty}^{b} \int_{\mathbb{R}^n} (1 - \cos(rf(s)u\cdot\ell))\,\mu(du)ds \\ &\geq c \inf_{\ell\in\mathbb{S}^n} \mu_\ell(V_{q,\ell}) \ln(qr).\end{aligned} \tag{3.15}$$

Since

$$\inf_{\ell\in\mathbb{S}^n} \mu(V_{q,\ell}) = \inf_{\ell\in\mathbb{S}^n} \mu_\ell([q,\infty)) > 0, \tag{3.16}$$

by (N_3') we derive from the above inequality that $\Theta(r, \mathbb{R}_+) \geq C \ln r$ for any C large enough, which implies $(A3'')$.

In the case $i = 4$, assumption (N_4') implies the existence of $q > 0$ for which (3.16) holds true. Since for $i = 4$ we can take c in (3.15) arbitrary large we again arrive at $(A3'')$. □

Lemma 3.2. *Conditions* (F_2)+(3.3) *imply* $(A4')$ *for* $q > 0$ *small enough.*

Proof. Without loss of generality assume that f is positive on $[a, b]$. Take $\rho > 0$ such that $\inf_{\ell\in\mathbb{S}^n} \mu(V_{\rho,\ell}) > 0$. Then, for $0 < q < \rho \min_{s\in(a,b)} f(s)$, we have by (3.13)

$$\begin{aligned}\Theta(r, [q, +\infty)) &\geq \inf_{\ell\in\mathbb{S}^n} \int_{V_{\rho,\ell}} \int_a^b (1 - \cos(rf(s)u\cdot\ell))ds\mu(du) \\ &\geq c \inf_{\ell\in\mathbb{S}^n} \int_{V_{\rho,\ell}} \Big(r^2(u\cdot\ell)^2 \wedge 1\Big)\mu(du) \geq c \inf_{\ell\in\mathbb{S}^n} \mu(V_{\rho,\ell})\Big((\rho r)^2 \wedge 1\Big),\end{aligned}$$

which implies the required estimate. □

Analogously to the one-dimension case we say that the measure ν satisfies the Cramer's condition, if for any $\varepsilon > 0$

$$\sup_{\|z\|\geq\varepsilon} \Big| \int_{\mathbb{R}^n} e^{iy\cdot z} \nu(dy) \Big| < \nu(\mathbb{R}^n).$$

Under the assumption that ν has finite second moment this condition leads to

$$\Xi(\varepsilon) := \inf_{\|z\|\geq\varepsilon} \int_{\mathbb{R}^n} (1 - \cos z\cdot y)\nu(dy) \geq c(\varepsilon^2 \wedge 1) \quad \text{for all } \varepsilon > 0. \tag{3.17}$$

Lemma 3.3. *Assume that* f *satisfies assumption* (F_1), *and for some* $\rho > 0$

$$\inf_{|h|\geq\varepsilon} \inf_{\ell\in\mathbb{S}^n} \int_{V_{\rho,\ell}} (1 - \cos(h\ell\cdot u))\mu(dy) \geq c(\varepsilon^2 \wedge 1) \quad \textit{for all } \varepsilon > 0. \tag{3.18}$$

Then $(A4')$ *holds true for some* $q > 0$ *small enough.*

Proof. Take $q < \gamma F\rho$ with $F = \operatorname{esssup}_{s\in I} f(s)$ and some $\gamma \in (0, 1)$. Then

$$\begin{aligned}\Theta(h, [q, \infty)) &= \inf_{\ell\in\mathbb{S}^n} \iint_{\substack{f(s)u\cdot\ell>q \\ (s,u)\in I\times\mathbb{R}^n}} (1 - \cos(hf(s)u\cdot\ell))\,ds\mu(du) \\ &\geq \inf_{\ell\in\mathbb{S}^n} \iint_{f(s)>\gamma F,\, u\in V_{\rho,\ell}} (1 - \cos(hf(s)u\cdot\ell))\,ds\mu(du) \\ &\geq \int_{f(s)>\gamma F} \Big[\inf_{|hf(s)|\geq\gamma F\varepsilon} \inf_{\ell\in\mathbb{S}^n} \int_{u\in V_{\rho,\ell}} (1 - \cos(hf(s)u\cdot\ell))\,\mu(du)\Big] ds \\ &\geq c((\gamma F\varepsilon)^2 \wedge 1) \int_{f(s)>\gamma F} ds,\end{aligned}$$

where we used (3.18) in the third line. Since the set $\{s : f(s) > \gamma F\}$ has positive Lebesgue measure, we obtain the required estimate. □

