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UNIFORM LIMIT THEOREMS UNDER LENGTH-BIASED SAMPLING AND TYPE I CENSORING

In recent years, in view of theory of empirical processes, authors have become more interested in the uniform analogue of the three fundamental theorems: the uniform law of large numbers of Glivenko-Cantelli type, the uniform central limit theorem for Donsker type and the functional law of the iterated logarithm (LIL). In this paper, under the bracketing entropy conditions, the uniform law of large numbers, uniform central limit theorem and the uniform LIL of Strassen type have been investigated in the case of length-biased and type I censoring.

1. **Introduction**

1.1. **Brief review.** In a classical probability theory, many literatures have focused on three famous limit theorems: the law of large numbers, the central limit theorem and the LIL. These topics are fundamental laws in the classical probability theory. Recently, there have been literatures concerning the uniform analogue of these three theorems which is related to the convergence of a particular type of random map called the empirical process.

Let $(S, \mathcal{B}, \mathbb{P})$ be any probability space. The usual *empirical measure* $\mathbb{P}_n$ of an independent and identically distributed random variables $X_1, \ldots, X_n$ with the same law $\mathbb{P}$, is the discrete random measure given by

$$\mathbb{P}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i},$$

where δ_{X_i} is the Dirac measure of X_i. Given a collection $\mathcal{F}$ of real-valued Borel measurable functions f, the uniform version of law of large numbers becomes

$$\sup_{f\in\mathcal{F}} (\mathbb{P}_n - \mathbb{P})f \longrightarrow 0,\ a.s., \tag{1.1}$$

where $\mathbb{P}f = \int f d\mathbb{P}$. The class $\mathcal{F}$ for which (1.1) is satisfied, is called $\mathbb{P}$-Glivenko-Cantelli class (see Van der Vaart and Wellner, 1996). Assuming a bracketing entropy on $\mathcal{F}$, DeHardt (1971) derived the uniform law of large numbers for a sequence of independent and identically distributed random variables.

Let $L^\infty(\mathcal{F})$ denotes the Banach space of bounded real-valued functions on $\mathcal{F}$ equipped with the supremum norm $\| \Psi \|_{\mathcal{F}} = \sup_{\varphi\in\mathcal{F}} |\Psi(\varphi)|$. The $\mathcal{F}$-indexed empirical process given by $\mathbb{G}_n f = \sqrt{n}(\mathbb{P}_n - \mathbb{P})f$ is an induced map from $\mathcal{F}$ to $\mathbb{R}$. The uniform version of central limit theorem states that for each $f \in \mathcal{F}$, $\mathbb{G}_n f$ converge in law to a Gaussian process in $L^\infty(\mathcal{F})$ which is called the $\mathbb{P}$-Brownian bridge process indexed by $\mathcal{F}$. The limit process $\{\mathbb{G}f,\ f \in \mathcal{F}\}$ must be a zero-mean Gaussian process with covariance function

$$E(\mathbb{G}f_1\mathbb{G}f_2) = \mathbb{P}f_1f_2 - \mathbb{P}f_1\mathbb{P}f_2. \tag{1.2}$$

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A class $\mathcal{F}$ for which this is true is called a $\mathbb{P}$-Donsker class (see Van der Vaart and Wellner, 1996). Assuming a metric entropy integrability condition, Ossiander (1987) stablished the uniform central limit theorem of $\mathbb{P}$-Donsker type.

Let $\mathcal{F} \subset L^2(\mathbb{P}) = \{f : \int |f|^2 d\mathbb{P}\}$. Class $\mathcal{F}$ is called a Strassen log log class for $\mathbb{P}$, or equivalently that $\mathcal{F}$ satisfies the compact LIL, if and only if with probability one, the sequence

$$\{(\frac{n}{2\log\log n})^{1/2}(\mathbb{P}_n - \mathbb{P})f, f \in \mathcal{F}\},$$

is relatively compact in $(L^\infty(\mathcal{F}), \|\ .\ \|_{\mathcal{F}})$. Clearly, every Strassen log log class is a log log class, the converse, of course, is false. Kuelbs and Dudley (1980) showed that the compact LIL of Strassen's type for empirical measures holds uniformly on a Donsker class $\mathcal{F}$. The Dudley and Philipp (1983) strengthened the related Kuelbs-Dudley's results and provided the most streamlined methods for obtaining LIL properties.

In survival analysis, dealing with statistical inference problems based on incomplete data and the random censorship model, this topic has been studied in various directions. Under the random censorship model, the uniform version of the LIL, the uniform law of large numbers and the uniform central limit theorem for the function-indexed Kaplan-Meier integral process were obtained by Bae and Kim (2003a), (2003b) and (2003c), respectively.

This paper focuses on the incomplete observations suffering from length-biased sampling and type I censoring. Here, the uniform law of large numbers for the function-indexed integral process is obtained by using the bracketing method of DeHardt (1971). The results are stated for convergence in the mean as well as almost sure convergence. Hence, it may be used in nonparametric statistical inference in verifying uniform consistency, see Van de Geer (2000) for applications. In the present paper, the uniform central limit theorem and the uniform LIL are established by using the idea of Ossiander (1987) and Kuelbs and Dudley (1980) respectively, which are expedient in nonparametric statistical inference.

1.2. **Length-biased sampling and type I censoring.** Lifetime data are often affected by sampling issues such as truncation and censoring. Under left-truncation and right-censoring, one observes the random vector (T, Z, δ) if and only if $Z \geq T$, where $Z = \min(Y, C)$ and $\delta = I\{Y \leq C\}$. Here Y is the lifetime of ultimate interest, C is the right censoring time and T is the truncation time. In some applications, the distribution function L of the truncation variable T may be assumed to take a given form. In many applications, such as renewal processes, marketing, epidemiologic studies, econometrics and genome-wide linkage studies, it has been found motivations for stationarity (or length-bias) assumption, i.e. assuming L to have uniform distribution. See for example Winter and Földes (1988), Lancaster (1990), Wang (1991), and Van Es et al. (2000). Put F for distribution function of Y. Then, the so-called length-biased df (of F) is

$$F^*(y) = P(Y \leq y | Z \geq T) = \mu^{-1} \int_0^y u dF(u), \tag{1.3}$$

where $\mu < \infty$ denotes the expectation of Y. Using equation (1.3), F can be easily shown in terms of F^* by

$$F(y) = \frac{\int_0^y u^{-1} dF^*(u)}{\int_0^\infty u^{-1} dF^*(u)}. \tag{1.4}$$

Let τ_F be the upper bound of the support of F and τ is a fixed positive constant. Assume that

(i): Y is independent of T.
(ii): $T \sim$Uniform$(0, \tau_L)$ for some $\tau_L \geq \tau_F$.

