OLGA V. ARYASOVA AND ANDREY YU. PILIPENKO

ON THE STRONG UNIQUENESS OF A SOLUTION TO SINGULAR STOCHASTIC DIFFERENTIAL EQUATIONS

We prove the existence and uniqueness of a strong solution for an SDE on a semi-axis with singularities at the point 0. The result obtained yields, for example, the strong uniqueness of non-negative solutions to SDEs governing Bessel processes.

INTRODUCTION

We consider a stochastic process the state space of which is a non-negative semi-axis. Assume that up to the first hitting time of zero the process \( (x(t))_{t \geq 0} \) satisfies an SDE

\[
x(t) = x_0 + \int_0^t a(x(s))ds + \int_0^t \sigma(x(s))dw(s),
\]

where \( x_0 \geq 0, a, \sigma \) are supposed to be locally Lipshitz continuous on \((0, \infty), (w(t))_{t \geq 0}\) is a Wiener process. Possible singularities of the coefficients generate different types of behavior of the process in a neighborhood of zero. As a consequence, the integral representation of \((x(t))_{t \geq 0}\) may acquire various forms.

As an example let us consider the following SDE

\[
\rho(t) = \rho(0) + w(t) + \frac{\beta - 1}{2} \int_0^t \frac{1}{\rho(s)}ds, \quad \rho(0) \geq 0.
\]

It is known that \( \beta \)-dimensional Bessel process with \( \beta > 1 \) is a unique non-negative strong solution to (1) (cf. [2]). Note that this equation possesses no additional terms. Otherwise, an additional summand can be represented by the local time \((l(t))_{t \geq 0}\) of unknown process \((x(t))_{t \geq 0}\) at the point 0 like in Skorokhod equation

\[
x(t) = x_0 + \int_0^t a(x(s))ds + \int_0^t b(x(s))dw(s) + l(t),
\]

or by principal values of some functionals of the unknown process like in the following representation for a \( \beta \)-dimensional Bessel process with \( \beta \in (0, 1) \) (cf. [13], Ch. XI)

\[
\rho(t) = \rho(0) + w(t) + \frac{\beta - 1}{2} k(t), \quad \rho(0) \geq 0.
\]

Here \( k(t) = V.P. \int_0^t \rho^{-1}(s)ds \) which, by definition, is equal to \( \int_0^\infty a^{\beta - 2}(L^\rho(t) - L^\rho_0(t))da \), \( L^\rho_0(t) \) is a local time of the process \( \rho(t) \) at the point \( a \).

It seems improbable to describe all possible forms of integral representations. Let \( f \) be a twice continuously differentiable function on \([0, \infty)\) which is a constant in a neighborhood of zero. Applying Itô formula (additional tricks are needed in some cases) we
see that the stochastic differential of the function $f(x(t))$ has identical form for solutions of equations (1)-(3). Namely,

$$(4) \quad f(x(t)) = f(x_0) + \int_0^t \left( a(x(s))f'(x(s)) + \frac{1}{2} \sigma^2(x(s))f''(x(s)) \right) ds + \int_0^t \sigma(x(s))f'(x(s)) dw(s).$$

The differential has no singularities. Intuitively, the singularities at the point 0 are killed by zero derivatives of the function $f$. We use this fact to formulate the problem as an analogue of a martingale problem (see Section 1). The main result of this paper is as follows: the existence of a weak non-negative solution for equation (4) spending zero time at the point 0 implies the existence and uniqueness of a non-negative strong solution spending zero time at 0. The pathwise uniqueness is obtained by method of Le Gall [5] based on the fact that the maximum of two solution also solves the equation.

The formulation of a martingale problem involving a class of functions that are constant in a neighborhood of possible singularities was used by many authors (see, for example, [12], [15]).

The notations and definitions used are collected in Section 1. We prove the main Theorem in Section 2. In Section 3 some examples are represented.

1. Notations and definitions

Let $a, \sigma$ be real-valued Borel-measurable functions defined on $[0, \infty)$. From now on we assume that the following condition is valid

**Condition A.** Suppose that the functions $a$ and $\sigma$ are locally Lipschitz continuous on $(0, \infty)$, i.e. for each $\varepsilon > 0$ there exist constants $C_\varepsilon > 0$ such that for all $\{x, y\} \subset [\varepsilon, \infty)$

$$|a(x) - a(y)| + |\sigma(x) - \sigma(y)| \leq C_\varepsilon |x - y|.$$  

The set of continuous functions $x : [0, \infty) \rightarrow [0, \infty)$ is denoted by $C^+([0, \infty))$. Let $C^+_t \equiv \sigma(x(s) : 0 \leq s \leq t, \ x \in C^+([0, +\infty)))$, and $G \equiv \sigma(x(s) : 0 \leq s < \infty, \ x \in C^+([0, +\infty)))$ be $\sigma$-algebras on $C^+([0, +\infty))$. The set of real-valued functions which are twice continuously differentiable on $[0, \infty)$ and constant in a neighborhood of zero is denoted by $C^+_2([0, +\infty))$. Given a probability measure $P$ on $(C^+([0, +\infty)), G)$, the family of continuous, square integrable local $G_t$-martingales is denoted by $\mathcal{M}^{0, \text{loc}}_2(P)$.

**Definition 1.** Given $x_0 \geq 0$, a solution to the martingale problem $M(a, \sigma, x_0)$ is a probability measure $P_{x_0}$ on $(C^+([0, +\infty)), G)$ such that

1. $P_{x_0}(x(0) = x_0) = 1$.
2. For each $f \in C^+_2([0, +\infty))$,  

$$Y_f(t) = f(x(t)) - f(x_0) - \int_0^t \left[ a(x(s))f'(x(s)) + \frac{1}{2} \sigma^2(x(s))f''(x(s)) \right] ds \in \mathcal{M}^{0, \text{loc}}_2(P_{x_0}).$$
3. $E^{P_{x_0}} \int_0^\infty 1_{[0)}(x(s)) ds = 0$.

**Definition 2.** The martingale problem is well-posed if for each $x_0 \geq 0$ there is exactly one solution to the martingale problem starting from $x_0$.

**Definition 3.** Given $x_0 \geq 0$, let a pair $(x(t), w(t))_{t \geq 0}$ of continuous adapted processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be such that

1. $(w(t))_{t \geq 0}$ is a standard $(\mathcal{F}_t)$-Brownian motion,
2. the process $(x(t))_{t \geq 0}$ takes values on $[0, \infty)$,
(iii) for each $t \geq 0$, and $f \in C^2_b([0, \infty))$, the equality

\begin{equation}
(5) \quad f(x(t)) = f(x_0) + \int_0^t \left( a(x(s))f'(x(s)) + \frac{1}{2} \sigma^2(x(s))f''(x(s)) \right) ds \\
+ \int_0^t \sigma(x(s))f'(x(s))dw(s)
\end{equation}

holds true $P$-a.s.

Then the pair $(x, w)$ is called a weak solution to equation (5) with initial condition $x_0$.

**Remark 1.** Let $f \in C^2_b([0, \infty))$. Then there exists $\delta_f > 0$ such that $f' = f'' = 0$ on $[0, \delta_f]$. This and Condition A ensure the existence of all the integrals on the right-hand side of (5).