Now we are ready to formulate the fixed-time version of Theorem 2.1. Let

$$p(x) \equiv p_1(x),$$
$$\mathcal{D}(x) \equiv \mathcal{D}(1,x),$$
$$\mathcal{K}(x) = \mathcal{K}(1,x).$$

Theorem 3.1. *Suppose that μ satisfies assumptions $(A1')$ and $(A2')$. In addition, suppose that μ and f satisfy one of the assumptions (N_i') and (F_i), $i = 1,..,4$, respectively. In the case $i = 1$ we assume in addition that μ satisfies the Cramer's condition* (3.18).

Then

$$p(x) \sim \frac{1}{\sqrt{(2\pi)^n \mathcal{K}(x)}} e^{\mathcal{D}(x)}, \quad \|x\| \to \infty. \tag{3.19}$$

The proof follows from Lemmas 3.1–3.3 above.

Let us give two examples when the conditions $(A1')$ and $(A2')$ are satisfied. For simplicity we consider the two-dimensional case.

Recall that the function

$$\sigma_Q(\xi) := \sup\{\xi \cdot u, u \in Q\}$$

is called the support function (cf. [29]) of the set Q. By definition, $\sigma_Q(\xi)$ is positive homogeneous, i.e. $\sigma_Q(\alpha x) = \alpha \sigma_Q(x)$ for $\alpha \geq 0$.

Example 3.1. Suppose that $f(s) > 0$ for all $s \in I$, and the support Q of the Lévy measure μ is bounded. By (3.3), there exists a subset $Q_0 \subset Q$, such that and

$$\sigma_0 := \inf_{\ell \in \mathbb{S}^2} \sigma_{Q_0}(\ell) > 0. \tag{3.20}$$

Observe that for any $\varepsilon > 0$

$$e^{(1-\varepsilon)F\sigma_{Q_0}(\xi)} \ll \mathcal{M}_{i_1..i_k}(\xi) \ll e^{(1+\varepsilon)F\sigma_{Q_0}(\xi)}, \tag{3.21}$$

where F is the essential supremum of $f(s)$ on I (cf. (3.10)). Observe that the same asymptotic relations hold for $\lambda_{min}(\xi)$ and $\lambda_{max}(\xi)$:

$$e^{(1-\varepsilon)F\sigma_{Q_0}(\xi)} \ll \lambda_{min}(\xi) \leq \lambda_{max}(\xi) \ll e^{(1+\varepsilon)F\sigma_{Q_0}(\xi)},$$

which implies $(A1')$:

$$\max_{ijkl} |\mathcal{M}_{ijkl}(\xi)| \ll e^{(1+\varepsilon)F\sigma_{Q_0}(\xi)} \ll e^{(2-4\varepsilon)F\sigma_{Q_0}(\xi)} \ll \lambda^3_{min}(\xi)\lambda^{-1}_{max}(\xi).$$

Further, for any $\varepsilon > 0$

$$\ln\left(\left(\frac{\max_{ijkl}|\mathcal{M}_{ijkl}(\xi)|}{\lambda_{min}(\xi)}\right) \vee 1\right) + \ln\left((\ln \lambda_{max}(\xi) \vee 1)\right) \ll \varepsilon F \sigma_{Q_0}(\xi) + \ln(\sigma_{Q_0}(\xi)).$$

Since $\sigma_Q(\xi) = \|\xi\| \sigma_{Q_0}(e_\xi)$, by (3.20) we get $(A2')$.