(iii): $C = T + \tau$,

Under assumptions **(i)-(iii)**, De Uña-Álvarez (2004) introduced a nonparametric estimator of F by means of an appropriate estimator for its length-biased version F^*. He suggested estimating $F^*(y)$ through

$$\widehat{F}^*(y) = \frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i I\{Z_i \le y\}}{p(Z_i)} \equiv \sum_{i=1}^{n} W_i I\{Z_i \le y\},$$

where

$$p(y) = P(\delta = 1 | Y = y, Z \ge T), \tag{1.5}$$

and

$$W_i = \frac{\delta_i}{np(Z_i)} = \frac{1}{n} I\{Z_i \le \tau\} + \frac{\delta_i Z_i}{n\tau} I\{Z_i > \tau\}, \qquad 1 \le i \le n.$$

Using (1.4), the nonparametric estimator of F is therefore given by

$$\widehat{F}(y) = \frac{\int_0^y u^{-1} d\widehat{F}^*(u)}{\int_0^\infty u^{-1} d\widehat{F}^*(u)} \equiv \sum_{i=1}^{n} \widetilde{W}_i I\{Z_i \le y\}, \tag{1.6}$$

where

$$\widetilde{W}_i = \frac{W_i Z_i^{-1}}{\sum_{j=1}^{n} W_j Z_j^{-1}}, \qquad 1 \le i \le n.$$

The strong consistency as well as Glivenko-Cantelli law of large numbers and the central limit theorem of $\widehat{F}$ have been obtained by De Uña-Álvarez (2004).

The major aims of this paper are to establish the uniform version of three fundamental limit theorems for the function-indexed integral process

$$U_n(\varphi) = \int \varphi d\big(\widehat{F} - F\big), \quad \text{for } \varphi \in \mathcal{F}.$$

This work is organised as follows: some definitions and main results are provided in Section 2. Some important lemmas which are useful tools to prove the main results and their proofs are given in Section 3.

2. **Main Results**

We define the following version of metric entropy with bracketing to measure the size of the function space (see, for example,van der Vaart and Wellner, 1996 for more details.) Let $(\mathcal{F}, ||\cdot||_p)$ be a subset of a normed space $(L^p(F), ||\cdot||_p)$, where $L^p(F) := \{\varphi : \| \varphi \|_p < \infty\}$ and $\| \varphi \|_p = \Big(\int |\varphi|^p dF\Big)^{1/p}$.

Definition 2.1. *Given two functions l and u in $L^p(F)$, the bracket $[l, u]$ is the set of all functions φ with $l \le \varphi \le u$. An ϵ-bracket in $L^p(F)$ is a bracket $[l, u]$ with $\| u - l \|_p < \epsilon$. The bracketing number $N_{[\,]}(\epsilon, \mathcal{F}, \| \cdot \|_p)$ is the minimum number of ϵ-brackets in $(L^p(F), \| \cdot \|_p)$ needed to ensure that every $\varphi \in \mathcal{F}$ lies in at least one bracket. The logarithm of the bracketing number is the entropy with bracketing.*

Definition 2.2. *The bracketing entropy integral of a class of functions $\mathcal{F}$ is defined by*

$$J(\delta) := J_{[\,]}(\delta, \mathcal{F}, \| \cdot \|_p) = \int_0^\delta \sqrt{\ln N_{[\,]}(\epsilon, \mathcal{F}, \| \cdot \|_p)}\, d\epsilon. \tag{2.1}$$

The notion of entropy with bracketing has been introduced by Dudley (1978) and the importance of $L^2(F)$-entropy with bracketing has been pointed out by Ossiander (1987). In the rest of the paper, whenever unambiguous we write $N_{[\,]}(\epsilon)$ instead of $N_{[\,]}(\epsilon, \mathcal{F}, \| \cdot \|_p)$.

2.1. **The Uniform Law of Large Numbers.** In order to develop a uniform law of large numbers for the function-indexed process $U_n(\varphi)$, one needs to investigate that $\sup_{\varphi\in\mathcal{F}}|U_n(\varphi)|$ converges to zero in a certain sense under a certain entropy condition on the class $\mathcal{F}$.

The function-indexed process $U_n(\varphi)$ is treated as random elements of $L^\infty(\mathcal{F})$. In this subsection, we set $p = 1$ and hence, $\mathcal{F} \subseteq L^1(F)$. We shall use the metric defined by $d_1(f, g) := \int |f - g|dF(x)$. The following definition is conveniently collected by Van der Vaart and Wellner (1996) for almost sure convergence and convergence in the mean.

Definition 2.3. *A sequence of $L^\infty(\mathcal{F})$-valued random functions $\{\Psi_n\}$ converges almost surely to a constant c if*

$$P^*(\sup_{\varphi\in\mathcal{F}}|\Psi_n(\varphi)-c|\to 0) = P(\sup_{\varphi\in\mathcal{F}}|\Psi_n(\varphi)-c|^*\to 0) = 1. \tag{2.2}$$

Here, P^* denotes the outer probability, and $|\Psi_n(\varphi)-c|^*$ is the minimal measurable cover function of $|\Psi_n(\varphi)-c|$.

The following theorem gives the uniform law of large numbers of $U_n(\varphi)$ process.

