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ON A DISCRETE EXTREMAL PROBLEM WITH CONSTRAINTS

The results on the existence of solutions for some discrete extremal problems with constraints were established. As an application the existence of a solution of a non-linear eigenvalue problem was obtained.

Variational methods constitute effective means for studying nonlinear boundary value problems for differential equations. At the same time, there are practically no works where these methods are used for difference equations. In the case of discrete Schrödinger equations on the infinite lattice the first results of this type were obtained in [1]. A similar approach is applied in this paper to examine a variational problem with constraints associated with a discrete nonlinear eigenvalue problem of the type

$$(Au)(n) = \lambda f(n, u(n)), n \in \mathbb{Z}^d. \tag{1}$$

This problem is a discrete analogue of the problem considered in [2].

Let us formulate the main assumptions. The operator A has the following form:

$$(Au)(n) = \sum_{m\in\mathbb{Z}^d} a(n, m)u(m), \tag{2}$$

where $\{a(n, m)\}$ — is a real symmetric infinite matrix that satisfies the condition:

(a1) $sup|a(n, m)| < \infty$ and there is such positive integer N_0, that $a(n, m) = 0$ if $|n - m| > N_0$. The operator A is a bounded self-adjoint operator in $l^2 = l^2\left(\mathbb{Z}^d\right),\ 1 \leqslant p \leqslant \infty$, for the space of all real p-summable sequences, indexed with $\mathbb{Z}^d$ elements [1].

It is also assumed that:

(a2) operator A is positive definite, that is there is such $\alpha_0 > 0$, that $(Au)(n) \geqslant \geqslant \alpha_0\|u\|^2_{l^2}$ for all $u \in l^2$.

Let us formulate the rest of the conditions on the operator A. It is assumed that

$$(a)(n, m) = \overline{a}(n, m) + a_0(n, m), \tag{3}$$

(a3) there exists a vector $k_0 \in \mathbb{Z}^d$, such that $\overline{a}(n + k_0,\ m + k_0) = \overline{a}(n, m)$,

(a4) $a_0(n + k,\ m + k) \to 0$ at $k \to 0$ for all $n, m \in \mathbb{Z}^d$.

According to (3) the operator A can be written as $A = \overline{A} + A_0$, where $\overline{A}$ and A_0 correspond to the matrices $\{\overline{a}(n, m)\}$ and $\{a_0(n, m)\}$ respectively.

Let us proceed to the conditions on the nonlinear term $f(n, u)$. Let f be defined as $f(n, u) = \overline{f}(n, u) + f_0(n, u)$, where $\overline{f}$ and f_0 are continuous in $u \in \mathbb{R}$, and the following conditions are fulfilled:

(f1) $\overline{f}(n + k_0, u) = \overline{f}(n, u)$;

(f2) for any $l > 0$ $\lim\limits_{n\to\infty} f_0(n, u) = 0$ uniformly on $u \in [-l, l]$;

(f3) there are such $p > 0$ and $c > 0$, that $|\overline{f}(n, u)| + |f_0(n, u)| \leqslant c|u|^{p-1},\ |u| \leqslant 1$;

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(f4) for some $n_0 \in \mathbb{Z}^d$ there is a sequence $u_k \in \mathbb{R}$, $\lim\limits_{k\to\infty} u_k = 0$, such that $f(n_0, u_k) > > 0$.

Let us set $F(n,u) = \int\limits_0^u f(n,y)dy$, $\overline{F}(n,u) = \int\limits_0^u \overline{f}(n,y)dy$ and determine the functionals of the class C^1 for l^1:

$$E(u) = \tfrac{1}{2}(Au,u), \qquad \overline{E}(u) = \tfrac{1}{2}(\overline{A}u,u),$$
$$\Phi(u) = \sum_{n\in\mathbb{Z}^d} F(n,u(n)), \quad \overline{\Phi}(u) = \sum_{n\in\mathbb{Z}^d} \overline{F}(n,u(n))$$

Let us consider the extremal problems for $q \geq 0$:

$$I_q = \inf\left\{E(u) : u \in l^2, \Phi(u) = q\right\}, \tag{4}$$

$$\overline{I}_q = \inf\left\{\overline{E}(u) : u \in l^2, \overline{\Phi}(u) = q\right\}. \tag{5}$$

It can be shown that for any $r \in [0,q]$ the following inequalities are valid:
$I_q \leqslant I_r + \overline{I}_{q-r}$, $\overline{I}_q \leqslant \overline{I}_r + \overline{I}_{q-r}$.

Furthermore, $I_0 = \overline{I}_0 = 0$ and I_q, $\overline{I}_q$ are continuous functions of $q \geqslant 0$.