**Remark 2.** It is not hard to verify that the existence of a weak solution $(x(t), w(t))_{t \geq 0}$ to equation (5) on a probability space $(\Omega, \mathcal{F}, P)$ with initial condition $x_0$ is equivalent to the existence of a probability measure $P$ on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ satisfying conditions (i) and (ii) of Definition 1. The process $(x(t))_{t \geq 0}$ induces the measure $P$ on $(C^+([0, \infty)), \mathcal{F})$, namely $P = P x^{-1}$.

For the proof see Appendix.

**Definition 4.** The weak uniqueness holds for equation (5) if, for any two weak solutions $(x, w)$ and $(\tilde{x}, \tilde{w})$ (which may be defined on different probability spaces) with a common initial value, i.e. $x(0) = \tilde{x}(0) = x_0 P$-a.s., the laws of processes $x$ and $\tilde{x}$ coincide.

**Definition 5.** The pathwise uniqueness holds for equation (5), if for any two weak solutions $(x, w)$ and $(\tilde{x}, \tilde{w})$ on the same probability space $(\Omega, \mathcal{F}, P)$ with common Brownian motion and common initial value, i.e. $x(0) = \tilde{x}(0) = x_0$ $P$-a.s., the equality $x(t) = \tilde{x}(t), t \geq 0$, fulfils $P$-a.s.

Denote by $(\mathcal{F}_t^x)$ the filtration of $w$ completed with respect to $P$.

**Definition 6.** Given a process $(w(t))_{t \geq 0}$, and $x_0 \geq 0$, we say that the process $(x(t))_{t \geq 0}$ is a strong solution to equation (5) with initial condition $x_0$ if it is adapted to the filtration $(\mathcal{F}_t^x)$ and conditions (i)-(iii) of Definition 3 hold.

**Definition 7.** The strong uniqueness holds for equation (5) if there exists a strong solution to equation (5) and the pathwise uniqueness is valid for equation (5).

2. The main result

Unfortunately, we are not able to write equation (5) for the process $(x(t))_{t \geq 0}$ itself because the function $f(x) = x$ does not belong to $C^2_b([0, \infty))$. Instead, in the next Lemma we obtain an SDE for the process $\zeta_\delta(x(t)) = x(t) \vee \delta$, $t \geq 0$, which will often be used in the sequel.

**Lemma 1.** Given $\delta > 0$, put $\zeta_\delta(x) = x \vee \delta$, $x \in [0, \infty)$. Suppose $(x(t))_{t \geq 0}$ is a weak solution to equation (5). Then the equality

\begin{equation}
(6) \quad \zeta_\delta(x(t)) = \zeta_\delta(x(0)) + \int_0^t a(x(s))1_{(\delta, \infty)}(x(s))ds \\
+ \int_0^t \sigma(x(s))1_{(\delta, \infty)}(x(s))dw(s) + \frac{1}{2} L^x_\delta(t)
\end{equation}

holds true $P$-a.s.
is valid for all $t \geq 0$. Here $(L_{\delta}^n(t))_{t \geq 0}$ is a local time of the process $(x(t))_{t \geq 0}$ at the point $\delta$ defined by the formula

\begin{equation}
L_{\delta}^n(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[\delta, \delta+\varepsilon]}(x(s)) ds, \quad t \geq 0.
\end{equation}

**Proof.** We make use of a standard approximation of non-smooth function by smooth ones like the construction in the proof of Tanaka's formula (see, for example, Theorem 4.1, Ch. 3 of [8]). For $a > 0$, let us approximate the function $\xi_a(x) = x \vee a$ by twice continuously differentiable functions. Put

$$
\psi(x) = \begin{cases} 
C \exp \left( \frac{(x-1)^2}{2(x-2)} \right), & 0 < x < 2, \\
0, & \text{otherwise}, 
\end{cases}
$$

where $C$ is a constant such that $\int_{-\infty}^{\infty} \psi(x) dx = 1$, and define, for $n \geq 1$

$$
u_n(x) = \int_{-\infty}^x dy \int_{-\infty}^y n\psi(nz) dz.
$$

Then the function $\nu_n$ is twice continuously differentiable and $\nu_n(x-a) \to \xi_a(x)$, as $n \to \infty$. Further,

$$
\nu'_n(x-a) \to \mathbf{1}_{(a, \infty)}(x), \quad n \to \infty,
$$

and

\begin{equation}
\int_{-\infty}^{\infty} \nu''_n(x) \varphi(x) dx \to \varphi(a), \quad n \to \infty
\end{equation}

for any continuous and bounded function $\varphi$.

Let $\eta_t \in C^2([0, \infty))$ be a non-decreasing and such that $\eta_t(x) = x$ on $[\delta/2, \infty)$. The process $(\eta_t(x(t)))_{t \geq 0}$ can be represented in the form (5), and thus it is a semimartingale. By Itô formula we have

\begin{equation}
u_n(\eta_n(x(t)) - \delta) + \delta = \nu_n(\eta_n(x(0)) - \delta) + \delta
+ \int_0^t \left[ a(x(s)) \eta_n'(x(s)) + \frac{1}{2} \sigma^2(x(s)) \eta_n''(x(s)) \right] \nu'_n(\eta_n(x(s)) - \delta) ds
+ \int_0^t \sigma(x(s)) \eta_n'(x(s)) \nu'_n(\eta_n(x(s)) - \delta) dw(s)
+ \frac{1}{2} \int_0^t \sigma^2(x(s))(\eta_n'(x(s)))^2 \nu''_n(\eta_n(x(s)) - \delta) ds.
\end{equation}

By occupation times formula (cf. [11], Corollary 1, p. 216), the last integral on the right-hand side of (9) is equal to

$$
\int_0^t \nu''_n(\eta_n(x(s)) - \delta) d(\eta_n(x))(s) = \int_{-\infty}^{+\infty} \nu''_n(a-\delta) L_{\delta}^{n_0}(a) da \to L_{\delta}^{\eta_0}(x)(t),
$$

as $n \to \infty$.

Here $L_{\delta}^{\eta_0}(x)(t)$ is a local time of the process $(\eta_n(x(t)))_{t \geq 0}$ at the point $\delta$.

Note that $\zeta_0(x) = x \vee \delta = \eta_0(x) \vee \delta$, $x \in [0, \infty)$. Passing to the limit in (9) as $n \to \infty$ and taking into account that $\eta_n(x) = x$, $x > \delta/2$, we arrive at the equation (6). \(\square\)

The main result of the paper is the following theorem.

**Theorem 1.** Suppose $a, \sigma$ satisfy Condition A and $\sigma(x) \neq 0, \ x \geq 0$. If for each $x_0 \geq 0$ there exists a solution to the martingale problem $M(a, \sigma, x_0)$, then for each $x_0 \geq 0$ there exists a strong solution to equation (5) with initial condition $x_0$ spending zero time at the point 0 and the strong uniqueness holds in the class of solutions spending zero time at 0.
We split the proof of Theorem into two steps. At the first one we show that the existence of a solution to the martingale problem provides well-posedness. At the second one the pathwise uniqueness is obtained from weak uniqueness. These two steps are formulated as Lemmas in the following way.