Theorem 2.1. *Suppose* (**i**)-(**iii**) *are satisfied and $J_{[\,]}(\infty, \mathcal{F}, d_1) < \infty$. Then, as $n \to \infty$,*

$$\sup_{\varphi\in\mathcal{F}}|U_n(\varphi)| \longrightarrow 0, \quad a.s. \tag{2.3}$$

Proof. Let $\epsilon > 0$ is given. By definition of the bracketing number, we can find finitely many ϵ-brackets $[l_i, u_i]$ whose union contains $\mathcal{F}$ and such that $\int \big(u_i - l_i\big)dF < \epsilon$ for every $i = 1, \ldots, N_{[]}(\epsilon)$. Then, for every $\varphi \in \mathcal{F}$, there is bracket such that

$$\begin{aligned} U_n(\varphi) &= \int \varphi d\hat{F} - \int \varphi dF \\ &\leq \int u_i d\big(\hat{F} - F\big) + \int \big(u_i - l_i\big)dF. \end{aligned}$$

Therefore,

$$\sup_{\varphi\in\mathcal{F}} U_n(\varphi) \leq \max_{1\leq i\leq N_{[]}(\epsilon)} \int u_i d\big(\hat{F} - F\big) + \epsilon.$$

According to Proposition 3.2

$$\limsup_{n\to\infty}\sup_{\varphi\in\mathcal{F}} U_n(\varphi) \leq \epsilon \ a.s.$$

Similarly, one can proved that

$$\inf_{\varphi\in\mathcal{F}} U_n(\varphi) \geq -\epsilon, a.s.,$$

and hence

$$\limsup_{n\to\infty}\sup_{\varphi\in\mathcal{F}} \big|U_n(\varphi)\big|^* \leq \epsilon \ a.s.,$$

for every $\epsilon > 0$. Take a sequence $\epsilon_m \downarrow 0$ to see that lim sup must actually be zero almost surely.

□

The uniform law of large numbers for the $\mathbb{R}$-indexed U_n process is stated in the following result.

Corollary 2.1. *Suppose that the assumptions of Theorem* 2.1 *hold. Then, as $n \to \infty$*

$$\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} \varphi(x)d\big(\hat{F}(x) - F(x)\big)\right| \longrightarrow 0, \quad a.s. \tag{2.4}$$

Proof. It is enough to apply Theorem 2.1 to $\mathcal{F} = \{\varphi.I_{(-\infty,t]}, t \in \mathbb{R}\}$, for which the condition $J_{[\ ]}(\infty, \mathcal{F}, d_1) < \infty$ is certainly satisfied. □

2.2. **The Uniform Central Limit Theorem.** In this subsection, we obtain a uniform central limit theorem for the process $U_n^*(\varphi) := \sqrt{n}U_n(\varphi)$. Set $p = 2$ and hence, $\mathcal{F} \subseteq L^2(F)$, we shall use the metric defined by $L^2(F)$, i.e. $d_2(f, g) =\| f - g \|_2$, for $f, g \in \mathcal{F}$. The following definition of weak convergence which is originally due to Hoffman-Jörgensen (1991) will be used.

Definition 2.4. *A sequence of $L^\infty(\mathcal{F})$-valued random functions $\{\Psi_n\}$ converges in law to a $L^\infty(\mathcal{F})$-valued random function Ψ whose law concentrates on a separable subset of $L^\infty(\mathcal{F})$ if*

$$Eg(\Psi) = \lim_{n\to\infty} E^* g(\Psi_n) \quad \forall g \in U(L^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}}),$$

where $U(L^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$ is the set of all bounded, uniformly continuous functions from $(L^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$ into $\mathbb{R}$. Here, E^* denotes the upper expectation with respect to the outer probability P^*. We denote this convergence by $\Psi_n \Rightarrow \Psi$.

The following assumption is imposed throughout this subsection to achieve Donsker type theorem for $U_n^*(\varphi)$.
(**iv**) : Assume that $\exists$ a $\gamma > 0$, for which $F(y) = 0$, $\forall y < \gamma$.

Remark 2.1. *In most practical situations the assumption that survival is certain for some small length of time is not restrictive* (see e.g. Asgharian et al., 2002).

Remark 2.2. *Let f be the density function F. It is possible to replace* (**iv**) *with $E(Y^{-2}) < \infty$ and $\sup_x f(x) < \infty$, and hence the proofs can be rewritten by this new conditions.*

We are ready to state the uniform central limit theorem for the process U_n^*.

Theorem 2.2. *Let $\mathcal{F}$ be the class of functions for which $J(1) < \infty$. Suppose that the assumptions* (**i**)- (**iv**) *are satisfied and $E(Y^{-1}) < \infty$. Then $U_n^* \Rightarrow W$ as elements of $L^\infty(\mathcal{F})$, where, $\{W(\varphi) : \varphi \in \mathcal{F}\}$, is a mean zero Gaussian process which is uniformly continuous in φ with respect to a metric and its covariance function is given by*

$$\mathrm{Cov}(W(\varphi), W(\phi)) = \mathrm{Cov}(\zeta(\varphi), \zeta(\phi)),$$

where

$$\zeta(\varphi) = \frac{\delta\mu}{p(Z)} Z^{-1}\big[\varphi(Z) - \int \varphi dF\big]. \tag{2.5}$$

Proof. Let $\{\zeta_i(\varphi)\}$ be i.i.d. copies of the random variable $\zeta(\varphi)$. Define the partial sum process $\{V_n(\varphi); \varphi \in \mathcal{F}\}$, where

$$V_n(\varphi) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \zeta_i(\varphi). \tag{2.6}$$

According to Lemma 3.3, we can decompose $U_n^*(\varphi)$ into

$$U_n^*(\varphi) = V_n(\varphi) + n^{\frac{1}{2}} R_n(\varphi),$$

where

$$n^{1/2} R_n(\varphi) = o_p(1),$$

Condition $J(1) < \infty$ implies that the metric space $(\mathcal{F}, d_2)$ is totally bounded and hence, the metric entropy integrability condition, stated in Lemma 3.2, is satisfied. Thus, we can apply Theorem 3.1 and Theorem 3.3 of Ossiander (1987) in to the process $\{V_n(\varphi); \varphi \in \mathcal{F}\}$. According to Lemma 3.5, the finite dimentional distributions of $\{U_n^*(\varphi); \varphi \in \mathcal{F}\}$ converge to those of a mean zero Gaussian process $\{W(\varphi);\varphi\in\mathcal{F}\}$. Furthermore, Lemma 3.9 states that U_n^* is tight in the sense that for every $\epsilon>0$, there exists $\delta>0$, such that

$$\limsup_{n\longrightarrow\infty} P^*\{\sup_{d_2(\varphi,\phi)<\delta}\left|U_n^*(\varphi)-U_n^*(\phi)\right|>\epsilon\}<\epsilon.$$

