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ON THE STRONG UNIQUENESS OF A SOLUTION TO SINGULAR STOCHASTIC DIFFERENTIAL EQUATIONS

We prove the existence and uniqueness of a strong solution for an SDE on a semi-axis with singularities at the point 0. The result obtained yields, for example, the strong uniqueness of non-negative solutions to SDEs governing Bessel processes.

Introduction

We consider a stochastic process the state space of which is a non-negative semi-axis. Assume that up to the first hitting time of zero the process $(x(t))_{t\geq 0}$ satisfies an SDE

$$x(t) = x_0 + \int_0^t a(x(s))ds + \int_0^t \sigma(x(s))dw(s),$$

where $x_0 \geq 0, a, \sigma$ are supposed to be locally Lipshitz continuous on $(0,\infty)$, $(w(t))_{t\geq 0}$ is a Wiener process. Possible singularities of the coefficients generate different types of behavior of the process in a neighborhood of zero. As a consequence, the integral representation of $(x(t))_{t\geq 0}$ may acquire various forms.

As an example let us consider the following SDE

$$\rho(t) = \rho(0) + w(t) + \frac{\beta-1}{2}\int_0^t \frac{1}{\rho(s)}ds,\ \rho(0)\geq 0. \tag{1}$$

It is known that β-dimensional Bessel process with $\beta > 1$ is a unique non-negative strong solution to (1) (cf. [2]). Note that this equation possesses no additional terms. Otherwise, an additional summand can be represented by the local time $(l(t))_{t\geq 0}$ of unknown process $(x(t))_{t\geq 0}$ at the point 0 like in Skorokhod equation

$$x(t) = x_0 + \int_0^t a(x(s))ds + \int_0^t b(x(s))dw(s) + l(t), \tag{2}$$

or by principal values of some functionals of the unknown process like in the following representation for a β-dimensional Bessel process with $\beta \in (0,1)$ (cf. [13], Ch. XI)

$$\rho(t) = \rho(0) + w(t) + \frac{\beta-1}{2}k(t),\ \rho(0)\geq 0. \tag{3}$$

Here $k(t) = V.P.\int_0^t \rho^{-1}(s)ds$ which, by definition, is equal to $\int_0^\infty a^{\beta-2}(L_a^\rho(t) - L_0^\rho(t))da$, $L_a^\rho(t)$ ia a local time of the process $\rho(t)$ at the point a.

It seems improbable to describe all possible forms of integral representations. Let f be a twice continuously differentiable function on $[0,\infty)$ which is a constant in a neighborhood of zero. Applying Itô formula (additional tricks are needed in some cases) we

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see that the stochastic differential of the function $f(x(t))$ has identical form for solutions of equations (1)-(3). Namely,

$$f(x(t)) = f(x_0) + \int_0^t \left(a(x(s))f'(x(s)) + \frac{1}{2}\sigma^2(x(s))f''(x(s)) \right) ds + \int_0^t \sigma(x(s))f'(x(s))dw(s). \tag{4}$$

The differential has no singularities. Intuitively, the singularities at the point 0 are killed by zero derivatives of the function f. We use this fact to formulate the problem as an analogue of a martingale problem (see Section 1). The main result of this paper is as follows: the existence of a weak non-negative solution for equation (4) spending zero time at the point 0 implies the existence and uniqueness of a non-negative strong solution spending zero time at 0. The pathwise uniqueness is obtained by method of Le Gall [5] based on the fact that the maximum of two solution also solves the equation. The formulation of a martingale problem involving a class of functions that are constant in a neighborhood of possible singularities was used by many authors (see, for example, [12], [15]).

The notations and definitions used are collected in Section 1. We prove the main Theorem in Section 2. In Section 3 some examples are represented.

1. Notations and definitions

Let a, σ be real-valued Borel-measurable functions defined on $[0, \infty)$. From now on we assume that the following condition is valid

Condition A. Suppose that the functions a and σ are locally Lipschitz continuous on $(0, \infty)$, i.e. for each $\varepsilon > 0$ there exist constants $C_\varepsilon > 0$ such that for all $\{x, y\} \subset [\varepsilon, \infty)$

$$|a(x) - a(y)| + |\sigma(x) - \sigma(y)| \le C_\varepsilon |x - y|.$$

The set of continuous functions $x : [0, \infty) \to [0, \infty)$ is denoted by $C^+([0, +\infty))$. Let $\mathfrak{G}_t \equiv \sigma\{x(s) : 0 \le s \le t,\ x \in C^+([0, +\infty))\}$, and $\mathfrak{G} \equiv \sigma\{x(s) : 0 \le s < \infty,\ x \in C^+([0, +\infty))\}$ be σ-algebras on $C^+([0, +\infty))$. The set of real-valued functions which are twice continuously differentiable on $[0, \infty)$ and constant in a neighborhood of zero is denoted by $C_c^2([0, +\infty))$. Given a probability measure P on $(C^+([0, +\infty)), \mathfrak{G})$, the family of continuous, square integrable local $\mathfrak{G}_t$-martingales is denoted by $\mathcal{M}_2^{c,loc}(P)$.

Definition 1. Given $x_0 \ge 0$, *a solution to the martingale problem* $M(a, \sigma, x_0)$ is a probability measure P_{x_0} on $(C^+([0, +\infty)), \mathfrak{G})$ such that

(i) $P_{x_0}(x(0) = x_0) = 1$.

(ii) For each $f \in C_c^2([0, +\infty))$,

$$Y_f(t) = f(x(t)) - f(x_0) - \int_0^t \left[a(x(s))f'(x(s)) + \frac{1}{2}\sigma^2(x(s))f''(x(s)) \right] ds \in \mathcal{M}_2^{c,loc}(P_{x_0}).$$

(iii)

$$E^{P_{x_0}} \int_0^\infty \mathbb{1}_{\{0\}}(x(s))ds = 0.$$

Definition 2. The martingale problem is *well-posed* if for each $x_0 \ge 0$ there is exactly one solution to the martingale problem starting from x_0.

Definition 3. Given $x_0 \ge 0$, let a pair $(x(t), w(t))_{t\ge 0}$ of continuous adapted processes on a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), P)$ be such that

(i) $(w(t))_{t\ge 0}$ is a standard $(\mathfrak{F}_t)$-Brownian motion,

(ii) the process $(x(t))_{t\ge 0}$ takes values on $[0, \infty)$,

(iii) for each $t \geq 0$, and $f \in C_c^2([0,\infty))$, the equality

$$(5) \quad f(x(t)) = f(x_0) + \int_0^t \left(a(x(s))f'(x(s)) + \frac{1}{2}\sigma^2(x(s))f''(x(s)) \right) ds + \int_0^t \sigma(x(s))f'(x(s))dw(s)$$

holds true P-a.s.

Then the pair (x,w) is called *a weak solution to equation (5) with initial condition* x_0.

Remark 1. Let $f \in C_c^2([0,\infty))$. Then there exists $\delta_f > 0$ such that $f' = f'' = 0$ on $[0,\delta_f]$. This and Condition A ensure the existence of all the integrals on the right-hand side of (5).