**Lemma 2** (weak uniqueness). Suppose \(a, \sigma\) satisfy the conditions of Theorem 1. Let for each \(x_0 \geq 0\), there exists a solution to martingale problem \(M(a, \sigma, x_0)\). Then the weak uniqueness holds for equation (5).

**Proof.** We would like to get a law of the process \((x(t))_{t \geq 0}\). But we don’t know an integral representation for \((x(t))_{t \geq 0}\) itself. Instead, we consider the process \((x(t) \lor \delta)_{t \geq 0}\). Applying a space transformation and change of time to the process \((x(t) \lor \delta)_{t \geq 0}\) we will see that the law of the process obtained coincides with that of the Wiener process with reflection at the point \(\delta\).

Similarly to Theorem 12.2.5 of [14] it can be shown that the existence of solution to the martingale problem for each \(x_0 \geq 0\) implies the existence of a strong Markov, time homogeneous measurable Markov family \(\{P_{x_0} : x_0 \in [0, \infty]\}\) such that for each \(x_0 \in [0, \infty)\), \(\tilde{P}_{x_0}\) is a solution to the martingale problem starting from \(x_0\). And by Theorem 12.2.4 of [14] to prove the uniqueness of a solution to the martingale problem it is sufficient to prove the uniqueness only for the family of strong Markov, time homogeneous solutions. If \(\tilde{P}_{x_0}\) is such a solution starting from \(x_0\), then according to Remark 2 there exists a pair \((x, w)\) on some probability space \((\Omega, \mathcal{F}, P_{x_0})\) which is a weak solution to equation (5) and \(P_{x_0} x^{-1} = \tilde{P}_{x_0}\). This yields that \((x(t))_{t \geq 0}\) is a strong Markov and time homogeneous process.

Note that if the process \((x(t))_{t \geq 0}\) does not hit zero the assertion of Lemma is trivial. So from now on we suppose that starting from \(x_0\) the process \((x(t))_{t \geq 0}\) hits zero \(P\text{-a.s.}\).

We follow the proof of Theorem 2.12 of [1]. Denote
\[
\rho(x) = \exp \left( \int_x^1 \frac{2a(y)}{\sigma^2(y)} dy \right), \quad x \in (0, 1],
\]
\[
s(x) = \begin{cases} \int_0^t \rho(y)dy & \text{if } \int_0^1 \rho(y)dy < \infty, \\ -\int_x^1 \rho(y)dy & \text{if } \int_0^1 \rho(y)dy = \infty. \end{cases}
\]

Let \(\zeta_t(x(t)) = x(t) \lor \delta, \quad t \geq 0\). Then by Lemma 1
\[
\zeta_t(x(t)) = \zeta_t(x(0)) + \int_0^t a(x(s))1_{(x, \infty)}(x(s))ds \\
+ \int_0^t \sigma(x(s))1_{(x, \infty)}(x(s))dw(s) + \frac{1}{2} L_{\delta}^2(t), \quad t \geq 0.
\]

Set \(\Delta = s(\delta), y(t) = s(x(t) \lor \delta) = s(x) \lor \Delta, \quad t \geq 0\). By Itô-Tanaka formula applied to the function \(x \mapsto s(x) \lor \Delta\), we have
\[
y(t) = y(0) + \int_0^t \rho(x(s))\sigma(x(s))1_{(x, \infty)}(x(s))dM(s) + \frac{1}{2} \rho(\delta)L_{\Delta}^2(t),
\]
where \(M(s) = \int_0^s \sigma(x(s))dw(s)\). Applying Itô-Tanaka formula to the function \(y \mapsto y \lor \Delta\), we get
\[
y(t) = y(t) \lor \Delta = y(0) + \int_0^t \rho(s^{-1}(y(s)))\sigma(x(s))1_{(\Delta, \infty)}(y(s))dM(s) + \frac{1}{2} \rho(\delta)L_{\Delta}^2(t) \\
= y(0) + N(t) + \frac{1}{2} \rho(\delta)L_{\Delta}^2(t),
\]
where \(N(t) = \int_0^t \rho(s^{-1}(y(s)))\sigma(x(s))1_{(\Delta, \infty)}(y(s))dM(s)\).
Consider $D_t = \int_0^t 1_{(\Delta, +\infty)}(y(s))ds$. Let us show that $D_t \to \infty$ as $t \to \infty$. Set for $a, b > 0$, $T_{a,b} = \inf\{t > 0 : x(t) = a \text{ or } b\}$. Define
\[
\mu_0 = \inf\{t > 0 : x(t) = 0\}, \text{ and for } k = 1, 2, \ldots,
\mu_k = \inf\{t > 0 : t \geq \mu_{k-1} + 1, x(t) \geq 2\delta \text{ or } x(t) = 0\}
\]
If the process can hit zero in finite time then for all $y \in [0, 2\delta]$, $P_y(T_{0, 2\delta} < \infty) = 1$ (cf. [1], Section 2). Then
\[
P_0(\mu_1 < \infty) = P_0(x(1) > 2\delta) + \int_{[0, 2\delta]} P_y(T_{0, 2\delta} < \infty)P_0(x(1) \in dy)
= P_0(x(1) > 2\delta) + P_0(x(1) \in [0, 2\delta]) = 1.
\]
Let
\[
\tau_1 = \inf\{t \geq 0 : x(t) = 2\delta\}, \text{ for } k = 1, 2, \ldots, \\
\kappa_k = \inf\{t > \tau_k : x(t) = \delta\}, \text{ and for } k = 2, 3, \ldots, \\
\tau_k = \inf\{t > \kappa_{k-1} : x(t) = 2\delta\}.
\]
Equality (10) yields $P_0(\tau_1 < \infty) = 1$. Indeed, note that $P_0(x(\mu_1) = 0) = \alpha \in (0, 1)$. Then, by strong Markov property
\[
P_0(\tau_1 = \infty) \leq P_0(\tau_1 \geq \mu_n) \leq P_0(\bigcap_{k=1}^n (x(\mu_k) = 0)) = (P_0(x(\mu_1) = 0))^n = \alpha^n \to 0, \text{ as } n \to \infty.
\]
Thus $P_0(\tau_1 < \infty) = 1$. Let for $k = 1, 2, \ldots$, $\kappa_k = \kappa_0 - \tau_k$. Then $\{\kappa_k : k \geq 1\}$ is a sequence of positive independent identically distributed random variables. Consequently, $D_\infty \geq \sum_{k=1}^n \kappa_k \to \infty$, as $n \to \infty$. So $\lim_{t \to +\infty} D_t = +\infty$.