Example 3.2. Assume that $f(s) > 0$ for all $s \in I$, and $\mu(du) = e^{\Phi(u)} 1_{B^c(0,1)} du$, where $B^c(0,1) := \mathbb{R}^2 \backslash B(0,1)$, and $\Phi(u)$ is strictly convex on $B^c(0,1)$, satisfying

$$\Phi(u) \gg \|u\|^{1+\varepsilon} \quad \text{for some } \varepsilon > 0 \text{ as } \|u\| \to \infty. \tag{3.22}$$

Denote by

$$\Lambda(z) := \sup_{u \in B^c(0,1)} \{z \cdot u - \Phi(u)\}$$

the Legendre–Fenchel transform of Φ.

We have

$$\mathcal{M}_{i_1,..,i_k}(\xi) \sim \int_{f(s)>F-\varepsilon} \int_{B^c(0,1)} f^k(s) u_{i_1} ... u_{i_k} e^{f(s)\xi\cdot u - \Phi(u)} du\, ds, \quad k \geq 0, \tag{3.23}$$

(for $k = 0$ the left-hand side expression is just $\Psi(i\xi)$). The integral on the right-hand side can be estimated by the multi-dimensional version of the Laplace method (see [14], expression (4.29)), which leads to the asymptotic behaviour

$$\mathcal{M}_{i_1,..,i_k}(\xi) \sim \frac{(2\pi)^{\frac{n}{2}} F^k u_{i_1} ... u_{i_k}}{\sqrt{|\det \nabla^2 \Phi(u)|}}\Big|_{u=u_0(F\xi)} e^{\Lambda(F\xi)}, \quad \|\xi\| \to \infty, \tag{3.24}$$

where

$$u_0(\xi) := arg \max_{u \in B^c(0,1)} \{\xi \cdot u - \Phi(u)\}. \tag{3.25}$$

Thus, for $1 \leq k \leq 4$ there exists some polynomial $B(\xi)$ such that

$$\frac{1}{B(\xi)} e^{\Lambda(F\xi)} \leq \mathcal{M}_{i_1..i_k}(\xi) \leq B(\xi) e^{\Lambda(F\xi)}. \tag{3.26}$$

As is the proof of the previous proposition, the same (up to constants) inequalities hold for the eigenvalues of $\mathbb{M}$. By (3.26)

$$\max_{ijkl} |\mathcal{M}_{ijkl}(\xi)| \leq B(\xi) e^{\Lambda(F\xi)} \ll \frac{e^{2\Lambda(F\xi)}}{B^4(\xi)} \leq \lambda^3_{min}(\xi) \lambda^{-1}_{max}(\xi),$$

and since by (3.22) $\ln \Lambda(\xi) \ll \|\xi\|$ as $\|\xi\| \to \infty$, we have

$$\ln\Big(\Big(\frac{\max_{ijkl}|\mathcal{M}_{ijkl}(\xi)|}{\lambda_{min}(\xi)}\Big) \vee 1\Big) + \ln\left((\ln \lambda_{max}(\xi) \vee 1)\right)$$
$$\leq 2 \ln B(\xi) + \ln\big(\Lambda(F\xi) + \ln B(\xi)\big) \ll \|\xi\|, \quad \|\xi\| \to \infty.$$

Thus, $(A1')$ and $(A2')$ are satisfied.

3.2. **General case: self-similar kernel.** In this subsection we consider the general case when $t \in \mathbb{T}$ is not fixed, and assume that $f(t,s)$ satisfies the self-similarity assumption:

$$f(t,s) = \chi(t) f\left(\frac{s}{\theta(t)}\right), \quad t \in \mathbb{T}, \quad s \in I, \tag{3.27}$$

with some functions $f : \mathbb{R} \to \mathbb{R}$ and $\chi, \theta : \mathcal{T} \to (0, +\infty)$. Assumption (3.27) is satisfied for particularly interesting processes like the Lévy process and the fractional Lévy motion. In these cases we have, respectively,

$$f(s) = \mathbf{1}_{[0,1]}(s), \quad \chi(t) = 1, \quad \theta(t) = t; \tag{3.28}$$

$$f(s) = \frac{1}{\Gamma(H+1/2)} \left[(1-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right], \quad \chi(t) = t^{H-1/2}, \quad \theta(t) = t. \tag{3.29}$$