Remark 2. It is not hard to verify that the existence of a weak solution $(x(t),w(t))_{t\geq 0}$ to equation (5) on a probability space $(\Omega,\mathfrak{F},P)$ with initial condition x_0 is equivalent to the existence of a probability measure $\tilde{P}$ on some probability space $(\tilde{\Omega},\tilde{\mathfrak{F}},\tilde{P})$ satisfying conditions (i) and (ii) of Definition 1. The process $(x(t))_{t\geq 0}$ induces the measure $\tilde{P}$ on $(C^+([0,\infty)),\mathfrak{G})$, namely $\tilde{P} = Px^{-1}$.

For the proof see Appendix.

Definition 4. *The weak uniqueness* holds for equation (5) if, for any two weak solutions (x,w) and $(\tilde{x},\tilde{w})$ (which may be defined on different probability spaces) with a common initial value, i.e. $x(0) = x_0$ P-a.s., $\tilde{x}(0) = x_0$ $\tilde{P}$-a.s. the laws of processes x and $\tilde{x}$ coincide.

Definition 5. *The pathwise uniqueness* holds for equation (5), if for any two weak solutions (x,w) and $(\tilde{x},w)$ on the same probability space $(\Omega,\mathfrak{F},P)$ with common Brownian motion and common initial value, i.e. $x(0) = \tilde{x}(0) = x_0$ P-a.s., the equality $x(t) = \tilde{x}(t),\ t \geq 0$, fulfils P-a.s.

Denote by $\left(\overline{\mathfrak{F}}_t^w\right)$ the filtration of w completed with respect to P.

Definition 6. Given a process $(w(t))_{t\geq 0}$, and $x_0 \geq 0$, we say that the process $(x(t))_{t\geq 0}$ is *a strong solution* to equation (5) with initial condition x_0 if it is adapted to the filtration $\left(\overline{\mathfrak{F}}_t^w\right)$ and conditions (i)-(iii) of Definition 3 hold.

Definition 7. *The strong uniqueness* holds for equation (5) if there exists a strong solution to equation (5) and the pathwise uniqueness is valid for equation (5).

2. The main result

Unfortunately, we are not able to write equation (5) for the process $(x(t))_{t\geq 0}$ itself because the function $f(x) = x$ does not belong to $C_c^2([0,\infty))$. Instead, in the next Lemma we obtain an SDE for the process $\zeta_\delta(x(t)) = x(t) \vee \delta,\ t \geq 0$, which will often be used in the sequel.

Lemma 1. *Given $\delta > 0$, put $\zeta_\delta(x) = x \vee \delta,\ x \in [0,\infty)$. Suppose $(x(t))_{t\geq 0}$ is a weak solution to equation (5). Then the equality*

$$(6) \quad \zeta_\delta(x(t)) = \zeta_\delta(x(0)) + \int_0^t a(x(s))\mathbb{1}_{(\delta,+\infty)}(x(s))ds + \int_0^t \sigma(x(s))\mathbb{1}_{(\delta,+\infty)}(x(s))dw(s) + \frac{1}{2}L_\delta^x(t)$$

is valid for all $t \ge 0$. *Here* $(L_\delta^x(t))_{t\ge 0}$ *is a local time of the process* $(x(t))_{t\ge0}$ *at the point* δ *defined by the formula*

$$L_\delta^x(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[\delta,\delta+\varepsilon)}(x(s))ds,\ t\ge 0. \tag{7}$$

Proof. We make use of a standard approximation of non-smooth function by smooth ones like the construction in the proof of Tanaka's formula (see, for example, Theorem 4.1, Ch. 3 of [8]). For $a>0$, let us approximate the function $\xi_a(x) = x\vee a$ by twice continuously differentiable functions. Put

$$\psi(x) = \begin{cases} C\exp\left(\frac{1}{(x-1)^2-1}\right), & 0<x<2,\\ 0, & \text{otherwise,}\end{cases}$$

where C is a constant such that $\int_{-\infty}^{\infty}\psi(x)dx = 1$, and define, for $n\ge 1$

$$u_n(x) = \int_{-\infty}^x dy\int_{-\infty}^y n\psi(nz)dz.$$

Then the function u_n is twice continuously differentiable and $u_n(x-a)+a\to\xi_a(x)$, as $n\to\infty$. Further,

$$u_n'(x-a)\to \mathbb{1}_{(a,\infty)}(x),\ n\to\infty,$$

and

$$\int_{-\infty}^{\infty} u_n''(x-a)\varphi(x)dx \to \varphi(a),\ n\to\infty \tag{8}$$

for any continuous and bounded function φ.

Let $\eta_\delta\in C_c^2([0,\infty))$ be a non-decreasing and such that $\eta_\delta(x) = x$ on $[\delta/2,\infty)$. The process $(\eta_\delta(x(t)))_{t\ge0}$ can be represented in the form (5), and thus it is a semimartingale. By Itô formula we have

$$\begin{aligned}
u_n(\eta_\delta(x(t))-\delta)+\delta &= u_n(\eta_\delta(x(0))-\delta)+\delta\\
&+\int_0^t\left[a(x(s))\eta_\delta'(x(s)) + \frac12\sigma^2(x(s))\eta_\delta''(x(s))\right]u_n'(\eta_\delta(x(s))-\delta)ds\\
&+\int_0^t \sigma(x(s))\eta_\delta'(x(s))u_n'(\eta_\delta(x(s))-\delta)dw(s)\\
&+\frac12\int_0^t\sigma^2(x(s))(\eta_\delta'(x(s)))^2u_n''(\eta_\delta(x(s))-\delta)ds.
\end{aligned} \tag{9}$$

By occupation times formula (cf. [11], Corollary 1, p. 216), the last integral on the right-hand side of (9) is equal to

$$\int_0^t u_n''(\eta_\delta(x(s))-\delta)d\langle\eta_\delta(x)\rangle(s) = \int_{-\infty}^{+\infty}u_n''(a-\delta)L_a^{\eta_\delta(x)}(t)da \to L_\delta^{\eta_\delta(x)}(t), \quad \text{as } n\to\infty.$$

Here $L_\delta^{\eta_\delta(x)}(t)$ is a local time of the process $(\eta_\delta(x(t)))_{t\ge0}$ at the point δ.

Note that $\zeta_\delta(x) = x\vee\delta = \eta_\delta(x)\vee\delta,\ x\in[0,\infty)$. Passing to the limit in (9) as $n\to\infty$ and taking into account that $\eta_\delta(x) = x,\ x>\delta/2$, we arrive at the equation (6). □

The main result of the paper is the following theorem.

Theorem 1. *Suppose* a,σ *satisfy Condition A and* $\sigma(x)\ne 0,\ x\ge 0$. *If for each* $x_0\ge 0$ *there exists a solution to the martingale problem* $M(a,\sigma,x_0)$, *then for each* $x_0\ge0$ *there exists a strong solution to equation (5) with initial condition* x_0 *spending zero time at the point* 0 *and the strong uniqueness holds in the class of solutions spending zero time at* 0.

We split the proof of Theorem into two steps. At the first one we show that the existence of a solution to the martingale problem provides well-posedness. At the second one the pathwise uniqueness is obtained from weak uniqueness. These two steps are formulated as Lemmas in the following way.