Then, the result follows immediately from Theorem 10.2 of Pollard (1990). □

2.3. **The Uniform LIL.** Again, we assume that $\mathcal{F}$ is a class of real-valued measurable functions on $\mathbb{R}$ such that $\mathcal{F}\subseteq L^2(F):=\{\varphi:\int\varphi^2(x)dF(x)<\infty\}$. It is known that $\mathcal{F}$ satisfies an empirical LIL of Strassen type under certain metric entropy integrability condition. This problem consists of showing the relative compactness of

$$\Big\{\frac{\sum_{i=1}^n\varphi(X_i)}{\sqrt{2n\log\log n}},\varphi\in\mathcal{F},n\geq 3\Big\}, \tag{2.7}$$

and specifying the set of its limit points, see for example Kuelbs and Dudley (1980). In this subsection, we state an empirical LIL for the process $\big\{U_n^*:\varphi\in\mathcal{F}\big\}$.

Theorem 2.3. *Under the stated assumption of Theorem* 2.2*, the process*

$$\left\{\frac{U_n^*(\varphi)}{\sqrt{2\log\log n}}:\varphi\in\mathcal{F},n\geq 3\right\}$$

is relatively compact with respect to $\|\cdot\|_{\mathcal{F}}$ *with probability* 1*, and the set of its limit points is*

$$\mathcal{U}(\mathcal{F}):=\left\{\varphi\to\int\zeta(\varphi).\zeta(g)dF:\varphi\in\mathcal{F},g\in\mathcal{U}\right\},$$

where

$$\mathcal{U}:=\left\{g\in\mathcal{M}:\|\zeta(g)\|^2=\int(\zeta(g))^2dF\leq 1\right\},$$

is the unit ball of Hilbert space $\mathcal{M}:=\big\{\varphi\in L^2(F):\int\zeta(\varphi)dF=0\big\}$.

Proof. Let $\{W(\varphi):\varphi\in\mathcal{F}\}$ be a Gaussian process with bounded and continuous sample paths whose mean is zero and covariance function is

$$EW(\varphi)W(\psi)=E(\zeta(\varphi)\zeta(\psi)). \tag{2.8}$$

Apply Lemma 3.11 to choose a sequence $\{W_i:i\geq 1\}$ of *i.i.d.* copies of $\{W(\varphi):\varphi\in\mathcal{F}\}$ and a measurable sequence $\{Y_n\}$ such that

$$\Big|\Big|\frac{U_n^*-\tilde{W}_n}{\sqrt{2\log\log n}}\Big|\Big|\leq Y_n=o(1),\quad a.s., \tag{2.9}$$

where

$$\tilde{W}_n(\varphi)=\frac{1}{\sqrt{n}}\sum_{i=1}^n W_i.$$

By Proposition 3.1, we satisfy the empirical LIL, that is,

$$\left\{\frac{\sum_{i=1}^n W_i(\varphi)}{\sqrt{2n\log\log n}}:\varphi\in\mathcal{F},n\geq 3\right\}$$

is relatively compact with respect to $\|\cdot\|$ with probability 1, and the set of its limit points is

$$\Im(\mathcal{F}):=\Big\{\varphi\to EW(\varphi)W(g):\varphi\in\mathcal{F},g\in\Im\Big\},$$

where

$$\Im:=\Big\{g\in\mathcal{L}^2(F):EW^2(g)\leq 1\Big\}.$$

This, together with (2.8) and (2.9), completes the proof. □

3. **Proofs**

In this section, we will present some useful lemmas to prove Theorem 2.2 and Theorem 2.3. Put $\mathcal{G} := \{\zeta(\varphi) : \varphi \in \mathcal{F}\}$. Consider $(\mathcal{G}, d_2)$ as the metric space. For the family $\mathcal{F} \subseteq L^2(F)$, we define an envelope by $\Phi(\cdot) = \sup_{f\in\mathcal{F}} |\varphi(\cdot)|$. The following regularity result is satisfied for Φ.

Lemma 3.1. *Suppose that* $J(1) < \infty$. *Then* $\int \Phi^2(x)dF(x)$ *is finite.*

Proof. By definition of $N_{[]}(1)$, there exists $\Big\{[\varphi^l_{0,1}, \varphi^u_{0,1}], \cdots, [\varphi^l_{N_{[]}(1),1}, \varphi^u_{N_{[]}(1),1}]\Big\}$ so that for every $\varphi \in \mathcal{F}$ there exists $0 \le i \le N_{[]}(1)$ satisfying $\varphi^l_{i,1} \le \varphi \le \varphi^u_{i,1}$ and $d_2(\varphi^l_{i,1}, \varphi^u_{i,1}) < 1$. So, it can be easily observed that $\Phi(\cdot) \le \sum_{i=0}^{N_{[]}(1)} \big(|\varphi^l_{i,1}(\cdot)| + |\varphi^u_{i,1}(\cdot)|\big)$. Hence, Since $N_{[]}(1) < \infty$, one can write

$$\int \Phi^2(x)dF(x) < \infty.$$

□

Lemma 3.2. *Let* $J(1) < \infty$. *Suppose the assumptions* (**i**)- (**iv**) *are satisfied and* $E(Y^{-1}) < \infty$. *Then the metric entropy integrability condition*

$$\int_0^1 \big[\ln N_{[]}(\epsilon, \mathcal{G}, d_2)\big]^{1/2} d\epsilon < \infty,$$

holds so that, Theorem 3.1 *and Theorem* 3.3 *of Ossiander* (1987) *are applicable to the process* $\{V_n(\varphi) : \varphi \in \mathcal{F}\}$.