For the Lévy measure μ we assume (2.1) and (1.3) to hold true, as before. In addition we assume that

$$\begin{aligned} &\theta(t) \to +\infty, \quad \ln(\ln \chi(t)) \vee 1) \ll \ln \theta(t), \quad t \to +\infty, \\ &\liminf_{t\to\infty} \chi(t) > 0. \end{aligned} \tag{3.30}$$

In the proof of the theorem below we will use the notation

$$H(x,z) := H(1,x,z), \quad \mathcal{M}_{i_1..i_k}(\xi) := \mathcal{M}_{i_1..i_k}(1,\xi), \quad \lambda_i(\xi) := \lambda_i(1,\xi). \tag{3.31}$$

Put $\tau(t) := \chi(t)\theta(t)$.

Theorem 3.2. *Suppose that μ satisfies assumptions $(A1')$ and $(A2')$. In addition, suppose that μ and f satisfy one of assumptions (N_i') and (F_i), $i = 1,..,4$, respectively. In the case $i = 1$ we assume in addition that μ satisfies the Cramer's condition* (3.17).

Then

(3.32)
$$p_t(x) \sim \frac{1}{\tau(t)}\sqrt{\frac{\theta(t)}{(2\pi)^n \mathcal{K}(x/\tau(t))}} e^{\theta(t)\mathcal{D}(x/\tau(t))}, \quad t + \|x\| \to \infty, \quad (t,x) \in [t_0, \infty) \times \mathbb{R}^n.$$

Proof. By the self-similarity assumption (3.27) we have

(3.33)
$$H(t,x,z) = \theta(t) H\left(\frac{x}{\tau(t)}, \chi(t)z\right), \quad \mathcal{M}_{i_1..i_k}(t,\xi) = \chi^k(t)\theta(t)\mathcal{M}_{i_1..i_k}(\chi(t)\xi), \quad k \geq 1.$$

Denote by $\zeta(y)$ the solution to

$$grad_\zeta H(y, i\zeta) = 0. \tag{3.34}$$

The equality (3.33) for $H(t,x,z)$ implies that the equation $grad_\xi H(t,x,i\xi) = 0$ can be rewritten as

$$\chi(t)\theta(t) grad_\zeta H\left(\frac{x}{\tau(t)}, i\zeta\right)\Big|_{\zeta = \chi(t)\xi} = 0,$$

from where we conclude that $\xi(t,x)$ satisfies

$$\xi(t,x) = \chi^{-1}(t)\zeta\left(\frac{x}{\tau(t)}\right),$$

By the equality for $\mathcal{M}_{ij}$ in (3.33) we have

$$\lambda_i(t, \xi(t,x)) = \chi^2(t)\theta(t)\lambda_i(\zeta(\frac{x}{\tau(t)})), \tag{3.35}$$

implying

$$\mathcal{D}(t,x) = \theta(t)\mathcal{D}\Big(\frac{x}{\tau(t)}\Big), \quad \det \mathbb{M}(t, \xi(t,x)) = \chi^2(t)\theta(t) \det \mathbb{M}\Big(\zeta\Big(\frac{x}{\tau(t)}\Big)\Big).$$

Thus (3.32) would follow from (2.16) with $\mathcal{A} = [t_0, +\infty) \times \mathbb{R}^n$, provided that conditions $(A1) - (A4)$ are verified.

By (3.30),

$$t + \|\xi\| \to \infty \quad \text{implies} \quad \theta(t) \to +\infty \quad \text{or} \quad \chi(t)\|\xi\| \to +\infty. \tag{3.36}$$

Let

$$\mathcal{B} = \{(t,\xi) \colon t \geq t_0\} \equiv \{(t, \xi(t,x)) \colon t \geq t_0\}.$$

By the right-hand side relation in (3.33) and (3.35) we get

$$\frac{\max_{ijkl} |\mathcal{M}_{ijkl}(t,\xi)|}{\lambda^3_{min}(t,\xi)\lambda^{-1}_{max}(t,\xi)} = \frac{\max_{ijkl} |\mathcal{M}_{ijkl}(\chi(t)\xi)|}{\theta(t)\lambda^3_{min}(\chi(t)\xi))\lambda^{-1}_{max}(\chi(t)\xi)}. \tag{3.37}$$

Hence, $(A1)$ follows from $(A1')$.