Lemma 2 (weak uniqueness)**.** *Suppose a, σ satisfy the conditions of Theorem 1. Let for each $x_0 \geq 0$, there exists a solution to martingale problem $M(a, \sigma, x_0)$. Then the weak uniqueness holds for equation (5).*

Proof. We would like to get a law of the process $(x(t))_{t\geq 0}$. But we don't know an integral representation for $(x(t))_{t\geq 0}$ itself. Instead, we consider the process $(x(t) \vee \delta)_{t\geq 0}$. Applying a space transformation and change of time to the process $(x(t) \vee \delta)_{t\geq 0}$ we will see that the law of the process obtained coincides with that of the Wiener process with reflection at the point δ.

Similarly to Theorem 12.2.5 of [14] it can be shown that the existence of solution to the martingale problem for each $x_0 \geq 0$ implies the existence of a strong Markov, time homogeneous measurable Markov family $\{\tilde{P}_{x_0} : \ x_0 \in [0, \infty)\}$ such that for each $x_0 \in [0, \infty)$, $\tilde{P}_{x_0}$ is a solution to the martingale problem starting from x_0. And by Theorem 12.2.4 of [14] to prove the uniqueness of a solution to the martingale problem it is sufficient to prove the uniqueness only for the family of strong Markov, time homogeneous solutions. If $\tilde{P}_{x_0}$ is such a solution starting from x_0, then according to Remark 2 there exists a pair (x, w) on some probability space $(\Omega, \mathcal{F}, P_{x_0})$ which is a weak solution to equation (5) and $P_{x_0}x^{-1} = \tilde{P}_{x_0}$. This yields that $(x(t))_{t\geq 0}$ is a strong Markov and time homogeneous process.

Note that if the process $(x(t))_{t\geq 0}$ does not hit zero the assertion of Lemma is trivial. So from now on we suppose that starting from x_0 the process $(x(t))_{t\geq 0}$ hits zero P-a.s.

We follow the proof of Theorem 2.12 of [1]. Denote

$$\begin{aligned} \rho(x) &= \exp\left(\int_x^1 \frac{2a(y)}{\sigma^2(y)} dy\right), \ x \in (0, 1], \\ s(x) &= \begin{cases} \int_0^x \rho(y)dy & \text{if } \int_0^1 \rho(y)dy < \infty, \\ -\int_x^1 \rho(y)dy & \text{if } \int_0^1 \rho(y)dy = \infty. \end{cases} \end{aligned}$$

Let $\zeta_\delta(x(t)) = x(t) \vee \delta, \ t \geq 0$. Then by Lemma 1

$$\zeta_\delta(x(t)) = \zeta_\delta(x(0)) + \int_0^t a(x(s))\mathbb{1}_{(\delta,+\infty)}(x(s))ds + \int_0^t \sigma(x(s))\mathbb{1}_{(\delta,+\infty)}(x(s))dw(s) + \frac{1}{2}L_\delta^x(t), \ t \geq 0.$$

Set $\Delta = s(\delta), y(t) = s(x(t) \vee \delta) = s(x) \vee \Delta, \ t \geq 0$. By Itô-Tanaka formula applied to the function $x \mapsto s(x) \vee \Delta$, we have

$$y(t) = y(0) + \int_0^t \rho(x(s))\sigma(x(s))\mathbb{1}_{(\delta,+\infty)}(x(s))dM(s) + \frac{1}{2}\rho(\delta)L_\delta^x(t),$$

where $M(s) = \int_0^t \sigma(x(s))dw(s)$. Applying Itô-Tanaka formula to the function $y \mapsto y \vee \Delta$, we get

$$\begin{aligned} y(t) = y(t) \vee \Delta = y(0) + \int_0^t \rho(s^{-1}(y(s)))\sigma(x(s))\mathbb{1}_{(\Delta,+\infty)}(y(s))dM(s) + \frac{1}{2}\rho(\delta)L_\Delta^y(t) \\ = y(0) + N(t) + \frac{1}{2}\rho(\delta)L_\Delta^y(t), \end{aligned}$$

where $N(t) = \int_0^t \rho(s^{-1}(y(s)))\sigma(x(s))\mathbb{1}_{(\Delta,+\infty)}(y(s))dM(s)$.

Consider $D_t = \int_0^t 1\!\!1_{(\Delta,+\infty)}(y(s))ds$. Let us show that $D_t \to \infty$ as $t \to \infty$. Set for $a, b > 0$, $T_{a,b} = \inf\{t > 0 : x(t) = a \text{ or } b\}$. Define

$$\begin{aligned}\mu_0 &= \inf\{t > 0 : x(t) = 0\}, \text{and for } \ k = 1, 2, \dots,\\ \mu_k &= \inf\{t > 0 : t \geq \mu_{k-1} + 1, \ x(t) \geq 2\delta \text{ or } x(t) = 0\}\end{aligned}$$

If the process can hit zero in finite time then for all $y \in [0, 2\delta]$, $P_y(T_{0,2\delta} < \infty) = 1$ (cf. [1], Section 2). Then

$$\begin{aligned}(10) \quad P_0(\mu_1 < \infty) = P_0(x(1) > 2\delta) + \int_{[0,2\delta]} P_y(T_{0,2\delta} < \infty) P_0(x(1) \in dy)\\ = P_0(x(1) > 2\delta) + P_0(x(1) \in [0, 2\delta]) = 1.\end{aligned}$$

Let

$$\begin{aligned}\tau_1 &= \inf\{t \geq 0 : x(t) = 2\delta\}, \text{ for } k = 1, 2, \dots,\\ \varkappa_k &= \inf\{t > \tau_k : x(t) = \delta\}, \text{ and for } k = 2, 3, \dots,\\ \tau_k &= \inf\{t > \varkappa_{k-1} : x(t) = 2\delta\}.\end{aligned}$$

Equality (10) yields $P_0(\tau_1 < \infty) = 1$. Indeed, note that $P_0(x(\mu_1) = 0) = \alpha \in (0, 1)$. Then, by strong Markov property

$$\begin{aligned}P_0(\tau_1 = \infty) \leq P_0(\tau_1 \geq \mu_n) \leq P_0(\bigcap_{k=1}^n (x(\mu_k) = 0))\\ = (P_0(x(\mu_1) = 0))^n = \alpha^n \to 0, \text{ as } n \to \infty.\end{aligned}$$

Thus $P_0(\tau_1 < \infty) = 1$. Let for $k = 1, 2, \dots,$ $\zeta_k = \varkappa_k - \tau_k$. Then $\{\zeta_k : k \geq 1\}$ is a sequence of positive independent identically distributed random variables. Consequently, $D_\infty \geq \sum_{k=1}^n \zeta_k \to \infty$, as $n \to \infty$. So $\lim_{t\to+\infty} D_t = +\infty$

Put

$$\varphi_\delta(t) = \inf\{s \geq 0 : D(s) > t\},$$

and

$$(11) \qquad U(t) = y(\varphi_\delta(t)) = U(0) + N(\varphi_\delta(t)) + \frac{1}{2}\rho(\delta) L^y_\Delta(t), \ t \geq 0.$$