Proof. This result is an easy consequence of Jensen's inequality. Fix $\epsilon >$. From definition of $N_{[]}(\epsilon)$, one can see that, there exists

$$\Big\{\big[\varphi^l_{0,\epsilon}, \varphi^u_{0,\epsilon}\big], \dots, \Big[\varphi^l_{N_{[]}(\epsilon),\epsilon}, \varphi^u_{N_{[]}(\epsilon),\epsilon}\Big]\Big\}$$

so that for every $\varphi \in \mathcal{F}$ there exists $0 \le i \le N_{[]}(\epsilon)$ satisfying $\varphi^l_{i,\epsilon} \le \varphi \le \varphi^u_{i,\epsilon}$ and $d_2(\varphi^l_{i,\epsilon}, \varphi^u_{i,\epsilon}) < \epsilon$. Let $g \in \mathcal{G}$. Then $g = \zeta(\varphi)$ for some $\varphi \in \mathcal{F}$. Set the following brackets for the class $\mathcal{G}$:

$$g^l_{j,\epsilon} := \frac{\delta\mu Z^{-1}}{p(Z)}\big(\varphi^l_{j,\epsilon}(Z) - \int \varphi^u_{j,\epsilon}dF\big), \quad j = 0, \dots, N_{[]}(\epsilon),$$

and

$$g^u_{j,\epsilon} := \frac{\delta\mu Z^{-1}}{p(Z)}\big(\varphi^u_{j,\epsilon}(Z) - \int \varphi^l_{j,\epsilon}dF\big), \quad j = 0, \dots, N_{[]}(\epsilon).$$

One can easily see that $g^l_{j,\epsilon} \le g \le g^u_{j,\epsilon}$. Using Jensen's inequality, we have

$$d_2^2(g^l_{j,\epsilon}, g^u_{j,\epsilon}) \le I_1 + I_2 + I_3,$$

where

$$I_1 = \int \frac{\mu^2}{x^2p^2(x)}\big(\varphi^l_{j,\epsilon}(x) - \varphi^u_{j,\epsilon}(x)\big)^2 dH_1^*(x),$$

$$I_2 = \Big(\int \frac{\mu^2x^{-2}}{p^2(x)}dH_1^*(x)\Big)d_2(\varphi^u_{j,\epsilon}, \varphi^l_{j,\epsilon}),$$

$$I_3 = 2\Big(\int \frac{\mu^2x^{-2}}{p^2(x)}\big(\varphi^u_{j,\epsilon}(x) - \varphi^l_{j,\epsilon}(x)\big)dH_1^*(x)\Big)\Big(\int (\varphi^u_{j,\epsilon} - \varphi^l_{j,\epsilon})dF\Big),$$

and $H_1^*(u) = P\big(Z \leq u, \delta = 1 | Z \geq T\big)$. Under (**i**)-(**iii**), one can check that

$$H_1^*(x) = \mu^{-1} \int_0^x up(u) dF(u), \tag{3.1}$$

where $p(y)$ is defined in (1.5). Under assumptions (**i**)-(**iii**) it is easily seen that

$$p(y) = I\{y \leq \tau\} + \frac{\tau}{y} I\{y > \tau\}.$$

Hence, a straightforward calculation shows that

$$\begin{aligned}
I_1 &= \int \frac{\mu}{xp(x)} \big(\varphi^l_{j,\epsilon}(x) - \varphi^u_{j,\epsilon}(x)\big)^2 dF(x) \\
&= \int_{x \leq \tau} \frac{\mu}{x} \big(\varphi^l_{j,\epsilon}(x) - \varphi^u_{j,\epsilon}(x)\big)^2 dF(x) \\
&+ \int_{x > \tau} \frac{\mu}{\tau} \big(\varphi^l_{j,\epsilon}(x) - \varphi^u_{j,\epsilon}(x)\big)^2 dF(x) \\
&\leq \frac{\mu}{\gamma} d_2^2(\varphi^l_{j,\epsilon}, \varphi^u_{j,\epsilon}) + \frac{\mu}{\tau} d_2^2(\varphi^l_{j,\epsilon}, \varphi^u_{j,\epsilon}) \\
&= \big[\frac{\mu}{\gamma} + \frac{\mu}{\tau}\big] d_2^2(\varphi^l_{j,\epsilon}, \varphi^u_{j,\epsilon}),
\end{aligned}$$

and similarly

$$I_2 \leq \big(\mu E(Y^{-1}) + \frac{\mu}{\tau}\big) d_2^2(\varphi^u_{j,\epsilon}, \varphi^l_{j,\epsilon}).$$

Also, by Jensen inequality, one may write

$$I_3 \leq \big(\frac{2\mu}{\gamma} + \frac{2\mu}{\tau}\big) d_2^2(\varphi^l_{j,\epsilon}, \varphi^u_{j,\epsilon}).$$

Hence, there exists a constant C satisfying

$$d_2^2(g^l_{j,\epsilon}, g^u_{j,\epsilon}) \leq C d_2^2(\varphi^l_{j,\epsilon}, \varphi^u_{j,\epsilon}).$$

Now Condition $J(1) < \infty$ implies that the integrability condition

$$\int_0^1 \big[\ln N_{[]}(\epsilon, \mathcal{G}, d)\big]^{1/2} d\epsilon < \infty,$$

holds. □

Lemma 3.3. *Suppose the assumption* (**i**)-(**iv**) *and* $E(Y^{-1}) = \int u^{-1} dF(u) < \infty$ *are satisfied. Then for each fixed* $\varphi \in \mathcal{F}$, *one can see that*

$$U_n^*(\varphi) = n^{-1/2} \sum_{i=1}^n \zeta_i(\varphi) + n^{1/2} R_n(\varphi),$$

where

$$n^{1/2} R_n(\varphi) = o_p(1),$$

and $\zeta_i(\varphi)$ *are i.i.d. copies of the random variable* $\zeta(\varphi)$ *given by* (2.5).

