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ON A LIMIT BEHAVIOUR OF A RANDOM WALK PENALISED IN THE LOWER HALF-PLANE

We consider a random walk $\tilde{S}$ which has different increment distributions in positive and negative half-planes. In the upper half-plane the increments are mean-zero i.i.d. with finite variance. In the lower half-plane we consider two cases: increments are positive i.i.d. random variables with either a slowly varying tail or with a finite expectation. For the distributions with a slowly varying tails, we show that $\{\frac{1}{\sqrt{n}}\tilde{S}(nt)\}$ has no weak limit in $\mathcal{D}[0,1]$; alternatively, the weak limit is a reflected Brownian motion.

1. Introduction and main results

Let random variables $\{\xi_n\}_{n\geq 1}$ be i.i.d. with a generic copy ξ and $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = \sigma^2$; $\{\eta_n\}_{n\geq 1}$ are positive i.i.d., independent from the previous ones and with a generic copy η. We consider a random walk $\tilde{S}$:

$$\tilde{S}(0) = 0,\ \tilde{S}(n+1) = \begin{cases} \tilde{S}(n) + \xi_{n+1} \text{ if } \tilde{S}(n) > 0, \\ \tilde{S}(n) + \eta_{n+1} \text{ if } \tilde{S}(n) \leq 0. \end{cases} \tag{1}$$

If $\tilde{S}(n) \in (0,\infty)$, then the random walk has increments distributed as ξ and on $(-\infty, 0]$ as η. Jumps $\{\eta_n\}$ pushes the random walk up from the negative half-plane. This can be interpreted as some non-immediate reflection to the positive half-plane. Similar approach for stochastic differential equations is called the penalization method and leads to a reflected diffusion if penalizations below zero are big enough, see for example [7] or [8, §1.4].

An interpretation of this model comes from the evaporation of gas from liquid in physics, for example, a model of ballistic evaporation [5]. The usual form of speed distribution of a particle in a medium is Maxwell distribution. But in case when there is a medium change (a particle bounces off liquid) several papers have indicated that super- or sub-Maxwell distributions could apply [5], [4], [6].

Another interpretation is the following. Consider an online shop with a warehouse. Assume that the warehouse is replenished every day (e.g. under a long-term contract) and also that the expectation of supply equals the expectation of demand. So, the increments of the number of goods in the warehouse are mean-zero random variables $\{\xi_k\}$ if the warehouse is non-empty. In case when the virtual amount of goods in stock is negative, i.e., the warehouse is empty and we have unsatisfied orders, the shop seeks to refill the warehouse by buying batches of goods until it has a positive balance. In our model this refilling corresponds to the sequence $\{\eta_k\}$. If we consider a regular shop or a traditional queueing model, then the unsatisfied orders would be discarded and the walk

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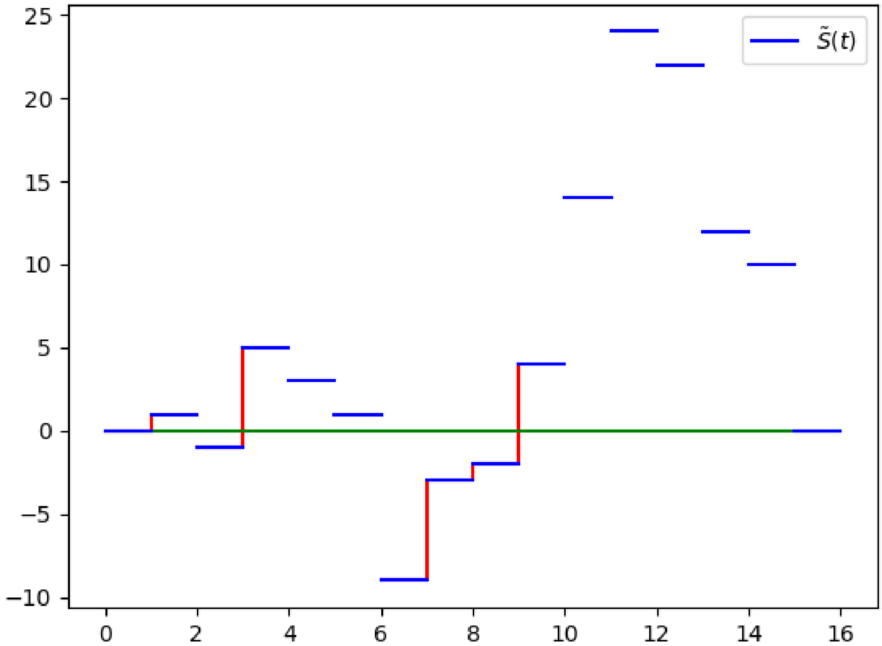

FIGURE 1. Sample path of $\tilde{S}$ (up to interpolation). Red vertical lines show when the jump is distributed as η.

couldn't overjump a zero-level. While the shortage is not overcome (which may take some time), the shop does not sell any goods.