It can be seen that the process $K(t) = N(\varphi_\delta(t))$ is a martingale and

$$\langle K \rangle(t) = \int_0^t \varkappa^2(U(s))ds, \ t \geq 0,$$

where $\varkappa(x) = \rho(s^{-1}(x))\sigma(s^{-1}(x)), \ x > 0$. By Itô-Tanaka formula we have

(12)

$$U(t) = U(0) + \int_0^t 1\!\!1_{(\Delta,+\infty)}(U(s))dK(s) + \frac{1}{2}\int_0^t 1\!\!1_{(\Delta,+\infty)}(U(s))dL^y_\Delta(\varphi_\delta(t)) + \frac{1}{2}L^U_\Delta(t), \ t \geq 0.$$

Making use change of variables in Lebesgue-Stiltjes integrals and taking into account that measure $dL^y_\Delta(\varphi_\delta(t))$ increases only on the set $\{t \geq 0 : y(\varphi_\delta(t)) = \Delta\}$, we arrive at the equality

$$\int_0^t 1\!\!1_{(\Delta,+\infty)}(U(s))dL^y_\Delta(\varphi_\delta(s)) = \int_{\varphi_\delta(0)}^{\varphi_\delta(t)} 1\!\!1_{(\Delta,+\infty)}(y(s))dL^y_\Delta(s) = 0.$$

Comparing (11) with (12) we get from the uniqueness of the semimartingale decomposition of U that

$$\int_0^t 1\!\!1_{(\Delta,+\infty)}(U(s))dK(s) = K(t), \ t \geq 0,$$

and

$$U(t) = U(0) + K(t) + \frac{1}{2} L^U_\Delta(t),\ t \geq 0.$$

Consider

$$A(t) = \int_0^t \varkappa^2(U(s))ds,$$

and put

$$A(\infty) = \lim_{t\to\infty} \varkappa^2(U(s))ds,$$

$$\tau(t) = \inf\{s \geq 0 : A(s) > t\},\ 0 \leq t < A(\infty).$$

Arguing as above we arrive at the equation

$$V(t) = U(\tau(t)) = V(0) + J(t) + \frac{1}{2} L^V_\Delta(t),\ 0 \leq t < A(\infty),$$

where $J(t) = K(\tau(t)),\ 0 \leq t < A(\infty)$, and $\langle J\rangle(t) = \int_0^{\tau(t)} \varkappa^2(U(s))ds = t$. By Theorem 7.2, Ch.2 of [8] there exists a Brownian motion $(w(t))_{t\geq 0}$ (defined, possibly, on an enlarged probability space) such that J coincides with w on $[0, A(\infty))$. Skorokhod's lemma (cf. [13], Ch.VI, Lemma 2.1) and Lemma 2.3, Ch.VI of [13] allow us make the conclusion that the process V is a Brownian motion started at $V(0) = s(x(0)) \vee \Delta$, reflected at Δ. Thus the measure $P^\delta = Law(U(t) : t \geq 0) = Law(y(\varphi_\delta(t)) : t \geq 0)$ is determined uniquely and does not depend on the choice of a solution $\tilde{P}_{x_0}$. This entails that the law of the process $x(\varphi_\delta(t)) \vee \delta$ is uniquely defined. Note that item (iii) of Definition 1 provides that the process $(x(t))_{t\geq 0}$ spends zero time at the point 0 P_{x_0}-a.s. Then for each $T > 0$, $\varphi_\delta(t) \rightrightarrows t$ on $[0, T]$ and, consequently, $x(\varphi_\delta(t)) \rightrightarrows x(t)$, as $\delta \downarrow 0$ P_{x_0}-a.s. Therefore, $Law(x(t) : t \geq 0)$ is defined uniquely and does not depend on the choice of the solution $\tilde{P}_{x_0}$. Then according to Remark 2 the weak uniqueness holds for equation (5). □

Lemma 3 (pathwise uniqueness)**.** *Let the weak uniqueness hold for equation (5). Then the pathwise uniqueness holds true for (5).*

Proof. Let $(x_1(t))_{t\geq 0}, (x_2(t))_{t\geq 0}$ be processes defined on the same probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), P)$ and let each of them is a weak solution to equation (5). The idea of the proof is as follows. We will see that the process $((x_1 \vee x_2)(t))_{t\geq 0}$ is also a weak solution to equation (5).

By Theorem IV-68 of [11] we have

$$\begin{aligned}(13)\quad (\zeta_\delta(x_1) \vee \zeta_\delta(x_2))(t) &= \zeta_\delta(x_1(t)) + (\zeta_\delta(x_2(t)) - \zeta_\delta(x_1(t))^+ \\ &= \zeta_\delta(x_0) + \int_0^t \mathbb{1}_{\zeta_\delta(x_2(s))-\zeta_\delta(x_1(s))>0} d\zeta_\delta(x_2(s)) + \int_0^t \mathbb{1}_{\zeta_\delta(x_2(s))-\zeta_\delta(x_1(s))\leq 0} d\zeta_\delta(x_1(s)) \\ &\quad + \frac{1}{2} L_0^{\zeta_\delta(x_1)-\zeta_\delta(x_2)}(t),\end{aligned}$$

where $L_0^{\zeta_\delta(x_1)-\zeta_\delta(x_2)}(t)$ is a local time of the process $(\zeta_\delta(x_1(t)) - \zeta_\delta(x_2(t)))_{t\geq 0}$ at 0. Then

$$\begin{aligned}(14)\quad (\zeta_\delta(x_1) \vee \zeta_\delta(x_2))(t) &= \zeta_\delta(x_0) + \int_0^t a((x_1 \vee x_2)(s)) \mathbb{1}_{(\delta,+\infty)}((x_1 \vee x_2)(s))ds \\ &\quad + \int_0^t \sigma((x_1 \vee x_2)(s)) \mathbb{1}_{(\delta,+\infty)}((x_1 \vee x_2)(s))dw(s) \\ &\quad + \frac{1}{2} L_\delta^{x_1\vee x_2}(t) + \frac{1}{2} L_0^{\zeta_\delta(x_1)-\zeta_\delta(x_2)}(t).\end{aligned}$$

Consider the last summand in the right-hand side of (14).

The properties of the local time (cf. Theorem 69, [11], p. 214) implies that $L_0^{\zeta_\delta(x_1)-\zeta_\delta(x_2)}(\cdot)$ increases only on $\{t: \zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t))\}$. We prove that increases only on $\{t: \zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t)) = \delta\}$.