Proof. We start by writing

$$\int \varphi d\widehat{F} - \int \varphi dF = \mu \int \varphi^0 d\widehat{F}^* + R_n(\varphi), \tag{3.2}$$

where

$$\varphi^0(u) = u^{-1} \big[\varphi(u) - \int \varphi dF\big],$$

and

$$R_n(\varphi) = -\mu\Big(\int \varphi d\widehat{F} - \int \varphi dF\Big)\big(\frac{1}{\tilde{\mu}} - \frac{1}{\mu}\big), \tag{3.3}$$

where $\tilde{\mu} = \Big(\int_0^\infty u^{-1}d\widehat{F}^*(u)\Big)^{-1}$. According to (1.3) and the stated assumption of this lemma, we can conclude easily that $\int u^{-2}d(F^*(u)) = \frac{1}{\mu}\int u^{-1}dF(u) < \infty$. Hence, we can apply the Central Limit Theorem to get

$$\begin{aligned}\sqrt{n}\big(\frac{1}{\tilde{\mu}} - \frac{1}{\mu}\big) &= \int u^{-1}d\widehat{F}^*(u) - \int u^{-1}dF^*(u) \\ &= O_p(1). \end{aligned}\tag{3.4}$$

Direct algebra gives

$$\int \varphi d\widehat{F} - \int \varphi dF := I_n(\varphi) + II_n(\varphi), \tag{3.5}$$

where

$$I_n(\varphi) = \frac{1}{\int_0^\infty u^{-1}dF^*(u)}\Big\{\int_0^\infty \varphi(u)u^{-1}d\widehat{F}^*(u) - \int_0^\infty \varphi(u)u^{-1}dF^*(u)\Big\},$$

and

$$II_n(\varphi) = \frac{\int_0^\infty \varphi(u)u^{-1}d(\widehat{F}^*(u))}{(\int_0^\infty u^{-1}dF^*(u))(\int_0^\infty u^{-1}d\widehat{F}^*(u))}\Big\{\int_0^\infty u^{-1}dF^*(u) - \int_0^\infty u^{-1}d\widehat{F}^*(u)\Big\}.$$

The Central limit Theorem implies

$$\sqrt{n}\Big\{\int_0^\infty \varphi(u)u^{-1}d\widehat{F}^*(u) - \int_0^\infty \varphi(u)u^{-1}dF^*(u)\Big\} = O_p(1), \tag{3.6}$$

and

$$\sqrt{n}\Big\{\int_0^\infty u^{-1}d\widehat{F}^*(u) - \int_0^\infty u^{-1}dF^*(u)\Big\} = O_p(1), \tag{3.7}$$

provided that $\int \varphi^2(u)u^{-2}dF^*(u)$ and $\int u^{-2}dF^*(u)$ exist respectively. According to (1.3), it follows that

$$\int \varphi^2(u)u^{-2}dF^*(u) = \mu\int \varphi^2(u)u^{-1}dF(u) < \frac{\mu}{\gamma}\int_\gamma^\infty \varphi^2(u)dF(u),$$

and

$$\int u^{-2}dF^*(u) = \mu\int u^{-1}dF(u) < \infty.$$

The condition of F-integrability of φ^2 and the assumption $\int u^{-1}dF(u) < \infty$, are enough to conclude consistency. Putting together with (3.5)-(3.7), we have

$$\sqrt{n}\Big(\int \varphi d\widehat{F} - \int \varphi dF\Big) = O_p(1).$$

This last result and (3.4) ensure

$$\sqrt{n}R_n(\varphi) = o_p(1). \tag{3.8}$$

The result follows from (3.2) and (3.8). □

Remark 3.1. *With the choice of* $\varphi(x) = I_{(-\infty,x]}$, $x \in \mathbb{R}$, *Lemma* 3.3 *will be reduced to Theorem* 3.3. *of De Uña-Álvarez* (2004).

The following lemma will be used to prove the finite dimensional distribution convergence of the process $U_n^*(\varphi)$ for $\varphi \in \mathcal{F}$.

Lemma 3.4. *Under conditions of Lemma* 3.3, *for each fixed* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, *for which* $\int \varphi^2(x)dF(x) < \infty$, *we have*

$$\sqrt{n}\int \varphi d(\hat{F}-F) \xrightarrow{d} N(0, \mathrm{Var}(\zeta(\varphi))).$$

Proof. The given representation in Lemma 3.3 together with the Central Limit Theorem gives the result. □

Let $\{W(\varphi) : \varphi \in \mathcal{F}\}$ be the mean zero Gaussian process with

$$\mathrm{Cov}(W(\varphi), W(\phi)) = \mathrm{Cov}(\zeta(\varphi), \zeta(\phi)).$$

Lemma 3.5. *Assume that* $E(Y^{-1}) < \infty$. *Then under* (**i**)-(**iv**), *the finite dimensional distributions of* U_n^* *converge to those of* W.

Proof. First note that by Lemma 3.3, one can rewrite $U_n^*(\varphi)$ as

$$U_n^*(\varphi) = V_n(\varphi) + n^{1/2}R_n(\varphi), \quad for \quad \varphi \in \mathcal{F},$$

where $V_n(\varphi)$ is defined in (2.6) and $R_n(\varphi)$ is given by (3.3). In order to get the one dimensional central limit theorem for U_n^*, one can use Lemma 3.4 and Slutsky Theorem for each fixed $\varphi \in \mathcal{F}$. Then, the Cramér-Wold device concludes the result. □

Let us to define the partial-sum process

$$\begin{aligned} S_n(\varphi) &= n^{1/2}\Big(\int_0^\infty \varphi(u)u^{-1}d\widehat{F}^*(u) - \int_0^\infty \varphi(u)u^{-1}dF^*(u)\Big) \\ &= n^{-\frac{1}{2}}\sum_{i=1}^n {\zeta_i}'(\varphi) \quad for \quad \varphi \in \mathcal{F}, \end{aligned}$$

where

$$\zeta^{'}(\varphi) = \frac{\varphi(Z)\delta}{Zp(Z)} - \int_0^\infty \varphi(u)u^{-1}dF^*(u),$$

and the metric space $(\mathcal{G}^{'}, d_2)$ with

$$\mathcal{G}^{'} := \{\zeta^{'}(\varphi) : \varphi \in \mathcal{F}\}.$$

The following lemma allows us to apply Theorem 3.3 of Ossiander (1987) to the process $\{S_n(\varphi) : \varphi \in \mathcal{F}\}$.