Put
\[
\varphi_\delta(t) = \inf\{s \geq 0 : D(s) > t\},
\]
and
\[
U(t) = y(\varphi_\delta(t)) = U(0) + N(\varphi_\delta(t)) + \frac{1}{2} \rho(\delta) L^y_\Delta(t), \quad t \geq 0.
\]
It can be seen that the process $K(t) = N(\varphi_\delta(t))$ is a martingale and
\[
\langle K \rangle(t) = \int_0^t \sigma^2(U(s))ds, \quad t \geq 0,
\]
where $\sigma(x) = \rho(s^{-1}(x))\sigma(s^{-1}(x))$, $x > 0$. By Itô-Tanaka formula we have
\[
U(t) = U(0) + \int_0^t 1_{(\Delta, +\infty)}(U(s))dK(s) + \frac{1}{2} \int_0^t 1_{(\Delta, +\infty)}(U(s))dL^y_\Delta(\varphi_\delta(t)) + \frac{1}{2} L^U_\Delta(t), \quad t \geq 0.
\]
Making use change of variables in Lebesgue-Stiltjes integrals and taking into account that measure $dL^y_\Delta(\varphi_\delta(t))$ increases only on the set $\{t \geq 0 : y(\varphi_\delta(t)) = \Delta\}$, we arrive at the equality
\[
\int_0^t 1_{(\Delta, +\infty)}(U(s))dL^y_\Delta(\varphi_\delta(s)) = \int_{\varphi_\delta(0)}^{\varphi_\delta(t)} 1_{(\Delta, +\infty)}(y(s))dL^y_\Delta(s) = 0.
\]
Comparing (11) with (12) we get from the uniqueness of the semimartingale decomposition of $U$ that
\[
\int_0^t 1_{(\Delta, +\infty)}(U(s))dK(s) = K(t), \quad t \geq 0,
\]
and
\[ U(t) = U(0) + K(t) + \frac{1}{2}L^V_x(t), \quad t \geq 0. \]

Consider
\[ A(t) = \int_0^t \varkappa^2(U(s))ds, \]
and put
\[ A(\infty) = \lim_{t \to \infty} \int_0^t \varkappa^2(U(s))ds, \]
\[ \tau(t) = \inf\{s \geq 0 : A(s) > t\}, \quad 0 \leq t < A(\infty). \]

Arguing as above we arrive at the equation
\[ V(t) = U(\tau(t)) = V(0) + J(t) + \frac{1}{2}L^V_x(t), \quad 0 \leq t < A(\infty), \]
where \( J(t) = K(\tau(t)), \quad 0 \leq t < A(\infty), \) and \( J(t) = \int_{\tau(t)}^t \varkappa^2(U(s))ds = t. \) By Theorem 7.2, Ch.2 of [8] there exists a Brownian motion \((w(t))_{t \geq 0}\) (defined, possibly, on an enlarged probability space) such that \( J \) coincides with \( w \) on \([0, A(\infty))\). Skorokhod’s lemma (cf. [13], Ch.VI, Lemma 2.1) and Lemma 2.3, Ch.VI of [13] allow us make the conclusion that the process \( V \) is a Brownian motion started at \( V(0) = s(x(0)) \vee \Delta \), reflected at \( \Delta \). Thus the measure \( P^\delta = \text{Law}(U(t) : t \geq 0) = \text{Law}(\varphi_\delta((t)) : t \geq 0) \) is determined uniquely and does not depend on the choice of a solution \( \bar{P}_{x_0} \). This entails that the law of the process \( x(\varphi_\delta(t)) \) is uniquely defined. Note that item \((iii)\) of Definition 1 provides that the process \((x(t))_{t \geq 0}\) spends zero time at the point \( 0 \) \( P_{x_0} \)-a.s. Then for each \( T > 0 \), \( \varphi_\delta(t) \equiv t \) on \([0, T]\) and, consequently, \( x(\varphi_\delta(t)) \equiv x(t) \), as \( \delta \downarrow 0 \) \( P_{x_0} \)-a.s. Therefore, \( \text{Law}(x(t) : t \geq 0) \) is determined uniquely and does not depend on the choice of the solution \( \bar{P}_{x_0} \). Then according to Remark 2 the weak uniqueness holds for equation (5). □

**Lemma 3** (pathwise uniqueness). Let the weak uniqueness hold for equation (5). Then the pathwise uniqueness holds true for (5).

**Proof.** Let \((x_1(t))_{t \geq 0}, (x_2(t))_{t \geq 0}\) be processes defined on the same probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) and let each of them is a weak solution to equation (5). The idea of the proof is as follows. We will see that the process \((x_1 \vee x_2(t))_{t \geq 0}\) is also a weak solution to equation (5).

By Theorem IV-68 of [11] we have
\[ (\zeta_\delta(x_1) \vee \zeta_\delta(x_2))(t) = \zeta_\delta(x_1(t)) + (\zeta_\delta(x_2(t)) - \zeta_\delta(x_1(t)))^+ \]
\[ = \zeta_\delta(x_0) + \int_0^t \mathbf{1}_{[\zeta_\delta(x_1(s)) < \zeta_\delta(x_2(s)) > 0]} d\zeta_\delta(x_2(s)) + \int_0^t \mathbf{1}_{[\zeta_\delta(x_2(s)) - \zeta_\delta(x_1(s)) \leq 0]} d\zeta_\delta(x_1(s)) + \frac{1}{2}L_0^{\zeta_\delta(x_1) - \zeta_\delta(x_2)}(t), \]
where \( L_0^{\zeta_\delta(x_1) - \zeta_\delta(x_2)}(t) \) is a local time of the process \((\zeta_\delta(x_1(t)) - \zeta_\delta(x_2(t)))_{t \geq 0}\) at 0. Then
\[ (\zeta_\delta(x_1) \vee \zeta_\delta(x_2))(t) = \zeta_\delta(x_0) + \int_0^t a((x_1 \vee x_2)(s)) \mathbf{1}_{[\delta, +\infty)}((x_1 \vee x_2)(s))ds \]
\[ + \int_0^t \sigma((x_1 \vee x_2)(s)) \mathbf{1}_{[\delta, +\infty)}((x_1 \vee x_2)(s))dw(s) \]
\[ + \frac{1}{2}L_0^{\zeta_\delta(x_1) - \zeta_\delta(x_2)}(t) + \frac{1}{2}L_0^{x_1 \vee x_2}(t) + \frac{1}{2}L_0^{\zeta_\delta(x_1) - \zeta_\delta(x_2)}(t). \]

Consider the last summand in the right-hand side of (14).
The properties of the local time (cf. Theorem 69, [11], p. 214) implies that
$L_0^\zeta(x_1(\cdot) - \zeta(x_2(\cdot)))$ increases only on \{ $t: \zeta_d(x_1(t)) = \zeta_d(x_2(t))$\}. We prove that increases only on \{ $t: \zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t)) = \delta$\}.