Let $q \in [0,\infty)\cap\mathbb{Q}$ be such that $\zeta_\delta(x_1(q)) > \delta, \zeta_\delta(x_2(q)) > \delta$.Define

$$\begin{aligned} a_q &= \sup\{t < q : (\zeta_\delta(x_1)\wedge\zeta_\delta(x_2))(t) = \delta\}, \\ b_q &= \inf\{t > q : (\zeta_\delta(x_1)\wedge\zeta_\delta(x_2))(t) = \delta\}, \end{aligned}$$

and $I_q = (a_q, b_q)$. Suppose $\zeta_\delta(x_1)(q) = \zeta_\delta(x_2)(q)$. Then by Theorem on homeomorphisms of flows (cf. Theorem V-46, [11]) applied to equation (6), we have $\zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t)),\ t \in [q, b_q)$. On the other hand, if there exists $r < q$ such that $\zeta_\delta(x_1(r)) \neq \zeta_\delta(x_2(r))$, by the same theorem $\zeta_\delta(x_1(q)) \neq \zeta_\delta(x_2(q))$. Thus $\zeta_\delta(x_1(q)) = \zeta_\delta(x_2(q))$ implies $\zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t)), t \in I_q$, and $(\zeta_\delta(x_1)\vee\zeta_\delta(x_2))(t) = \zeta_\delta(x_1(t)), t \in I_q$. Comparison (14) with (6) permits the conclusion that $L_0^{\zeta_\delta(x_1)-\zeta_\delta(x_2)}(t) = 0, t \in I_q$. In the case of $\zeta_\delta(x_1(q)) \neq \zeta_\delta(x_2(q))$ we get $\zeta_\delta(x_1(t)) \neq \zeta_\delta(x_2(t)),\ t \in I_q$. So for every $[\alpha,\beta] \in I_q$ there exists $\varepsilon_0 > 0$ such that $|\zeta_\delta(x_1(t)) - \zeta_\delta(x_2(t))| > \varepsilon_0, t \in [\alpha,\beta]$. Then for all $\varepsilon \in [0,\varepsilon_0),\ \mathbb{1}_{[0,\varepsilon]}|\zeta_\delta(x_2(t)) - \zeta_\delta(x_1(t))| = 0,\ t \in [\alpha,\beta]$. From Corollary 3 of [11], p.225 we obtain

$$\begin{aligned} &L_0^{\zeta_\delta(x_1)-\zeta_\delta(x_2)}(t) \\ &\quad= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}|\zeta_\delta(x_2(s)) - \zeta_\delta(x_1(s))| d\langle\zeta_\delta(x_1) - \zeta_\delta(x_2)\rangle(s) = 0,\ t \in [\alpha,\beta]. \end{aligned}$$

Therefore, $L_0^{\zeta_\delta(x_1)-\zeta_\delta(x_2)}(\cdot)$ can increases only on $\{t:\ \zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t)) = \delta\}$, i.e. on $\{t:\ (\zeta_\delta(x_1)\vee\zeta_\delta(x_2))(t) = \delta\}$.

Let $f \in C_c^2([0,\infty))$. Then there exists $\delta > 0$ such that f is constant on $[0, 2\delta]$, and we have

$$f\left((x_1 \vee x_2)(t)\right) = f\left((\zeta_\delta(x_1)\vee\zeta_\delta(x_2))(t)\right),\ 0 \le t < +\infty.$$

We have seen that the local times $L_0^{\zeta_\delta(x_1)-\zeta_\delta(x_2)}(\cdot)$ and $L_\delta^{x_1\vee x_2}(\cdot)$ do not increase on $\{t : (x_1 \vee x_2)(t) > 2\delta\}$. Taking into account that $f'(x) = f''(x) = 0$ on $[0, 2\delta]$ and making use of Itô formula we obtain

$$\begin{aligned} &f((x_1\vee x_2)(t)) = f(x_0) + \int_0^t a((x_1\vee x_2)(s)) f'((x_1\vee x_2)(s))ds \\ &+ \int_0^t \sigma((x_1\vee x_2)(s)) f'((x_1\vee x_2)(s))dw(s) + \frac{1}{2}\int_0^t \sigma^2((x_1\vee x_2)(s)) f''((x_1\vee x_2)(s))ds. \end{aligned}$$

Therefore, the process $((x_1\vee x_2)(t))_{t\ge 0}$ satisfies equation (5). By the weak uniqueness for all $t \ge 0$, $E^P(x_1\vee x_2)(t) = E^P x_1(t) = E^P x_2(t)$. This yields $(x_1\vee x_2)(t) = x_1(t) = x_2(t),\ t \ge 0$ P-a.s. □

Proof of Theorem. Let for each $x_0 \ge 0$, there exists a solution to the martingale problem $M(a,\sigma,x_0)$. Then by Lemma 2 the weak uniqueness holds for equation (5). Then the assertion of Theorem follows from Lemma 3 similarly to Yamada-Watanabe theorem (cf. Theorem IV-1.1 of [8]). □

Remark 3. The statement of the Theorem holds true if $\sigma = 0$ on some set $B \subset (0,\infty)$. Indeed, let at first the set B does not have limit points in some neighborhood of 0. Suppose $x_0 \in B$ and $a(x_0) = 0$. Then a solution of equation (5) stays at the point x_0 forever. Suppose $a(x_0) \neq 0$. If for some $y \in B$ such that $y < x_0$, $a(y) = 0$, a solution starting from x_0 never hits the point y due to homeomorphic property of solutions of SDE (see [9], Ch. 4.4). If there exists $y \in B$ such that $y \le x_0$ and $a(y) > 0$, then a solution of (5) never attends the half-interval $[0, y)$. In two last cases the assertion of the Theorem is fulfilled because a, σ are Lipshitz continuous on $[y, \infty)$ (see, for example [7]). If for all $y \in B$ the inequality $y > x_0$ holds, we need to prove the uniqueness only up to the time of hitting B by a solution. The case when for all $y \in B$ such that $y \le x_0$, we have $a(y) < 0$ is reduced to the previous one. Thus, if the set B does not have limit points in some neighborhood of 0, then a solution to equation (5) either attends a point of B just once or does not attends it or lives in it forever. The assertion of Theorem holds true in this case.

Now, suppose that 0 is a limit point of the set B. Suppose there exists a subsequence $\{y_n : n \ge 1\} \subset B$ such that $y_n \to 0$ as $n \to \infty$ and $a(y_n) \ge 0$. Then starting away from 0, a solution never hits some neighborhood of 0 and the assertion of Theorem holds true. If such a subsequence does not exist, then there is a neighborhood of 0, say U, such that for all $y \in B \cap U$, $a(y) < 0$. In this case, starting from 0 a solution does not attend the interval $(y, +\infty)$ for all $y \in B \cap U$. Thus, if 0 is a limit point of B, a solution either hits 0 in finite time with positive probability or does not hit 0 a.s. In the former case a solution stays at the point 0 forever. But this contradicts with item (iii) of Definition 1. In the latter case the assertion of Theorem is obvious.

3. Examples

Example 1 (Skorokhod equation ([7], §23)). Let a, b be functions on $[0, \infty)$. Let $(w(t))_{t \ge 0}$ be a Wiener process, $x_0 \ge 0$. Recall the definition of a solution to Skorokhod problem.

Let (x, l) be a pair of continuous processes adapted to the filtration $(\overline{\mathfrak{F}}_t^w)$ and such that

(i) x is non-negative,
(ii) $l(0) = 0, l(\cdot)$ is nondecreasing,
(iii) $l(\cdot)$ increases only at those moments of time when $x(t) = 0$, i.e. for each $t \ge 0$,

$$\int_0^t \mathbb{1}_{\{0\}}(x(s))dl(s) = l(t), \tag{15}$$

(iv) for each $t \ge 0$, the relation

$$x(t) = x_0 + \int_0^t a(x(s))ds + \int_0^t b(x(s))dw(s) + l(t) \tag{16}$$

holds and all the integrals in the right-hand side of (16) are well-defined.