Further, by the right-hand side relation in (3.33) and (3.35)

$$\frac{\max_{ijkl} |\mathcal{M}_{ijkl}(t,\xi)|}{\lambda_{min}(t,\xi)} = \chi^2(t)\frac{\max_{ijkl} |\mathcal{M}_{ijkl}(\chi(t)\xi)|}{\lambda_{min}(\chi(t)\xi)} \tag{3.38}$$

which together with $(A2')$ and (3.36) gives

$$\ln\left(\Big(\chi^{-2}(t)\frac{\max_{ijkl} |\mathcal{M}_{ijkl}(t,\xi)|}{\lambda_{min}(t,\xi)}\Big) \vee 1\right) \ll \ln\theta(t) + \chi(t)\|\xi\|, \quad t + \|\xi\| \to +\infty, \quad (t,\xi) \in \mathcal{B}.$$

Similarly,

$$\ln\Big(\Big(\ln\lambda_{max}(t,\xi)\Big)\vee 1\Big) = \ln\Big(\ln\Big(\chi^2(t)\theta(t)\lambda_{max}(\chi(t)\xi)\Big)\vee 1\Big)$$
$$= \ln\Big(\Big(\ln\chi^2(t)+\ln\theta(t)+\ln\lambda_{max}(\chi(t)\xi)\Big)\vee 1\Big).$$

(By the second-line relation in (3.30) we can drop the term $(1\vee\chi^{-1}(t))$ in $(A2)$). By $(A2')$, (3.36) and (3.30) we have

$$\ln\Big(\Big(\ln\lambda_{max}(t,\xi)\Big)\vee 1\Big) \ll \ln\theta(t)+\chi(t)\|\xi\|, \quad t+\|\xi\|\to+\infty, \quad (t,\xi)\in\mathcal{B}.$$

This completes the proof of $(A2)$.

By $(A3'')$, for every $\varkappa>0$ there exists $Q>0$ such that

$$\Theta(r,\mathbb{R}_+)\ge\varkappa\ln r, \quad r\ge Q.$$

By the self-similarity assumption (3.27), we have

$$\Theta(t,r,A)=\theta(t)\Theta\left(\chi(t)r,\frac{1}{\chi(t)}A\right).$$

Denote $\theta_*=\inf_t\theta(t)$, $\chi_*=\inf_t\chi(t)$. Then taking

$$\varkappa=\frac{1+\delta}{\theta_*} \quad\text{and}\quad R=\chi_*^{-1}Q,$$

we obtain $(A3)$ from $(A3')$.

Finally, by $(A4')$ we have

$$\inf_{r\ge\varepsilon}\Theta(t,r,[q\chi(t),+\infty))=\theta(t)\inf_{r\ge\varepsilon}\Theta(\chi(t)r,[q,+\infty))$$
$$=\theta(t)\inf_{r'>\chi(t)\varepsilon}\Theta(r',[q,+\infty))\ge c\theta(t)\Big((\chi(t)\varepsilon)^2\wedge 1\Big),$$

implying $(A4)$.

Thus, conditions $(A1)$ – $(A4)$ are satisfied, implying that (3.32) follows from (2.16). □

4. Application

As an example of an application of Theorem 2.1 we prove the ratio limit theorem for the distribution density $p(x)$. In [18] the ratio limit theorem is proved in the one-dimensional case for the invariant distribution density of the Lévy-driven Ornstein-Uhlebeck process X, and further used in [24] in the proof of the spectral gap property of X. Namely, in [24] the proof of the existence of the spectral gap consists of two parts: it it shown that X and the dual process X^* satisfy a) the Doeblin condition, b) the Lyapunov type condition. The ratio limit theorem is essential to show b) for X^*; other parts can be deduced from [22], [23], [25].

Let

$$r_a(x):=\frac{p(x+a)}{p(x)}, \quad a\in\mathbb{R}^n. \tag{4.1}$$

Recall that $\zeta(x)$ denotes the critical point of $H(1,x,i\xi)$ on $\mathbb{R}$.