Let $q \in [0, \infty) \cap \mathbb{Q}$ be such that $\zeta_\delta(x_1(q)) > \delta, \zeta_\delta(x_2(q)) > \delta.$ Define

$$a_q = \sup\{t < q: (\zeta_\delta(x_1) \land \zeta_\delta(x_2))(t) = \delta\},$$

$$b_q = \inf\{t > q: (\zeta_\delta(x_1) \land \zeta_\delta(x_2))(t) = \delta\},$$

and $I_q = (a_q, b_q).$ Suppose $\zeta_\delta(x_1(q)) = \zeta_\delta(x_2(q)).$ Then by Theorem on homeomorphisms of flows (cf. Theorem V-46, [11]) applied to equation (6), we have $\zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t)), t \in [a_q, b_q].$ On the other hand, if there exists $r < q$ such that $\zeta_\delta(x_1(r)) \neq \zeta_\delta(x_2(r))$, by the same theorem $\zeta_\delta(x_1(q)) \neq \zeta_\delta(x_2(q)).$ Thus $\zeta_\delta(x_1(q)) = \zeta_\delta(x_2(q))$ implies $\zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t)), t \in I_q,$ and $(\zeta_\delta(x_1) \lor \zeta_\delta(x_2))(t) = \zeta_\delta(x_1(t)), t \in I_q$. Comparison (14) with (6) permits the conclusion that $L_0^\zeta(x_1(\cdot) - \zeta(x_2(\cdot)) = 0, t \in I_q.$

In the case of $\zeta_\delta(x_1(q)) \neq \zeta_\delta(x_2(q))$ we get $\zeta_\delta(x_1(t)) \neq \zeta_\delta(x_2(t)), t \in I_q.$ So for every $[\alpha, \beta] \subseteq I_q$ there exists $\varepsilon_0 > 0$ such that \{$\zeta_\delta(x_1(t)) - \zeta_\delta(x_2(t)) > \varepsilon_0, t \in [\alpha, \beta]$\}. Then for all $\varepsilon \in [0, \varepsilon_0], I_{[0, \varepsilon]}[\zeta_\delta(x_2(t)) - \zeta_\delta(x_1(t))] = 0, t \in [\alpha, \beta]$ From Corollary 3 of [11], p. 225 we obtain

$$L_0^\zeta(x_1(\cdot) - \zeta(x_2(\cdot))) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t [1_{[0, \varepsilon]}(\zeta_\delta(x_2(s)) - \zeta_\delta(x_1(s))]ds = 0, t \in [\alpha, \beta].$$

Therefore, $L_0^\zeta(x_1(\cdot) - \zeta(x_2(\cdot)))$ can increases only on \{ $t: \zeta_\delta(x_1(t)) = \zeta_\delta(x_2(t)) = \delta$\}, i.e. on \{ $t: (\zeta_\delta(x_1) \lor \zeta_\delta(x_2))(t) = \delta$\}.

Let $f \in C^2([0, \infty)),$ Then there exists $\delta > 0$ such that $f$ is constant on $[0, 2\delta], and we have

$f((x_1 \lor x_2)(t)) = f((\zeta_\delta(x_1) \lor \zeta_\delta(x_2))(t)), 0 \leq t < +\infty.$

We have seen that the local times $L_0^\zeta(x_1(\cdot) - \zeta(x_2(\cdot))$ and $L_\delta^Z(x_1(\cdot) \lor x_2(\cdot))$ do not increase on \{ $(x_1 \lor x_2)(t) > 2\delta$\}. Taking into account that $f'(x) = f''(x) = 0$ on $[0, 2\delta]$ and making use of Itô formula we obtain

$$f((x_1 \lor x_2)(t)) = f(x_0) + \int_0^t a((x_1 \lor x_2)(s))f'(((x_1 \lor x_2)(s)))ds$$

$$+ \int_0^t \sigma((x_1 \lor x_2)(s))f'((x_1 \lor x_2)(s))ds + \frac{1}{2} \int_0^t \sigma^2((x_1 \lor x_2)(s))f''((x_1 \lor x_2)(s)))ds.$$}

Therefore, the process $(x_1 \lor x_2)(t)$ satisfies equation (5). By the weak uniqueness for all $t \geq 0, E^P(x_1 \lor x_2)(t) = E^P(x_1)(t) = E^P(x_2)(t).$ This yields $(x_1 \lor x_2)(t) = x_1(t) = x_2(t), t \geq 0 \ P-a.s.$

**Proof of Theorem.** Let for each $x_0 \geq 0,$ there exists a solution to the martingale problem $M(a, \sigma, x_0).$ Then by Lemma 2 the weak uniqueness holds for equation (5). Then the assertion of Theorem follows from Theorem 3 similarly to Yamada-Watanabe theorem (cf. Theorem IV-1.1 of [8]).
It is known (see [13], p.446) that this process has a transition probability density (Skorokhod equation ([7], Example 1). In the latter case the assertion of Theorem is obvious. Example 2. Let \( (x, l) \) be a pair of the processes (\ref{eq:16}) and all the integrals in the right-hand side of (16) are well-defined. Then the pair \((x, l)\) is called a strong solution to equation (16).

Now, suppose that 0 is a limit point of the set \( B \). Suppose there exists a subsequence \( \{ y_n : n \geq 1 \} \subset B \) such that \( y_n \to 0 \) as \( n \to \infty \) and \( a(y_n) \geq 0 \). Then starting away from 0, a solution never hits some neighborhood of 0 and the assertion of Theorem holds true. If such a subsequence does not exist, then there is a neighborhood of 0, say \( U \), such that for all \( y \in B \cap U \), \( a(y) < 0 \). In this case, starting from 0 a solution does not attend the interval \((y, +\infty)\) for all \( y \in B \cap U \). Thus, if 0 is a limit point of \( B \), a solution either hits 0 in finite time with positive probability or does not hit 0 a.s. In the former case a solution stays at the point 0 forever. But this contradicts with item \((iii)\) of Definition 1. In the latter case the assertion of Theorem is obvious.

### 3. Examples

**Example 1** (Skorokhod equation ([7], §23)). Let \( a, b \) be functions on \([0, \infty)\). Let \((w(t))_{t \geq 0}\) be a Wiener process, \( x_0 \geq 0 \). Recall the definition of a solution to Skorokhod problem.

Let \((x, l)\) be a pair of continuous processes adapted to the filtration \((\bar{F}_t)\) and such that

(i) \( x \) is non-negative,

(ii) \( l(0) = 0 \), \( l(\cdot) \) is nondecreasing,

(iii) \( l(\cdot) \) increases only at those moments of time when \( x(t) = 0 \), i.e. for each \( t \geq 0 \),

\[
\int_0^t 1_{\{x(s)\}} dl(s) = l(t),
\]

(iv) for each \( t \geq 0 \), the relation

\[
x(t) = x_0 + \int_0^t a(x(s)) ds + \int_0^t b(x(s)) dw(s) + l(t)
\]

holds and all the integrals in the right-hand side of (16) are well-defined.

Then the pair \((x, l)\) is called a strong solution to equation (16).

If \((x, l)\) is such a solution, then for each \( f \in C^2([0, \infty)) \), by Itô formula for semimartingales, we have

\[
f(x(t)) = f(x_0) + \int_0^t f'(x(s)) b(x(s))dw(s) + \int_0^t f'(x(s)) a(x(s)) ds
\]

\[
+ \frac{1}{2} \int_0^t f''(x(s)) b^2(x(s)) ds + \int_0^t f'(x(s)) dl(s).
\]

According to (15), the last member in the right-hand side of (17) is equal to 0. Thus, if the pair of the processes \((x, l)\) is a strong solution to equation (16), then the process \( x \) is a strong solution to equation (5) in the sense of Definition 6.