Then the pair (x, l) is called a strong solution to equation (16).

If (x, l) is such a solution, then for each $f \in C_c^2([0, \infty))$, by Itô formula for semimartingales, we have

$$f(x(t)) = f(x_0) + \int_0^t f'(x(s))b(x(s))dw(s) + \int_0^t f'(x(s))a(x(s))ds + \frac{1}{2}\int_0^t f''(x(s))b^2(x(s))ds + \int_0^t f'(x(s))dl(s). \tag{17}$$

According to (15), the last member in the right-hand side of (17) is equal to 0. Thus, if the pair of the processes (x, l) is a strong solution to equation (16), then the process x is a strong solution to equation (5) in the sense of Definition 6.

Example 2 (β-dimensional Bessel processes). Let ρ be a Bessel process of dimension β. It is known (see [13], p.446) that this process has a transition probability density

$$p_t^\beta(x, y) = t^{-1}(y/x)^\nu y \exp\left(-(x^2 + y^2)/2t\right) I_\nu(xy/t) \text{ for } x > 0, t > 0,$$

and

$$p_t^\beta(0, y) = 2^{-\nu} t^{-(\nu+1)} \Gamma^{-1}(\nu + 1) y^{2\nu+1} \exp(-y^2/2t),$$

where $\nu = \beta/2 - 1$. If $0 < \beta < 2$ the point 0 is instantaneously reflecting and for $\beta \geq 2$ it is polar.

1) Let $\beta > 1$. In this case the process ρ is a semimartingale, which satisfies the SDE of the form (see [13], Ch.XI, §1).

$$\rho(t) = \rho(0) + w(t) + \frac{\beta - 1}{2} \int_0^t \frac{1}{\rho(s)} ds, \ \rho(0) \geq 0. \tag{18}$$

Cherny [2] has shown that there exists a unique non-negative strong solution to equation (18). Let ρ be a non-negative solution to (18). Applying Itô formula we get the equation

$$f(\rho(t)) = f(\rho(0)) + \int_0^t f'(\rho(s)) dw(s) + \frac{\beta - 1}{2} \int_0^t \frac{f'(\rho(s))}{\rho(s)} ds + \frac{1}{2} \int_0^t f''(\rho(s)) ds.$$

Thus, (ρ, w) is a weak solution to equation (5) in the sense of Definition 3. Then according to Theorem there exists a strong solution to (5) and the strong uniqueness holds. Therefore we obtain the result of Cherny from ours.

2) Let $0 < \beta < 1$ and let $(\rho(t))_{t\geq 0}$ be a Bessel process on some probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), P)$. Then the process ρ is not a semimartingale. Nonetheless it has the family of local times defined by the formula

$$\int_0^t \phi(\rho(s)) ds = \int_0^\infty \phi(x) L_x^\rho(t) x^{\beta - 1} dx. \tag{19}$$

valid for all $t > 0$ and for every positive measurable function ϕ on $[0, \infty)$ a.s. The Bessel process of dimension β is a weak solution to the following equation (cf. [13], Ch.XI, Ex. 1.26)

$$\rho(t) = \rho(0) + w(t) + \frac{\beta - 1}{2} k(t), \ \rho(0) \geq 0, \tag{20}$$

where $(w(t))_{t \geq 0}$ is an $(\mathfrak{F}_t)$-Wiener process, $k(t) = V.P. \int_0^t \rho^{-1}(s) ds$ which, by definition, is equal to $\int_0^\infty a^{\beta - 2}(L_a^\rho(t) - L_0^\rho(t)) da$.

Let us check that the pair (ρ, w) is a weak solution to equation (5) in the sense of Definition 3. Then the Theorem yields that there exists a unique strong solution to equation (20). To prove this we need the following Lemma.

Lemma 4. *Let $t_1, t_2 \in \mathbb{Q}, \ t_1 < t_2$. Then for almost all $\omega \in \Omega$ such that $\rho(t, \omega) > 0, \ t \in [t_1, t_2]$, the equality*

$$k(t_2) - k(t_1) = \int_{t_1}^{t_2} \frac{1}{\rho(s)} ds \tag{21}$$

holds.

Proof. There exists $\varepsilon > 0$ such that $\rho(t) \geq \varepsilon, \ t \in [t_1, t_2]$. The properties of the local time imply that for all $a < \varepsilon$, $L_a^\rho(t_2) = L_a^\rho(t_1)$. Then

$$\begin{aligned} k(t_2) - k(t_1) = \int_0^\infty a^{\beta - 2} \left[(L_a^\rho(t_2) - L_0^\rho(t_2)) - (L_a^\rho(t_1) - L_0^\rho(t_1)) \right] da \\ = \int_0^\infty \frac{\mathbb{1}_{[\varepsilon, \infty)}(a)}{a} a^{\beta - 1} (L_a^\rho(t_2) - L_a^\rho(t_1)) da. \end{aligned}$$

By (19) we get

$$k(t_2) - k(t_1) = \int_{t_1}^{t_2} \mathbb{1}_{[\varepsilon, \infty)}(\rho(s)) \frac{1}{\rho(s)} ds = \int_{t_1}^{t_2} \frac{1}{\rho(s)} ds.$$

$\square$

Because of the continuity of the process $(\rho(t))_{t\geq 0}$ there exists $\tilde{\Omega}\in\Omega,\ P(\tilde{\Omega})=1$, such that for all $\omega\in\tilde{\Omega}$, formula (21) holds true for all $t_1,t_2\geq 0$ satisfying $\rho(t)>0, t\in[t_1,t_2]$.

Let τ be a stopping time such that $\rho(\tau)\neq 0$ P-a.s. Put $\sigma=\inf\{s\geq\tau:\ \rho(s)=0\}$. We have

$$\rho(t)=\rho(\tau)+w(t)-w(\tau)+\frac{\beta-1}{2}\int_\tau^t\frac{ds}{\rho(s)},\ t\in[\tau,\sigma).$$

Let $f\in C_c^2([0,\infty))$. Itô formula for semimartingales yields

(22)
$$f(\rho(t))-f(\rho(\tau))=\int_\tau^t f'(\rho(s))dw(s)+\frac{\beta-1}{2}\int_\tau^t\frac{f'(\rho(s))}{\rho(s)}ds+\frac{1}{2}\int_\tau^t f''(\rho(s))ds,\ t\in[\tau,\sigma).$$

Choose $\delta>0$ such that f is constant on $[0,2\delta]$. Define

$$\begin{aligned}\tau_0&=0,\\ \text{for } i\geq 0,\ \varkappa_i&=\inf\{t>\tau_i:\rho(t)=\delta/2\},\\ \text{and for } i\geq 1,\ \tau_i&=\inf\{t>\varkappa_{i-1}:\rho(t)=\delta\},\end{aligned}$$

Then

$$f(\rho(t))=f(\rho(0))+\sum_{k=0}^\infty[f(\rho(\varkappa_k\wedge t))-f(\rho(\tau_k\wedge t))]+\sum_{k=0}^\infty[f(\rho(\tau_{k+1}\wedge t))-f(\rho(\varkappa_k\wedge t))].$$

The second sum in the right-hand side is equal to zero. If $f(\rho(0))<\delta/2$, then $f(\rho(\varkappa_0\wedge t))-f(\rho(\tau_0))=0$.