Our goal is to analyse the limit of the scaled sequence of the processes $\left\{\frac{1}{\sqrt{n}}\tilde{S}(nt),\ t \in [0,1]\right\}_{n\geq 1}$ as $n \to \infty$. Depending on the distribution of η we want to distinguish two cases:

(1) η has a finite expectation:

$$\mathbb{E}\eta < \infty.$$

(2) the distribution of η has a slowly varying tail

$$\mathbb{P}(\eta > x) \sim l(x),\ l \in R_0,$$

where R_0 a set of slowly varying functions (check definition 1.4.2 in [1]).

Remark 1. Another interesting case is when the tail of η is regularly varying, i.e., η belongs to the domain of attraction of an α-stable distribution. This case was studied in [9] with the restriction on ξ to be in $\mathbb{Z}$ and greater than or equal to -1. We conjecture that the limit process should be the same as in [9], although we postpone the proof for the future paper.

For the case E$\eta < \infty$ we assume that $\tilde{S}$ is linearly interpolated for $t \geq 0$:

$$\tilde{S}(t) = \tilde{S}(\lfloor t \rfloor) + (t - \lfloor t \rfloor)\tilde{S}(\lceil t \rceil).$$

Theorem 1. *A sequence of processes*

$$\Big(\frac{\tilde{S}(nt)}{\sqrt{n}},\ t \in [0,1]\Big)_{n\geq 1}$$

converges weakly in $\mathrm{C}[0,1]$ *to a reflected Brownian motion:*

$$W_{reflected}(t) := W(t) - \min_{s\leq t} W(s).$$

Remark 2. It is well-known that $W_{\text{reflected}}$ has the same distribution as the absolute value of a Brownian motion $|W|$, see for example Theorem 1.3.2 in [8].

For the second case we assume $\tilde{S}$ has a flat-right interpolation for $t \geq 0$:

$$\tilde{S}(t) = \tilde{S}(\lfloor t \rfloor).$$

Theorem 2. *For a sequence of processes*

$$\Big(\frac{\tilde{S}(nt)}{\sqrt{n}},\ t \in [0,1]\Big)_{n\geq 1}$$

there is no weak limit in $\mathcal{D}[0,1]$. *Moreover for any* $t > 0$

$$\max_{s\leq t} \frac{\tilde{S}(ns)}{\sqrt{n}} \xrightarrow{\mathbb{P}} \infty,\ n \to \infty. \tag{2}$$

Our results show that intuitively understandable things happen: when η has an expectation, the process behaves like a Brownian motion with a reflection; otherwise when η has a slowly varying tail, the process blows up at the beginning.

Note that model (1) is equivalent (up to recounting of ξ_i and η_i) to the following

$$\tilde{S}(0) = 0,\ \tilde{S}(n) = \sum_{i=1}^{n-T(n)} \xi_i + \sum_{i=1}^{T(n)} \eta_i,\ n \geq 1, \tag{3}$$

where $T(n)$ is the number of visits to $(-\infty, 0]$ before the time n:

$$T(n) = \#\{k < n : \tilde{S}(k) \leq 0\},\ n \geq 1. \tag{4}$$

We work only with this representation in the paper further.

Denote

$$S_\xi(n) = \sum_{i=1}^{n} \xi_i,\ S_\eta(n) = \sum_{i=1}^{n} \eta_i. \tag{5}$$

So

$$\tilde{S}(n) = S_\xi(n - T(n)) + S_\eta(T(n)). \tag{6}$$

We set by definition $S_\xi(t) := S_\xi(\lfloor t \rfloor),\ S_\eta(t) := S_\eta(\lfloor t \rfloor), T(t) := T(\lfloor t \rfloor)$ for all $t \geq 0$.

Although the cases we discuss are very different, we tried to pick out common traits in Section 2.