Theorem 1. *Let the conditions $(a1)-(a4)$ and $(f1)-(f4)$ be fulfilled. For every minimizing sequence of the problem (4) to be relatively compact in l^2, it is necessary and sufficient that the following inequality is satisfied:*

$$I_q < I_r + \overline{I}_{q-r}, \ \forall r \in [0,q) \tag{6}$$

For every minimizing sequence of the problem (5) to be relatively compact in l^2 with regard to shifts, it is necessary and sufficient that the following inequality is satisfied:

$$\overline{I}_q < \overline{I}_r + \overline{I}_{q-r}, \ \forall r \in (0,q). \tag{7}$$

In particular, the inequality (6) (respectively (7)) guarantees the existence of the problem's (4) (respectively (5)) solution.

Proof. Let u_k be a minimizing sequence for (6). Then u_k is bounded in l^2. Without loss of generality it can be assumed that $\|u_k\|_{l^2}^2 \to \gamma > 0$, as $k \to \infty$. By virtue of Lemma 1 [1] one of the following three cases must occur:

(i) there is such $m_k \in \mathbb{Z}^d$, that for any $\varepsilon > 0$ there will be such $r > 0$, that $\sum\limits_{n\in m_k+K_t} u_k^2(n) \leq \gamma - \varepsilon$;

(ii) $\lim\limits_{k\to\infty} \|u_k\|_{l^\infty} = 0$;

(iii) there are $\beta \in (0,\gamma)$ and such sequences with bounded support $u_{k,1}$, $u_{k,2}$, that $\lim\limits_{k\to\infty} \|u_k - (u_{k,1}+u_{k,2})\|_{l^2} = 0$, $\lim\limits_{k\to\infty} \|u_{k,1}\|_{l^2}^2 = \beta$, $\lim\limits_{k\to\infty} \|u_{k,2}\|_{l^2}^2 = \gamma - \beta$ and

$$dist(supp(u_{k,1}), supp(u_{k,2})) \to \infty \text{ as } k \to \infty.$$

In the case (ii) it follows from Lemma 2 [1] that $\lim\limits_{k\to\infty} u_k = 0$ in l^p, $2 < p \leqslant \infty$. Then by virtue of the condition $(f3)$, $\Phi(u_k) \to 0$, as $k \to \infty$, which contradicts the condition $\Phi(u_k) = q > 0$. Thus, the case (ii) is excluded.

In the case (iii) we have $q = \lim\limits_{k\to\infty} \Phi(u_{k,1}+u_{k,2}) = \lim\limits_{k\to\infty} \lfloor \Phi(u_{k,1}) + \Phi(u_{k,2}) \rfloor$, since the supports of $u_{k,1}$ and $u_{k,2}$ do not intersect for large k. Without loss of generality it can be assumed that the following limits exist: $\lim\limits_{k\to\infty} \Phi(u_{k,1}) = q_1$, $\lim\limits_{k\to\infty} \Phi(u_{k,2}) = q_2$ and

$q_1 + q_2 = q$. It is easy to check that $q_1 > 0$ and $q_2 > 0$. Passing to the subsequences (and interchanging, if necessary, the positions of $u_{k,1}$ and $u_{k,2}$) it can be assumed that
$\lim_{k\to\infty} \lfloor \Phi(u_{k,2}) - \overline{\Phi}(u_{k,2}) \rfloor = 0, \ \lim_{k\to\infty} \lfloor E(u_{k,2}) - \overline{E}(u_{k,2}) \rfloor \geqslant 0.$
Hence it follows that

$$I_q = \lim_{k\to\infty} E(u_k) = \lim_{k\to\infty} [E(u_{k,1}) + E(u_{k,2})] \geqslant I_{q1} + \lim_{k\to\infty} \overline{E}(u_{k,2}) = I_{q1} + I_{q2},$$

which contradicts (6).

Thus only the case (i) remains. In this case the sequence $u_k(\circ + m_k)$ is relatively compact in l^2. If the sequence m_k is bounded, then the sequence u_k itself is relatively compact. If m_k is unbounded then, like before, it follows that $I_q \geqslant \overline{I}_q$. The latter contradicts (6) for $r = 0$. This proves the sufficiency in the first theorem statement. The necessity is established in a similar way [3].

The second statement is proved in a similar way.

□

Now let us consider the periodic problem, similar to (5). Let us set
$Q_m = \left\{ n \in \mathbb{Z}^d : -\frac{m_j - 1}{2} < n_j \leqslant \frac{m_j - 1}{2}, \ j = 1, \ldots, d \right\}$ for the vector $m \in \mathbb{Z}^d$ with positive components.