Suppose $\rho(0)\geq\delta/2$. It follows from (22) that

$$\begin{aligned}(23)\quad f(\rho(t))=f(\rho(0))+\sum_{k=0}^\infty\int_{\tau_k\wedge t}^{\varkappa_k\wedge t}f'(\rho(s))dw(s)+\frac{\beta-1}{2}\sum_{k=0}^\infty\int_{\tau_k\wedge t}^{\varkappa_k\wedge t}\frac{f'(\rho(s))}{\rho(s)}ds\\ +\frac{1}{2}\sum_{k=0}^\infty\int_{\tau_k\wedge t}^{\varkappa_k\wedge t}f''(\rho(s))ds,\ t\geq 0.\end{aligned}$$

Note that for all $t\in[\varkappa_k,\tau_{k+1}],\ k\geq 0,\ f'(\rho(t))=f''(\rho(t))=0$. Then (23) can be rewritten in the form

(24)
$$f(\rho(t))=f(\rho(0))+\int_0^t f'(\rho(s))dw(s)+\frac{\beta-1}{2}\int_0^t\frac{f'(\rho(s))}{\rho(s)}ds+\frac{1}{2}\int_0^t f''(\rho(s))ds,\ t\geq 0.$$

If $\rho(0)<\delta/2$ equation (24) can be obtained similarly.

Hence, the pair (ρ,w) is a weak solution to equation (5) and, consequently, the strong existence and uniqueness hold for equation (20).

Example 3. Let the process $(x(t))_{t\geq 0}$ be a weak solution to an SDE of the form

$$(25)\qquad x(t)=x(0)+\int_0^t a(|x(s)|)ds+\int_0^t b(|x(s)|)dw(s),$$

where the coefficients a and b are locally Lipshitz continuous on $(0,\infty)$.

Then for each even function being constant in a neighborhood of zero according to Itô formula, we get

$$f(x(t)) = f(x(0)) + \int_0^t a(|x(s)|)f'(x(s))ds + \int_0^t b(|x(s)|)f'(x(s))dw(s) + \frac{1}{2}\int_0^t b^2(|x(s)|)f''(x(s))ds. \tag{26}$$

Note that if the process $(x(t))_{t\geq 0}$ is a weak solution to equation (25) spending zero time at the origin then the process $y(t) = |x(t)|,\ t \geq 0$, satisfies equality (26) for each $f \in C_c^2([0,+\infty))$. By Theorem 1 the process $y(t),\ t \geq 0$, is a unique non-negative strong solution to (26) spending zero time at the origin.

Consider an SDE which can be regarded as an example of equation of the form (25)

$$x(t) = x(0) + \int_0^t |x(s)|^\alpha dw(s),\ \alpha \in (0, 1/2). \tag{27}$$

It is known that there exists a weak solution to (27) spending zero time at the point 0 (cf. [10], 3.10b).

Remark 4. Girsanov [6] has shown that without additional assumption this equation has infinitely many weak solutions.

Remark 5. It can be proved (cf. [3]) that in the class of solutions spending zero time at the point 0 the pathwise uniqueness holds and a strong solution exists.

So, there exists a weak solution to the equation

$$f(x(t)) = f(x(0)) + \int_0^t (x(s))^\alpha f'(x(s))dw(s) + \frac{1}{2}\int_0^t (x(t))^{2\alpha} f''(x(s))ds \tag{28}$$

in the sense of Definition 3 spending zero time at the point 0. According to Theorem 1 there is a strong solution to (28) spending zero time at the point 0. Certainly, this solution coincides with the unique strong solution to the equation

$$x(t) = x(0) + \int_0^t (x(s))^\alpha dw(s) + dL_0^x(t),\ \alpha \in (0, 1/2),$$

spending zero time at the point 0 which was constructed by Bass and Chen (see [4]). Here $(L_0^x(t))_{t\geq 0}$ is a local time of the process $(x(t))_{t\geq 0}$ at the point 0 defined by formula (7).

Appendix

Proof of assertion of Remark 2. The "only if" assertion is trivial.

To prove the "if" assertion we can argue as in Prop.2.1, Ch.IV of [8]. Suppose P is a solution to the martingale problem $M(a, \sigma, x_0)$ on space $(C^+([0,+\infty)), \mathfrak{G}), (\mathfrak{G}_t))$, and $f \in C_c^2([0,\infty))$. Then the process $Y_f(t)$ is a continuous, square integrable local martingale with respect to P. Applying condition (ii) of Definition 1 to the function f^2, we calculate the characteristics of the process $(Y_f(t))_{t\geq 0}$. Namely,

$$\langle Y_f\rangle(t) = \int_0^t \sigma^2(x(s))(f'(x(s)))^2 ds.$$

Consequently, there is a Brownian motion $(w_f(t))_{t\geq 0}$ defined on an extension of $(C^+([0,\infty)), \mathfrak{G}, (\mathfrak{G}_t), P)$ such that

$$Y_f(t) = \int_0^t \sigma(x(s))f'(x(s))dw_f(s).$$

We will show that it can be chosen the same Brownian motion for all $f \in C_c^2([0,\infty))$.

Similarly to the Proof of Lemma 1, for $k = 1, 2, \dots$, consider a non-decreasing function $\eta_k \in C_c^2([0,\infty))$ such that $\eta_k(x) = x,\ x > 1/k$, and η_k is a constant on $[0, \frac{1}{2k}]$.

Let us fix k and put

$$\tau_l = \inf\{t : \eta_k(x(t)) > l\},\ l = 1, 2, \dots.$$

Then, for all $l = 1, 2, \dots$,

$$\eta_k(x(t \wedge \tau_l)) - \eta_k(x_0) - \int_0^{t\wedge\tau_l} \left[a(x(s))\eta_k'(x(s)) + \frac{1}{2}\sigma^2(x(s))\eta_k''(x(s)) \right] ds$$

is a continuous, square integrable P-martingale. Then $Y_{\eta_k}(t) \in \mathcal{M}_2^{c,loc}(P)$, and there exists a Brownian motion $(w_k(t))_{t\geq 0}$ on an extension $(\Omega_k, \mathfrak{F}_k, P_k)$ of $(C^+([0,\infty)), \mathfrak{G}, (\mathfrak{G}_t), P)$ such that

$$\eta_k(x(t)) = \eta_k(x_0) + \int_0^t \left(a(x(s))\eta_k'(x(s)) + \frac{1}{2}\sigma^2(x(s))\eta_k''(x(s)) \right) ds + \int_0^t \sigma(x(s))\eta_k'(x(s)) dw_k(s). \tag{29}$$

Fix $m \geq 1$. Then for all $k \geq m$, $\eta_m(x) = \eta_k(x),\ x > 1/m$. Put

$$\tilde{w}_m(t) := \int_0^t \mathbb{1}_{(\frac{1}{m}, +\infty)}(x(s)) dw_m(s). \tag{30}$$