**Example 2** (\(\beta\)-dimensional Bessel processes). Let \( \rho \) be a Bessel process of dimension \( \beta \).

It is known (see [13], p.446) that this process has a transition probability density

\[
p^\rho_t(x, y) = t^{-\frac{1}{2}}(y/x)^{\nu} \exp\left(-\frac{x^2 + y^2}{2t}\right) I_\nu(xy/t) \text{ for } x > 0, t > 0,
\]

and

\[
p^\rho_t(0, y) = \frac{1}{2} t^{-\nu} \Gamma^{-1}(\nu + 1) y^{2\nu+1} \exp(-y^2/2t),
\]
where \( \nu = \beta / 2 - 1 \). If \( 0 < \beta < 2 \) the point 0 is instantaneously reflecting and for \( \beta \geq 2 \) it is polar.

1) Let \( \beta > 1 \). In this case the process \( \rho \) is a semimartingale, which satisfies the SDE of the form (see [13], Ch.XI, §1).

\[
\rho(t) = \rho(0) + w(t) + \frac{\beta - 1}{2} \int_0^t \frac{1}{\rho(s)} ds, \quad \rho(0) \geq 0.
\]

Cherny [2] has shown that there exists a unique non-negative strong solution to equation (18). Let \( \rho \) be a non-negative solution to (18). Applying Itô formula we get the equation

\[
f(\rho(t)) = f(\rho(0)) + \int_0^t f'(\rho(s))dw(s) + \frac{\beta - 1}{2} \int_0^t \frac{f'(\rho(s))}{\rho(s)} ds + \frac{1}{2} \int_0^t f''(\rho(s))ds.
\]

Thus, \((\rho, w)\) is a weak solution to equation (5) in the sense of Definition 3. Then according to Theorem there exists a strong solution to (5) and the strong uniqueness holds. Therefore we obtain the result of Cherny from ours.

2) Let \( 0 < \beta < 1 \) and let \( (\rho(t))_{t \geq 0} \) be a Bessel process on some probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\). Then the process \( \rho \) is not a semimartingale. Nonetheless it has the family of local times defined by the formula

\[
\int_0^t \phi(\rho(s))ds = \int_0^\infty \phi(x)L_\rho(x)^{\beta-1}dx.
\]

valid for all \( t > 0 \) and for every positive measurable function \( \phi \) on \([0, \infty)\) a.s. The Bessel process of dimension \( \beta \) is a weak solution to the following equation (cf. [13], Ch.XI, Ex. 1.26)

\[
\rho(t) = \rho(0) + w(t) + \frac{\beta - 1}{2} k(t), \quad \rho(0) \geq 0,
\]

where \((w(t))_{t \geq 0}\) is an \((\mathcal{F}_t)\)-Wiener process, \(k(t) = \text{V.P.} \int_0^t \rho^{-1}(s)ds\) which, by definition, is equal to \(\int_0^\infty a^{\beta-2}(L_\rho^a(t) - L_{\rho}^a(t)^2)da\).

Let us check that the pair \((\rho, w)\) is a weak solution to equation (5) in the sense of Definition 3. Then the Theorem yields that there exists a unique strong solution to equation (20). To prove this we need the following Lemma.

**Lemma 4.** Let \( t_1, t_2 \in \mathbb{Q}, \ t_1 < t_2 \). Then for almost all \( \omega \in \Omega \) such that \( \rho(t, \omega) > 0, \ t \in [t_1, t_2] \), the equality

\[
k(t_2) - k(t_1) = \int_{t_1}^{t_2} \frac{1}{\rho(s)} ds
\]

holds.

**Proof.** There exists \( \varepsilon > 0 \) such that \( \rho(t) \geq \varepsilon, \ t \in [t_1, t_2] \). The properties of the local time imply that for all \( a < \varepsilon, \ L_\rho^a(t_2) = L_\rho^a(t_1) \). Then

\[
k(t_2) - k(t_1) = \int_0^{\infty} a^{\beta-2} ([L_\rho^a(t_2) - L_\rho^a(t_1)] - [L_\rho^a(t_1) - L_\rho^a(t_1)]) da
\]

\[
= \int_0^{\infty} \frac{1_{[\varepsilon, \infty]}(a)}{a} a^{\beta-1}(L_\rho^a(t_2) - L_\rho^a(t_1))da.
\]

By (19) we get

\[
k(t_2) - k(t_1) = \int_{t_1}^{t_2} 1_{[\varepsilon, \infty]}(\rho(s)) \frac{1}{\rho(s)} ds = \int_{t_1}^{t_2} \frac{1}{\rho(s)} ds.
\]

\(\square\)
Because of the continuity of the process \((\rho(t))_{t \geq 0}\) there exists \(\tilde{\Omega} \in \Omega\), \(P(\tilde{\Omega}) = 1\), such that for all \(\omega \in \tilde{\Omega}\), formula (21) holds true for all \(t_1, t_2 \geq 0\) satisfying \(\rho(t) > 0, t \in [t_1, t_2]\).

Let \(\tau\) be a stopping time such that \(\rho(\tau) \neq 0\) \(P\text{-a.s.}\). Put \(\sigma = \inf\{s \geq \tau : \rho(s) = 0\}\). We have
\[
\rho(t) = \rho(\tau) + w(t) - w(\tau) + \frac{\beta - 1}{2} \int_\tau^t \frac{ds}{\rho(s)}, \quad t \in [\tau, \sigma).
\]

Let \(f \in C^2([0, \infty))\). Itô formula for semimartingales yields
\[
(22) \quad f(\rho(t)) - f(\rho(\tau)) = \int_\tau^t f'(\rho(s))dw(s) + \frac{\beta - 1}{2} \int_\tau^t \frac{f'(\rho(s))}{\rho(s)}ds + \frac{1}{2} \int_\tau^t f''(\rho(s))ds, \quad t \in [\tau, \sigma).
\]

Choose \(\delta > 0\) such that \(f\) is constant on \([0, 2\delta]\). Define
\[
\tau_0 = 0, \quad \tau_i = \inf\{t > \tau_{i-1} : \rho(t) = \delta/2\}, \quad \text{for } i \geq 0, \quad \tau_i = \inf\{t > \rho \tau_{i-1} : \rho(t) = \delta\}, \quad \text{for } i \geq 1.
\]

Then
\[
f(\rho(t)) = f(\rho(0)) + \sum_{k=0}^{\infty} [f(\rho(\tau_k \wedge t)) - f(\rho(\tau_k \wedge t))] + \sum_{k=0}^{\infty} [f(\rho(\tau_k+1 \wedge t)) - f(\rho(\tau_k \wedge t))].
\]

The second sum in the right-hand side is equal to zero. If \(f(\rho(0)) < \delta/2\), then \(f(\rho(\tau_0 \wedge t)) - f(\rho(\tau_0)) = 0\).