Theorem 4.1. *Assume that conditions of Theorem 3.1 are satisfied, and*

$$\lambda_{max}(\xi)\le c\lambda_{min}(\xi) \quad \textit{for all } \xi\in\mathbb{R}^n, \tag{4.2}$$

for some $c>0$, independent of ξ. Then

$$r_a(x)\sim e^{a\cdot\zeta(x)} \quad \textit{as } \|x\|\to\infty. \tag{4.3}$$

For the proof we need the auxiliary lemma.

Lemma 4.1. *Let $\mathbb{A}_n(x) = (a_{ij}(x))_{i,j=1}^n$ be a non-degenerate $n \times n$ matrix with $C^1(\mathbb{R}^n)$ elements. Then*

$$\|\nabla \det \mathbb{A}_n\| \le c_n \max_{ij} \|\nabla a_{ij}(x)\| \max_{i,j} |a_{ij}(x)|^{n-1}. \tag{4.4}$$

Proof. We prove the statement of the Lemma by induction. For $n = 1$ the statement is obvious. Suppose that the statement of the Lemma holds true for any non-degenerate $(n-1) \times (n-1)$ matrix with smooth elements. Denote by $\mathbb{A}_{n-1}(j)$, $j = 1, .., n$, the matrices obtained from $\mathbb{A}_n$ by deleting the first line and j-th row. Then

$$\begin{aligned}
\|\nabla \det \mathbb{A}_n(x)\| &\le \Big\|\nabla\big(\sum_{j=1}^n (-1)^{j+1} a_{11} \det \mathbb{A}_{n-1}(j)\big)\Big\| \\
&= \Big\|\sum_{j=1}^n (-1)^{j+1}\big(\nabla a_{1j} \cdot \det \mathbb{A}_{n-1}(j) + a_{1j} \nabla \det \mathbb{A}_{n-1}(j)\big)\Big\| \\
&\le n\big(\max_{ij} \|\nabla a_{ij}(x)\| \max_{ij} |a_{ij}|^{n-1} \\
&\quad + c_{n-1} \max_{ij} |a_{ij}(x)| \max_{ij} \|\nabla a_{ij}(x)\| \max_{ij} |a_{ij}|^{n-2}\big) \\
&\le c_n \max_{ij} \|\nabla a_{ij}(x)\| \max_{ij} |a_{ij}(x)|^{n-1}.
\end{aligned}$$

□

Proof of Theorem 4.1. From Theorem 3.1 we have

$$r_a(x) \sim \frac{\mathcal{K}(x)}{\mathcal{K}(x+a)} e^{\mathcal{D}(x+a)-\mathcal{D}(x)}, \quad \|x\| \to \infty.$$

We show that

$$\frac{\mathcal{K}(x)}{\mathcal{K}(x+a)} \to 1$$

and

$$\mathcal{D}(x+a) - \mathcal{D}(x) = a \cdot \zeta(x) + o(1) \quad \text{as } \|x\| \to \infty.$$

We have

$$\mathcal{D}(x+a) - \mathcal{D}(x) = a \cdot \zeta(x) - (x+a) \cdot (\zeta(x+a) - \zeta(x)) + \mathcal{M}_0(\zeta(x+a)) - \mathcal{M}_0(\zeta(x)).$$

Observe that ζ satisfies the equation

$$x = \nabla_\zeta \mathcal{M}_0(\zeta). \tag{4.5}$$

Differentiating with respect to x we get

$$\mathbb{I} = \nabla_x\big(\nabla_\zeta \mathcal{M}_0(\zeta(x))\big) = \nabla_\zeta^2 \mathcal{M}_0(\zeta) \nabla_x \zeta(x) = \mathbb{M}(\zeta(x)) \nabla_x \zeta(x).$$

Since $\mathbb{M}$ is non-degenerate, we have for $e_a := \frac{a}{\|a\|}$

$$\mathbb{M}^{-1}(\zeta(x)) e_a = \nabla \zeta(x) e_a =: \zeta_a'(x), \tag{4.6}$$

implying

$$\|\zeta_a'(x)\| \le \|\mathbb{M}^{-1}(\zeta(x))\| = \frac{1}{\lambda_{min}(\zeta(x))} \to 0, \quad \|x\| \to \infty. \tag{4.7}$$