As a consequence of the following simple Lemma we have that for each $m \geq 1$ the process $(\tilde{w}_m(t))_{t\geq 0}$ is adapted w.r.t. the filtration generated by the process $(x(t))_{t\geq 0}$ and for all $k \geq m$,

$$\int_0^t \mathbb{1}_{(\frac{1}{m}, +\infty)}(x(s)) dw_k(s) = \int_0^t \mathbb{1}_{(\frac{1}{m}, +\infty)}(x(s)) dw_m(s) = \int_0^t \mathbb{1}_{(\frac{1}{m}, +\infty)}(x(s)) d\tilde{w}_m(s) \quad \text{a.s.} \tag{31}$$

Lemma 5. *Let A be an open set in $\mathbb{R}$. Let $x_0 \in A$, $(x(t))_{t\geq 0}$ be a continuous adapted process on a probability space $(\Omega, \mathcal{F}_t, P)$. Let $(w(t))_{t\geq 0}$ be a Wiener process on some extension of the space $(\Omega, \mathcal{F}_t, P)$. Suppose a, b, f are continuous functions on $\mathbb{R}$, $b(x) \neq 0$ for $x \in A$, and for all $t \geq 0$, the equality*

$$f(x(t)) = f(x_0) + \int_0^t a(x(s)) ds + \int_0^t b(x(s)) dw(s)$$

holds. Put $\mathcal{F}_t^x = \sigma\{x(s) : 0 \leq s \leq t\}$. Then the process $\int_0^t \mathbb{1}_A(x(s)) dw(s), t \geq 0$, is adapted w.r.t. $(\mathcal{F}_t^x)$.

Moreover, suppose $(\bar{w}(t))_{t\geq 0}$ is a Wiener process on an extension of the probability space $(\Omega, \mathcal{F}_t, P)$, $\bar{a}, \bar{b}, \bar{f}$ are continuous functions on $\mathbb{R}$, $\bar{b}(x) \neq 0$ for $x \in A$, and the equality

$$\bar{f}(x(t)) = \bar{f}(x_0) + \int_0^t \bar{a}(x(s)) ds + \int_0^t \bar{b}(x(s)) d\bar{w}(s)$$

holds.

If $a(x) = \bar{a}(x), b(x) = \bar{b}(x), f(x) = \bar{f}(x)$ on A, then

$$\int_0^t \mathbb{1}_A(x(s)) dw(s) = \int_0^t \mathbb{1}_A(x(s)) d\bar{w}(s). \tag{32}$$

The proof is trivial.

The sequence $\{\tilde{w}_m : m \geq 1\}$ defined in (30) is fundamental in mean square on compact intervals. Indeed, for $k \geq m, T > 0$, using martingale inequality (cf. [8], Theorem I-6.10) and (31), we get

$$E\left[\sup_{t\in[0,T]}|\tilde{w}_k(t)-\tilde{w}_m(t)|\right]^2 \le 4E|\tilde{w}_k(T)-\tilde{w}_m(T)|^2 =$$

$$4E\left[\int_0^T \left(\mathbb{1}_{(\frac{1}{k},+\infty)} - \mathbb{1}_{(\frac{1}{m},+\infty)}\right)(x(s))dw_k(s)\right]^2 \le 4E\int_0^T \mathbb{1}_{(\frac{1}{k},\frac{1}{m}]}(x(s))ds \to 0, m\to+\infty.$$

Then the sequence $\{\tilde{w}_m : \ m\ge 1\}$ is uniformly convergent on compact intervals in probability. Denote the limit of the sequence $\{\tilde{w}_m : \ m\ge 1\}$ by $\tilde{w}$.

The process $(\tilde{w}(t))_{t\ge0}\in\mathcal{M}_2^{c,loc}(P)$ and

$$\langle\tilde{w}(t)\rangle(t) = \int_0^t \mathbb{1}_{(0,\infty)}(x(s))ds = t.$$

Here we used the fact that the process $(x(t))_{t\ge0}$ spends zero time at the point 0. Thus the process $(\tilde{w}(t))_{t\ge0}$ is a Wiener process. Besides, by construction,

$$\tilde{w}_k(t) = \int_0^t \mathbb{1}_{(\frac{1}{k},+\infty)}(x(s))d\tilde{w}(s).$$

Let $f\in C_c^2([0,\infty))$ be such that f is constant on $[0,1/k]$. Then there exists a Wiener process $(w_f(t))_{t\ge0}$ such that

$$f(x(t)) = f(x_0) + \int_0^t \left(a(x(s))f'(x(s)) + \frac{1}{2}\sigma^2(x(s))f''(x(s))\right)ds + \int_0^t \sigma(x(s))f'(x(s))dw_f(s). \tag{33}$$

By Itô formula, (29) yields

$$\begin{aligned} f(\eta_k(x(t))) = f(\eta_k(x_0)) &+ \int_0^t a(x(s))\eta_k'(x(s))f'(\eta_k(x(s)))ds \\ &+ \frac{1}{2}\int_0^t \sigma^2(x(s))\eta_k''(x(s))f'(\eta_k(x(s)))ds \\ + \int_0^t \sigma(x(s))\eta_k'(x(s))f'(\eta_k(x(s)))dw_k(s) &+ \frac{1}{2}\int_0^t \sigma^2(x(s))(\eta_k'(x(s)))^2 f''(\eta_k(x(s)))ds. \end{aligned} \tag{34}$$

The second integral in the right-hand side of (34) is equal to 0 because $\eta_k''(x)=0$ on $(1/k,+\infty)$. Taking into account that $\eta_k'(x)=x'=1$ on $(1/k,+\infty)$, and $f(\eta_k(x))=f(x)$ on $(1/k,+\infty)$, we arrive at the equation

$$f(\eta_k(x(t))) = f(\eta_k(x_0)) + \int_0^t a(x(s))f'(x(s))ds + \int_0^t \sigma(x(s))f'(x(s))dw_k(s) + \frac{1}{2}\int_0^t \sigma^2(x(s))f''(x(s))ds. \tag{35}$$

Note that

$$\int_0^t \sigma(x(s))f'(x(s))dw_k(s) = \int_0^t \sigma(x(s))f'(x(s))\mathbb{1}_{\{\sigma(x(s))f'(x(s))\ne0\}}dw_k(s).$$

Applying Lemma 5 to equations (33) and (35) we have

$$\int_0^t \mathbb{1}_{(1/k,+\infty)}(x(s))\mathbb{1}_{\{\sigma(x(s))f'(x(s))\neq 0\}}dw_f(s)$$
$$= \int_0^t \mathbb{1}_{(1/k,+\infty)}(x(s))\mathbb{1}_{\{\sigma(x(s))f'(x(s))\neq 0\}}dw_k(s)$$
$$= \int_0^t \mathbb{1}_{(1/k,+\infty)}(x(s))\mathbb{1}_{\{\sigma(x(s))f'(x(s))\neq 0\}}d\tilde{w}(s).$$

So for each $f \in C_c^2([0,+\infty))$ the equality

$$f(x(t)) = f(x_0) + \int_0^t \left(a(x(s))f'(x(s)) + \frac{1}{2}\sigma^2(x(s))f''(x(s)) \right) ds$$
$$+ \int_0^t \sigma(x(s))f'(x(s))d\tilde{w}(s)$$

is justified, and the pair $(x(t), \tilde{w}(t))_{t\geq 0}$ is a weak solution to equation (5). $\square$

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