Suppose \(\rho(0) \geq \delta/2\). It follows from (22) that
\[
(23) \quad f(\rho(t)) = f(\rho(0)) + \sum_{k=0}^{\infty} \int_{\tau_k \wedge t}^{\tau_{k+1} \wedge t} f'(\rho(s))dw(s) + \frac{\beta - 1}{2} \sum_{k=0}^{\infty} \int_{\tau_k \wedge t}^{\tau_{k+1} \wedge t} \frac{f'(\rho(s))}{\rho(s)}ds
\]
\[
+ \frac{1}{2} \sum_{k=0}^{\infty} \int_{\tau_k \wedge t}^{\tau_{k+1} \wedge t} f''(\rho(s))ds, \quad t \geq 0.
\]

Note that for all \(t \in [\tau_k, \tau_{k+1}], \quad k \geq 0, \quad f'(\rho(t)) = f''(\rho(t)) = 0\). Then (23) can be rewritten in the form
\[
(24) \quad f(\rho(t)) = f(\rho(0)) + \int_0^t f'(\rho(s))dw(s) + \frac{\beta - 1}{2} \int_0^t \frac{f'(\rho(s))}{\rho(s)}ds + \frac{1}{2} \int_0^t f''(\rho(s))ds, \quad t \geq 0.
\]

If \(\rho(0) < \delta/2\) equation (24) can be obtained similarly.

Hence, the pair \((\rho, w)\) is a weak solution to equation (5) and, consequently, the strong existence and uniqueness hold for equation (20).

**Example 3.** Let the process \((x(t))_{t \geq 0}\) be a weak solution to an SDE of the form
\[
(25) \quad x(t) = x(0) + \int_0^t a(|x(s)|)ds + \int_0^t b(|x(s)|)dw(s),
\]
where the coefficients \(a\) and \(b\) are locally Lipshitz continuous on \((0, \infty)\).
Then for each even function being constant in a neighborhood of zero according to Itô formula, we get
\begin{equation}
(26) \quad f(x(t)) = f(x(0)) + \int_0^t a(|x(s)|)f'(x(s))ds + \int_0^t b(|x(s)|)f''(x(s))dw(s) + \frac{1}{2}\int_0^t b^2(|x(s)|)f''(x(s))ds.
\end{equation}

Note that if the process \((x(t))_{t \geq 0}\) is a weak solution to equation (25) spending zero time at the origin then the process \(y(t) = |x(t)|, \ t \geq 0\), satisfies equality (26) for each \(f \in C_c^2([0, +\infty))\). By Theorem 1 the process \(y(t), \ t \geq 0\), is a unique non-negative strong solution to (26) spending zero time at the origin.

Consider an SDE which can be regarded as an example of equation of the form (25)
\begin{equation}
(27) \quad x(t) = x(0) + \int_0^t |x(s)|^\alpha dw(s), \ \alpha \in (0, 1/2).
\end{equation}

It is known that there exists a weak solution to (27) spending zero time at the point 0 (cf. [10], 3.10b).

**Remark 4.** Girsanov [6] has shown that without additional assumption this equation has infinitely many weak solutions.

**Remark 5.** It can be proved (cf. [3]) that in the class of solutions spending zero time at the point 0 the pathwise uniqueness holds and a strong solution exists.

So, there exists a weak solution to the equation
\begin{equation}
(28) \quad f(x(t)) = f(x(0)) + \int_0^t (x(s))^{\alpha}f'(x(s))dw(s) + \frac{1}{2}\int_0^t (x(t))^{2\alpha}f''(x(s))ds
\end{equation}
in the sense of Definition 3 spending zero time at the point 0. According to Theorem 1 there is a strong solution to (28) spending zero time at the point 0. Certainly, this solution coincides with the unique strong solution to the equation
\begin{equation}
x(t) = x(0) + \int_0^t (x(s))^{\alpha}dw(s) + dL_0^\alpha(t), \ \alpha \in (0, 1/2),
\end{equation}
spending zero time at the point 0 which was constructed by Bass and Chen (see [4]). Here \((L_0^\alpha(t))_{t \geq 0}\) is a local time of the process \((x(t))_{t \geq 0}\) at the point 0 defined by formula (7).

**Appendix**

**Proof of assertion of Remark 2.** The "only if" assertion is trivial.

To prove the "if" assertion we can argue as in Prop.2.1, Ch.IV of [8]. Suppose \(P\) is a solution to the martingale problem \(M(a, \sigma, x_0)\) on space \((C^+([0, +\infty)), \mathcal{G}_t), (\mathcal{F}_t)\), and \(f \in C_c^2([0, \infty))\). Then the process \(Y_f(t)\) is a continuous, square integrable local martingale with respect to \(P\). Applying condition (ii) of Definition 1 to the function \(f^2\), we calculate the characteristics of the process \((Y_f(t))_{t \geq 0}\). Namely,
\begin{equation}
\langle Y_f \rangle(t) = \int_0^t \sigma^2(x(s))(f'(x(s)))^2 ds.
\end{equation}

Consequently, there is a Brownian motion \((w_f(t))_{t \geq 0}\) defined on an extension of \((C^+([0, \infty)), \mathcal{G}_t), (\mathcal{F}_t), P\) such that
\begin{equation}
Y_f(t) = \int_0^t \sigma(x(s))f'(x(s))dw_f(s).
\end{equation}

We will show that it can be chosen the same Brownian motion for all \(f \in C_c^2([0, \infty))\).
Similarly to the Proof of Lemma 1, for \( k = 1, 2, \ldots \), consider a non-decreasing function \( \eta_k \in C^2_\sigma([0, \infty)) \) such that \( \eta_k(x) = x, \ x > 1/k \), and \( \eta_k \) is a constant on \([0, \frac{1}{k}]\).

Let us fix \( k \) and put
\[
\tau_1 = \inf\{t : \eta_k(x(t)) > l\}, \ l = 1, 2, \ldots.
\]

Then, for all \( l = 1, 2, \ldots \),
\[
\eta_k(x(t \wedge \tau_1)) - \eta_k(x_0) - \int_0^{t \wedge \tau_1} \left[a(x(s))\eta'_k(x(s)) + \frac{1}{2} \sigma^2(x(s))\eta''_k(x(s))\right] ds
\]
is a continuous, square integrable \( P \)-martingale. Then \( Y_{\eta_k}(t) \in \mathcal{M}^{c,loc}_2(P) \), and there exists a Brownian motion \((w_k(t))_{t \geq 0}\) on an extension \((\Omega_k, \mathcal{F}_k, P_k)\) of \((C^+(\{0, \infty\})), \mathcal{G}, (\mathcal{G}_t), P)\) such that
\[
\eta_k(x(t)) = \eta_k(x_0) + \int_0^t \left(a(x(s))\eta'_k(x(s)) + \frac{1}{2} \sigma^2(x(s))\eta''_k(x(s))\right) ds
\]
\[
+ \int_0^t \sigma(x(s))\eta'_k(x(s))dw_k(s).
\]

Fix \( m \geq 1 \). Then for all \( k \geq m \), \( \eta_m(x) = \eta_k(x), \ x > 1/m \). Put
\[
\tilde{w}_m(t) := \int_0^t 1_{(\frac{k}{m}, +\infty)}(x(s))dw_m(s).
\]