Therefore by the mean value theorem and (4.5) we have

$$
\begin{aligned}
(x+a)\cdot(\zeta(x+a)-\zeta(x)) - \mathcal{M}_0(\zeta(x+a)) - \mathcal{M}_0(\zeta(x)) &= (x+a)\int_0^1 \nabla\zeta(y)\big|_{y=x+sa}\cdot a ds \\
&\quad - \int_0^1 \nabla_\zeta \mathcal{M}_0(\zeta(y))\nabla\zeta(y)\cdot a\big|_{y=x+sa} ds \\
&= \|a\|\Big(\int_0^1 (x+a)\cdot\zeta_a'(x+sa)ds - \int_0^1 (x+sa)\cdot\zeta_a'(x+sa)\Big) \\
&= \|a\|^2\int_0^1 (1-s)e_a\cdot\zeta_a'(x+sa)ds.
\end{aligned}
$$

By (4.7) the norm of the right-hand side expression tends to 0 as $\|x\|\to\infty$, implying

$$
\mathcal{D}(x+a)-\mathcal{D}(x)\sim a\cdot\zeta(x)+o(1), \quad \|x\|\to\infty. \tag{4.8}
$$

Further,

$$
\frac{\mathcal{K}(x+a)}{\mathcal{K}(x)} = e^{\ln\frac{\mathcal{K}(x+a)}{\mathcal{K}(x)}} = e^{\int_0^1 \nabla(\ln\mathcal{K}(x+sa))\cdot a ds}.
$$

By (4.2) and Lemma 4.1

$$
\begin{aligned}
\big\|\nabla\ln\mathcal{K}(x)\big\| &\le c_1(n)\frac{\max_{ijk}|\mathcal{M}_{ijk}(\zeta(x))|}{\lambda_{min}^n(\zeta(x))}\lambda_{max}^{n-1}(\zeta(x))\|\nabla\zeta(x)\| \\
&\le c_2(n)\frac{\max_{ijk}|\mathcal{M}_{ijk}(\zeta(x))|}{\lambda_{min}^2(\zeta(x))} \\
&\ll \lambda_{min}^{-\frac{1}{2}}(\zeta(x))\to 0, \quad \|x\|\to\infty,
\end{aligned}
$$

where in the last line we used (2.28). Thus,

$$
\frac{\mathcal{K}(x+a)}{\mathcal{K}(x)}\to 1 \quad \text{as } \|x\|\to\infty,
$$

which together with (4.8) implies the statement of the theorem. □

Acknowledgement. The author thanks A. Kulik for many inspiring and fruitful discussions and remarks. Also, the author thanks the referee for careful reading of the paper and making many helpful comments and suggestions.

References

1. M. T. Barlow, R. B. Bass, Z.-Q. Chen and M.Kassmann. *Non-local Dirichlet forms and symmetric jump processes*, Trans. Amer. Math. Soc. **361** (2009), 1963–1999.
2. M. T. Barlow, A. Grigoryan and T. Kumagai. *Heat kernel upper bounds for jump processes and the first exit time*, J. Reine Angew. Math. **626** (2009), 135–157.
3. R. F. Bass and D. A. Levin. *Transition probabilities for symmetric jump processes*, Trans. Amer. Math. Soc. **354** (2002), 2933–2953.
4. R. F. Bass, T. Kumagai, *Symmetric Markov chains on $\mathbb{Z}^d$ with unbounded range*, Trans. Amer. Math. Soc. **360(4)** (2008), 2041–2075.
5. R. F. Bass, T. Kumagai and T. Uemura, *Convergence of symmetric Markov chains on Z^d*, Probab. Theory Relat. Fields. **148** (2010), 107–140.
6. S. V. Bodnarchuk and O. M. Kulyk. *Conditions for existence and smoothness of the distribution density for an Ornstein-Uhlenbeck process with Lévy noise*, Teor. Imovirn. Mat. Stat. **79** (2008), 20–33. (Ukrainian; English transl. in *Theor. Probab. Math. Stat.* **79** (2009), 23–38).
7. E. A. Carlen, S. Kusuoka and D. W. Stroock. *Upper bounds for symmetric Markov transition functions*, Ann. Inst. Poincaré. **2** (1987), 245–287.
8. Z.-Q. Chen, P. Kim and T. Kumagai, *Weighted Poincaré inequality and heat kernel estimates for finite range jump processes*, Math. Ann. **342(4)** (2008), 833–883.
9. Z.-Q. Chen, P. Kim and T. Kumagai. *On Heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces*, Acta Math. Sin. (Engl. Ser.) **25** (2009), 1067–1086.