Lemma 3.6. *Suppose the assumption* (**i**)-(**iv**), *and the condition* $J(1) < \infty$ *are satisfied. Then, the metric entropy integrability condition*

$$\int_0^1 \big[\ln N_{[]}(\epsilon, \mathcal{G}^{'}, d_2)\big]^{1/2} d\epsilon < \infty$$

holds.

Proof. Let $\epsilon > 0$. The proof of Lemma 3.6 is similar to that of Lemma 3.2 with the brackets

$$g_{j,\epsilon}^l := \frac{\varphi_{j,\epsilon}^l(Z)\delta}{Zp(Z)} - \int \varphi_{j,\epsilon}^u(u)u^{-1}dF^*(u), \quad j = 0, \ldots, N_{[]}(\epsilon),$$

and

$$g_{j,\epsilon}^u := \frac{\varphi_{j,\epsilon}^u(Z)\delta}{Zp(Z)} - \int \varphi_{j,\epsilon}^l(u)u^{-1}dF^*(u), \quad j = 0, \ldots, N_{[]}(\epsilon),$$

for the class $\mathcal{G}^{'}$ and is omitted. □

Lemma 3.7. *Suppose the assumption* (**i**)-(**iv**), *and the condition* $J(1) < \infty$ *are satisfied. Then, for* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\limsup_{n\to\infty} P^*\Big\{ \sup_{d_2(\varphi,\phi)<\delta} \big|S_n(\varphi) - S_n(\phi)\big| > \epsilon \Big\} < \epsilon.$$

Proof. According to Lemma 3.6, $\int_0^1 \big[\ln N_{[]}(\epsilon, \mathcal{G}', d_2)\big]^{1/2} d\epsilon < \infty$ so that, Theorem 3.3 of Ossiander (1987) can be applied to the process $\big\{S_n(\varphi) : \varphi \in \mathcal{F}\big\}$. □

To get a uniform tightness for $\big\{U_n^*(\varphi) : \varphi \in \mathcal{F}\big\}$, the next lemma will be crucial.

Lemma 3.8. *Suppose the stated assumption of Theorem* 2.2 *hold. Then for all* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\limsup_{n\to\infty} P^*\big\{ \sup_{d_2(\varphi,\phi)<\delta} n^{1/2}\big|R_n(\varphi) - R_n(\phi)\big| > \epsilon \big\} < \epsilon.$$

Proof. According to Lemma 3.3, $R_n(\varphi)$ can be written as

$$R_n(\varphi) = R_{n1}(\varphi) + R_{n2}(\varphi),$$

where

$$R_{n1}(\varphi) = -\mu[\frac{1}{\tilde{\mu}} - \frac{1}{\mu}]I_n(\varphi),$$

and

$$R_{n2}(\varphi) = -\mu[\frac{1}{\tilde{\mu}} - \frac{1}{\mu}]II_n(\varphi).$$

Notice that

$$\begin{aligned} n\big|R_n(\varphi) - R_n(\phi)\big| &\leq n\big|R_{n1}(\varphi) - R_{n1}(\phi)\big| + 2n\| R_{n2} \|_{\mathcal{F}} \\ &= \mu^2\sqrt{n}\Big|\frac{1}{\tilde{\mu}} - \frac{1}{\mu}\Big|\big|S_n(\varphi) - S_n(\phi)\big| + 2n \| R_{n2} \|_{\mathcal{F}} \end{aligned}$$

Therefore,

$$\text{(3.9)} \quad P^*\big(\sup_{d_2(\varphi,\phi)<\delta} n^{1/2}\big|R_n(\phi) - R_n(\varphi)\big| > \epsilon\big)$$
$$\leq P^*\big(\mu^2\sqrt{n}(\frac{1}{\tilde{\mu}} - \frac{1}{\mu}) \sup_{d_2(\varphi,\phi)<\delta} \big|S_n(\phi) - S_n(\varphi)\big| > \frac{\epsilon}{2}\big) + P^*\big(2n\big|\big|R_{n2}\big|\big|_{\mathcal{F}} > \frac{\epsilon}{2}\big).$$

Lemma 3.7 together with (3.4) yields

$$P^*\Big(\mu^2\sqrt{n}(\frac{1}{\tilde{\mu}} - \frac{1}{\mu}\Big) \sup_{d_2(\varphi,\phi)<\delta} \big|S_n(\phi) - S_n(\varphi)\big| > \frac{\epsilon}{2}\big) \leq \frac{\epsilon}{2}. \tag{3.10}$$

To deal with the term $P^*\big(2n\big|\big|R_{n2}\big|\big|_{\mathcal{F}} > \frac{\epsilon}{2}\big)$, one can observe that

$$\begin{aligned} 2n\big|\big|R_{n2}\big|\big|_{\mathcal{F}} &\leq 2\mu\sqrt{n}\Big|\frac{1}{\tilde{\mu}} - \frac{1}{\mu}\Big|\frac{\int_0^\infty \sup_{\varphi\in\mathcal{F}} |\varphi|u^{-1}d\big(\widehat{F}^*(u)\big)}{\big(\int u^{-1}dF^*(u)\big)\big(\int u^{-1}d\hat{F^*}(u)\big)} \\ &\times \sqrt{n}\Big|\int u^{-1}dF^*(u) - \int u^{-1}d\widehat{F}^*(u)\Big|. \end{aligned} \tag{3.11}$$

Now, Lemma 3.1 and the Strong law of large numbers give that

$$\int_0^\infty \sup_{\varphi\in\mathcal{F}} |\varphi|u^{-1}d\big(\widehat{F}^*(u)\big) \longrightarrow \int_0^\infty \sup_{\varphi\in\mathcal{F}} |\varphi|u^{-1}d\big(F^*(u)\big) < \infty. \tag{3.12}$$