As a consequence of the following simple Lemma we have that for each \( m \geq 1 \) the process \((\tilde{w}_m(t))_{t \geq 0}\) is adapted w.r.t. the filtration generated by the process \((x(t))_{t \geq 0}\) and for all \( k \geq m \),
\[
\int_0^t 1_{(\frac{k}{m}, +\infty)}(x(s))dw_k(s) = \int_0^t 1_{(\frac{k}{m}, +\infty)}(x(s))dw_m(s) = \int_0^t 1_{(\frac{k}{m}, +\infty)}(x(s))d\tilde{w}_m(s) \text{ a.s.}
\]

**Lemma 5.** Let \( A \) be an open set in \( \mathbb{R} \). Let \( x_0 \in A \), \( (x(t))_{t \geq 0} \) be a continuous adapted process on a probability space \( (\Omega, \mathcal{F}_t, P) \). Let \((w(t))_{t \geq 0}\) be a Wiener process on some extension of the space \((\Omega, \mathcal{F}_t, P)\). Suppose \( a, b, f \) are continuous functions on \( \mathbb{R} \), \( b(x) \neq 0 \) for \( x \in A \), and for all \( t \geq 0 \), the equality
\[
f(x(t)) = f(x_0) + \int_0^t a(x(s))ds + \int_0^t b(x(s))dw(s)
\]
holds. Put \( \mathcal{F}^t = \sigma\{x(s) : 0 \leq s \leq t\} \). Then the process \( \int_0^t 1_A(x(s))dw(s), t \geq 0 \), is adapted w.r.t. \( (\mathcal{F}^t) \).

Moreover, suppose \((\tilde{w}(t))_{t \geq 0}\) is a Wiener process on an extension of the probability space \((\Omega, \mathcal{F}_t, P)\), \( \tilde{a}, \tilde{b}, \tilde{f} \) are continuous functions on \( \mathbb{R} \), \( \tilde{b}(x) \neq 0 \) for \( x \in A \), and the equality
\[
\tilde{f}(x(t)) = \tilde{f}(x_0) + \int_0^t \tilde{a}(x(s))ds + \int_0^t \tilde{b}(x(s))d\tilde{w}(s)
\]
holds.

If \( a(x) = \tilde{a}(x), b(x) = \tilde{b}(x), f(x) = \tilde{f}(x) \) on \( A \), then
\[
\int_0^t 1_A(x(s))dw(s) = \int_0^t 1_A(x(s))d\tilde{w}(s).
\]

The proof is trivial.

The sequence \( \{\tilde{w}_m : m \geq 1\} \) defined in (30) is fundamental in mean square on compact intervals. Indeed, for \( k \geq m, T > 0 \), using martingale inequality (cf. [8], Theorem I-6.10)
and (31), we get
\[ E \left[ \sup_{t \in [0,T]} |\tilde{w}_k(t) - \tilde{w}_m(t)| \right]^2 \leq 4E|\tilde{w}_k(T) - \tilde{w}_m(T)|^2 = \]

\[ 4E \left\{ \int_0^T \left( 1_{(\frac{1}{k},+\infty)} - 1_{(\frac{1}{m},+\infty)} \right)(x(s))dw_k(s) \right\}^2 \leq 4E \int_0^T 1_{(\frac{1}{k},+\infty)}(x(s))ds \to 0, \ m \to +\infty. \]

Then the sequence \( \{\tilde{w}_m : m \geq 1\} \) is uniformly convergent on compact intervals in probability. Denote the limit of the sequence \( \{\tilde{w}_m : m \geq 1\} \) by \( \tilde{w} \).

The process \( \tilde{w}(t) \in \mathcal{M}_{2,loc}^c(P) \) and
\[
\langle \tilde{w}(t) \rangle(t) = \int_0^t 1_{(0,\infty)}(x(s))ds = t.
\]

Here we used the fact that the process \( (x(t))_{t \geq 0} \) spends zero time at the point 0. Thus the process \( \tilde{w}(t)_{t \geq 0} \) is a Wiener process. Besides, by construction,
\[
\tilde{w}_k(t) = \int_0^t 1_{(\frac{1}{k},+\infty)}(x(s))d\tilde{w}(s).
\]

Let \( f \in C^2_c([0,\infty)) \) be such that \( f \) is constant on \([0,1/k]\). Then there exists a Wiener process \( w_f(t)_{t \geq 0} \) such that
\[
(33) \quad f(x(t)) = f(x_0) + \int_0^t \left( a(x(s))f'(x(s)) + \frac{1}{2}\sigma^2(x(s))f''(x(s)) \right)ds + \int_0^t \sigma(x(s))f'(x(s))dw_f(s).
\]

By Itô formula, (29) yields
\[
(34) \quad f(\eta_k(x(t))) = f(\eta_k(x_0)) + \int_0^t a(x(s))\eta_k'(x(s))f'(\eta_k(x(s)))ds + \frac{1}{2}\int_0^t \sigma^2(x(s))\eta_k'(x(s))f'(\eta_k(x(s)))ds + \int_0^t \sigma(x(s))\eta_k'(x(s))f'(\eta_k(x(s)))dw_k(s) + \frac{1}{2}\int_0^t \sigma^2(x(s))\eta_k'(x(s))^2f''(\eta_k(x(s)))ds.
\]

The second integral in the right-hand side of (34) is equal to 0 because \( \eta_k'(x) = 0 \) on \((1/k, +\infty)\). Taking into account that \( \eta_k(x) = x' = 1 \) on \((1/k, +\infty)\), and \( f(\eta_k(x)) = f(x) \) on \((1/k, +\infty)\), we arrive at the equation
\[
(35) \quad f(\eta_k(x(t))) = f(\eta_k(x_0)) + \int_0^t a(x(s))f'(x(s))ds + \int_0^t \sigma(x(s))f'(x(s))dw_k(s) + \frac{1}{2}\int_0^t \sigma^2(x(s))f''(x(s))ds.
\]

Note that
\[
\int_0^t \sigma(x(s))f'(x(s))dw_k(s) = \int_0^t \sigma(x(s))f'(x(s))1_{(\sigma(x(s))f'(x(s)) \neq 0)}dw_k(s).
\]
Applying Lemma 5 to equations (33) and (35) we have
\[
\int_0^t \mathbf{1}_{(1/k, +\infty)}(x(s)) \mathbf{1}_{\{\sigma(x(s))f'(x(s)) \neq 0\}} dw_f(s)
\]
\[
= \int_0^t \mathbf{1}_{(1/k, +\infty)}(x(s)) \mathbf{1}_{\{\sigma(x(s))f'(x(s)) \neq 0\}} dw_k(s)
\]
\[
= \int_0^t \mathbf{1}_{(1/k, +\infty)}(x(s)) \mathbf{1}_{\{\sigma(x(s))f'(x(s)) \neq 0\}} d\tilde{w}(s).
\]
So for each \( f \in C^2_2([0, +\infty)) \) the equality
\[
f(x(t)) = f(x_0) + \int_0^t \left( a(x(s))f'(x(s)) + \frac{1}{2} \sigma^2(x(s))f''(x(s)) \right) ds
\]
\[
+ \int_0^t \sigma(x(s))f'(x(s))d\tilde{w}(s)
\]
is justified, and the pair \((x(t), \tilde{w}(t))_{t \geq 0}\) is a weak solution to equation (5). \(\Box\)

References