10. Z.-Q. Chen, P. Kim and T. Kumagai. *Global heat kernel estimates for symmetric jump processes*, Trans. Amer. Math. Soc. **363** (2011), 5021-5055.
11. Z.-Q. Chen and T. Kumagai. *Heat kernel estimates for stable-like processes on d-sets*, Stoch. Proc. Appl. **108** (2003), 27–62.
12. Z.-Q. Chen, T. Kumagai. *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Th. Rel. Fields. **140** (2008), no. 1-2, 277–317.
13. E. T. Copson. *Asymptotic expansions*, Cambridge Uni. Press, Cambridge, 1965.
14. M. V. Fedoryuk. *The saddle point method*, Nauka, Moscow, 1977. (In Russian).
15. I. I. Gikhman and A. V. Skorokhod. *Stochastic differential equations and their applications*, Naukova Dumka, Kiev, 1982. (In Russian).
16. N. Jacob, V. Knopova, S. Landwehr and R. L. Schilling. *A geometric interpretation of the transition density of a Lévy process*, To appear in: Science China: Mathematics.
17. P. Hartman and A.Wintner, *On the infinitesimal generators of integral convolutions*, Am. J. Math. **64** (1942), 273–298.
18. V. P. Knopova and A. M. Kulik. *Exact asymptotic for distribution densities of Lévy functionals*, Electronic J. Prob. **16** (2011), 1394–1433.
19. V. P. Knopova and A. M. Kulik, *Asymptotic behaviour of the distribution density of the fractional Lévy motion*, Preprint 2011, available at *http://arxiv.org/abs/1112.0497*.
20. V. P. Knopova and R. S. Schilling, *Transition Density Estimates for a Class of Lévy and Lévy-Type Processes*, To appear in J. Theor. Prob. DOI: 10.1007/s10959-010-0300-0.
21. V. P. Knopova and R. S. Schilling, *A note on the existence of transition probability densities of Lévy processes*, To appear in Forum Math. DOI: 10.1515/FORM.2011.108
22. A. M. Kulik, *Absolute continuity and convergence in variation for distributions of a functionals of Poisson point measure*, 2008. arXiv:0803.2389.
23. A. M. Kulik, *Exponential ergodicity of the solutions to SDE's with a jump noise*, *Stochastic Process. Appl.* **119** (2009), no. 2, 602–632.
24. A. M. Kulik, *Asymptotic and spectral properties of exponentially φ-ergodic Markov processes*, Stoch. Proc. Appl. **121** (2011), 1044–1075.
25. H. Masuda. *Ergodicity and exponential β-mixing bounds for multidimensional diffusions with jumps*, Stoch. Proc. Appl. **117** (2007), 35–56.
26. P. Lancaster. *Theory of matrices*, Academic Press, New York, 1969.
27. B. S. Rajput and J. Rosinski. *Spectral representations of infinitely divisible processes*, Prob. Th. Rel. Fields. **82** (1989), 451–487.
28. R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*, Springer, Berlin, 1997.
29. R. T. Rockafellar. *Convex Analysis*, Second ed., Princeton University Press, USA, 1972.
30. B. V. Shabat. *Introduction to Complex Analysis. Part II: Functions of several variables*, AMS, USA, 1992.

V.M. Glushkov Institute of Cybernetics National Academy of Science of Ukraine, 40, Acad. Glushkov Ave., 03187, Kiev, Ukraine

E-mail address: vic_knopova@gmx.de