Combining (3.11) and (3.12) together with (3.4) and (3.7), one can find

$$P^*\big(2n\big|\big|R_{n2}\big|\big|_{\mathcal{F}} > \frac{\epsilon}{2}\big) < \frac{\epsilon}{2}. \tag{3.13}$$

Equations (3.9), (3.10) and (3.13) complete the proof. □

Lemma 3.9. *Suppose the stated assumption of Theorem* 2.2 *hold. Then, for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\limsup_{n\longrightarrow\infty} P^*\big\{ \sup_{d_2(\varphi,\phi)<\delta} \big|U_n^*(\varphi) - U_n^*(\phi)\big| > \epsilon\big\} < \epsilon.$$

Proof. With a simple algebra

$$|U_n^*(\varphi) - U_n^*(\phi)| \leq |V_n(\varphi) - V_n(\phi)| + \sqrt{n}\big|R_n(\varphi) - R_n(\phi)\big|.$$

Hence, Theorem 3.3 of Ossiander (1987) together with Lemma 3.2 and Lemma 3.8, one can conclude that

$$\begin{aligned} P^*\big(\sup_{d_2(\varphi,\phi)<\delta} \big|U_n^*(\varphi) - U_n^*(\phi)\big| > 3\epsilon\big) &\leq P^*\big(\sup_{d_2(\varphi,\phi)<\delta} \big|V_n(\phi) - V_n(\varphi)\big| > \epsilon\big) \\ &+ P^*\big(\sup_{d_2(\varphi,\phi)<\delta} \big|R_n(\phi) - R_n(\varphi)\big| > 2\epsilon\big) \\ &< 3\epsilon \end{aligned}$$

□

In the following, we mention some auxiliary results to provide a proof for Theorem 2.3.

Lemma 3.10. *Assume that conditions of Theorem* 2.2 *hold.There exists a sequence* $W_1, W_2, \dots$ *i.i.d. copies of* $\{W(\varphi) : \varphi \in \mathcal{F}\}$ *such that*

$$n^{\frac{1}{2}} \sup_{\varphi\in\mathcal{F}} \left| \int \varphi d(\hat{F}_n - F - \overline{W}_n) \right| \xrightarrow{p} 0, \quad as \quad n \to \infty,$$

where

$$\overline{W}_n = \frac{W_1 + W_2 + \ldots + W_n}{n}.$$

The W_i*'s can also be chosen such that, for some measurable* Y_n

$$n^{\frac{1}{2}} \sup_{\varphi\in\mathcal{F}} \left| \int \varphi d(\hat{F}_n - F - \overline{W}_n) \right| \leq Y_n = o\left(\sqrt{\log\log n}\right) \; a.s.$$

Proof. This lemma is a restatement of Theorem 2.2. See also Theorem 1.3 of Dudley and Philipp (1983) and Theorem 4.1 of Ossiander (1987). □

Lemma 3.11. *Under the assumptions of Lemma* 3.10*, there exists a sequence* $\{\tilde{W}_n : n \geq 1\}$*, with bounded and continuous sample paths, of copies of a Gaussian process* $\{W(\varphi) : \varphi \in \mathcal{F}\}$ *defined on* $(\Omega, \mathcal{T}, P)$ *such that* $\| U_n^* - \tilde{W}_n \| \xrightarrow{p} 0 \quad as \quad n \to \infty$*. The* W_i*'s can also be chosen such that for some measurable* Y_n,

$$\| U_n^* - \tilde{W}_n \| \leq Y_n = o(\sqrt{\log\log n}) \; a.s.$$

Proof. According Lemma 3.10, we can define the process $\big\{\tilde{W}_n(\varphi); \varphi \in \mathcal{F}\big\}$, where

$$\tilde{W}_n(\varphi) = n^{\frac{1}{2}} \int \varphi(x)\overline{W}_n(dx) \qquad \varphi \in \mathcal{F}.$$

The process $\big\{\tilde{W}_n(\varphi); \varphi \in \mathcal{F}\big\}$ is a Gaussian process with mean zero and the covariance function

$$\mathrm{Cov}(\tilde{W}_n(\varphi), \tilde{W}_n(\psi)) = \mathrm{Cov}(\zeta(\varphi), \zeta(\psi)),$$

where $\zeta(.)$ is given in (2.5). Observe that $U_n^* \Rightarrow W$ as a random elements of $L^\infty(\mathcal{F})$. Observe also that

$$\| U_n^* - \tilde{W}_n \| = n^{1/2} \sup_{\varphi\in\mathcal{F}} \left| \int \varphi d(\widehat{F}_n - F - \overline{W}_n) \right|.$$

Since the Gaussian processes W and $\tilde{W}$ have the same mean and covariance structure, they have the same distribution. So, Theorem 2.2 implies the results. □

Proposition 3.1. (Theorem 4.3 of Pisier, 1975) *Suppose* $J(1) < \infty$. *Let* $\{W_i : i \geq 1\}$ *be a sequence of i.i.d. copies of a Gaussian process* $\{W(\varphi) : \varphi \in \mathcal{F}\}$ *defined on* $(\Omega, \mathcal{T}, P)$. *Suppose* $\{W(\varphi) : \varphi \in \mathcal{F}\}$ *has bounded and continuous sample paths with* $E(W(\varphi)) = 0$ *and* $E\| W \|^2 < \infty$. *Then* W *satisfies the empirical LIL. That is,*

$$\left\{ \frac{\sum_{i=1}^n W_i(\varphi)}{\sqrt{2n \log\log n}} : \varphi \in \mathcal{F}, n \geq 3 \right\}$$

is relatively compact with respect to $\| \cdot \|$ *with probability* 1, *and the set of its limit points is*

$$\Im(\mathcal{F}) = \Big\{ \varphi \to EW(\varphi)W(g) : \varphi \in \mathcal{F}, g \in \Im \Big\},$$

where

$$\Im = \Big\{ g \in L^2(F) : EW^2(g) \leq 1 \Big\}.$$

Proposition 3.2. (Theorem 3.2 of De Uña-Álvarez, 2004) *For each* F*-integrable* φ*, we have*

$$\int \varphi d\hat{F} \to \int \varphi dF \ a.s.$$

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