Furthermore, without loss of generality it can be assumed that the vector $k_0 \in \mathbb{Z}^d$ under the conditions $(a3), (f1)$ has positive components. Let $s > 0$ be an integer. Let us denote by H_s the finite-dimensional space of sk_0-periodic sequences with the scalar product $(u, v)_s = \sum_{n \in Q_{Sk_0}} u(n)v(n)$. Obviously, the operator $\overline{A}$ acts in H_s.

Let us set $E_s(u) = \dfrac{1}{2}(\overline{A}u, u)_s, \ \Phi_s(u) = \sum_{n \in Q_S k_0} \overline{F}(n, u(n))$.

It is easy to see that the finite-dimensional extremal problem
$I_q^{(s)} = inf \left\{ E_s(u) : u \in H_s, \ \Phi_s(u) = q \right\}, \ q \geqslant 0$, always has at least one solution u_s.

Theorem 2. *Let the conditions* $(a1)-(a2)$, $(f1)-(f4)$ *and the inequality (7) be fulfilled. Then there exist solutions of the problem (5), a sequence* $s_k \in \mathbb{N}$ *and a sequence* $m_k \in \mathbb{Z}^d$*, such that* $\lim_{j\to\infty} [\chi_{s_j} \cdot u_{s_j}(\circ + m_j) - u] = 0$ *in the space* l^2*, where* $\chi_s(n) = 1$ *if* $n \in Q_{sk_0}$ *and* $\chi_s(n) = 0$ *if* $n \notin Q_{sk_0}$.

The theorem is proved similarly to Theorem 1.

Let us apply Theorem 1 to the problem (1).

Theorem 3. *Let the conditions* $(a1) - (a2)$, $(f1) - (f4)$ *be fulfilled and one of the following inequalities hold:* $f(n,u)u > 0, \forall n \in \mathbb{Z}^d, \ u \neq 0$, *or* $f(n,u) \geqslant cF(n,u), \forall n \in \mathbb{Z}^d, \ u \in \mathbb{R}$*, where* $c > 0$*. If the inequality (6) is fulfilled, then there is a solution* $(u, \lambda), \ u \in l^2, \ u \neq 0, \ \lambda > 0$*, of the problem (1) with* $\Phi(u) = q$*. If* $a_0(n,m) \equiv 0, \ f_0(n,m) \equiv 0$*, then for the existence of such a solution it is sufficient that the inequality (7) is fulfilled.*

Proof. According to Theorem 1 there is a solution $u \neq 0$ of the problem (4). Obviously, $\bigtriangledown\Phi(u) = \{f(n, u(n))\}$. If $f(n,u)u > 0, \ u \neq 0$, then
$(\bigtriangledown\Phi(u), u) = \sum_{n \in \mathbb{Z}^d} f(n, u(n))u(n) > 0.$

If $f(n,u)u \geq 0, \ u \neq 0$, then
$(\bigtriangledown\Phi(u), u) = \sum_{n \in \mathbb{Z}^d} f(n, u(n))u(n) \geqslant c \sum_{n \in \mathbb{Z}^d} F(n, u(n)) = c\Phi(u) = cq > 0.$

In both cases $\bigtriangledown\Phi(u) \neq 0$. As $\Delta E(u) = Au$, then by virtue of the Lagrange multiplier rule (see, for instance, [4]), there is such $\lambda \in \mathbb{R}$, that
$(Au)(n) = \lambda f(n, u(n)).$

Multiplying this equation scalarly by u and considering that $(Au,u)>0,\ (f(n,u),u)>0$, we obtain that $\lambda>0$.

The theorem is proved. □

Let us consider a partial case $f(n,u)=K(n)|u|^{p-2}u$, where $p>0,\ K(n)=\overline{K}(n)+$ $+K_0(n)>0,\ \overline{K}(n)>0,\ K_0(n)$ is periodic, $\lim\limits_{n\to\infty}K_0(n)=0$.

In this case it is easy to see that $I_q=q^{\frac{2}{p}}I_1>0,\ \overline{I}_q=q^{\frac{2}{p}}\overline{I}_1>0$, hence it follows that the inequality (7) is always fulfilled, and the inequality (6) is fulfilled if and only if $I_1<\overline{I}_1$.

In particular, if $a_0(n,m)\equiv 0,\ K_0(n)\equiv 0$, then there is a solution $u\in l^2,\ u\neq 0$ of equation

$$Au=\lambda K(n)|u|^{p-2}u,\ \lambda>0.$$

Hence it follows that $\nu=\lambda^{\frac{1}{p-2}}u\neq 0$ is the equation's solution

$$A\nu=K(n)|\nu|^{p-2}\nu,\ \lambda>0.$$

The existence of solutions of the last equation also follows from the results of [1].

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