In Section 3 we prove Theorem 1. We aim to show that $\frac{1}{\sqrt{n}} S_\eta(T(nt))$ converges to $-\min_{s\leq t} W(s)$ and do so by showing that the latter lies neatly between $\frac{1}{\sqrt{n}} S_\eta(T(nt)-1)$ and $\frac{1}{\sqrt{n}} S_\eta(T(nt)) + \frac{1}{\sqrt{n}} \max_{i\leq nt} |\xi_i|$.

The proof of the second case can be found in Section 4 and is based on the observation that the overshoots above level n of the random walk S_η are much larger than n. This is a known property, proved by Rogozin, see for example Theorem 8.8.2 in [1]. Thus we need to show that this overshoot happens before $T(n)$ because only in this way it could affect $\tilde{S}$.

2. Proof: shared part

Let m be a (sign flipped) running minimum of S_ξ:

$$m(n) = -\min_{k\le n} S_\xi(k). \tag{7}$$

Lemma 1. *For each $n \ge 1$*

$$S_\eta(T(n)-1) \le m(n-T(n)) < S_\eta(T(n)) + \max_{i\le n} |\xi_i| \tag{8}$$

Proof. The left inequality is trivial when $T(n) = 1$. Assume that $T(n) > 1$ and that for some $n \ge 1$

$$S_\eta(T(n)-1) > m(n-T(n)). \tag{9}$$

Take $k < n$, such that $T(n) = T(k+1)$ and $T(k)+1 = T(n)$. This is always possible since $T(n) > 1$. Thus

$$\tilde{S}(k) = S_\xi(k-T(k)) + S_\eta(T(k)) > -m(k-T(k)) + S_\eta(T(n)-1).$$

From (9) we infer that $\tilde{S}(k) > 0$. Therefore $T(k) = T(k+1)$ which contradicts with our choice of k.

As for the right hand side inequality, observe that $\tilde{S}$ jumps downwards at $k \ge 1$ only when $\tilde{S}(k-1) > 0$ and $S_\xi(k-T(k))$ jumps downwards. Thus for every $k \ge 1$

$$-\max_{i\le k-T(k)} |\xi_i| \le \tilde{S}(k) - \tilde{S}(k-1) < \tilde{S}(k) = S_\xi(k-T(k)) + S_\eta(T(k)).$$

Hence

$$S_\eta(T(k)) + \max_{i\le k-T(k)} |\xi_i| \ge -S_\xi(k-T(k)).$$

Taking maximum over $k \le n$ of both sides of the last inequality we get

$$S_\eta(T(n)) + \max_{i\le n} |\xi_i| \ge -\min_{k\le n} S_\xi(k-T(k)) = -\min_{k\le n-T(n)} S_\xi(k) = m(n-T(n)).$$

■

By the Donsker theorem

$$\frac{1}{\sqrt{n}} S_\xi(n\cdot) \xrightarrow{w} W(\cdot),\ n\to\infty,$$

meaning a weak convergence in C[0, 1]. By the Skorokhod Representation theorem (Theorem 6.7 in [2]) we can construct a probability space which supports random elements $\hat{S}_\xi^{(n)}$ and $\hat{W}$ such that

$$\hat{S}_\xi^{(n)} \stackrel{d}{=} S_\xi,\ \hat{W} \stackrel{d}{=} W,\ n \ge 1$$

and the uniform convergence holds

$$\frac{1}{\sqrt{n}} \hat{S}_\xi^{(n)}(nt) \rightrightarrows \hat{W}(t) \text{ as } n\to\infty \text{ a.s.} \tag{10}$$

for $t \in [0, 1]$.

To emphasize that we work on this new space we will write a hat whenever we use random variables dependent on $\hat{S}_\xi^{(n)}$. Without loss of generality we may assume that this new probability space is reach enough and contains a copy of the sequence $\{\eta_k\}_{k\ge1}$. We leave the same notation and assume that the original sequence $\{\eta_k\}_{k\ge1}$ belongs to this probability space. Using sequences $\{\hat{S}_\xi^{(n)}(k)\}$ and $\{S_\eta(k)\}$ we construct copies $\{\hat{T}^{(n)}(k)\}$, $\{\hat{\tilde{S}}^{(n)}(k)\}$, $\{\hat{m}^{(n)}(k)\}$ of $\{T(k)\}$, $\{\tilde{S}(k)\}, \{m(k)\}$ similarly to formulas (4), (7), (6). Note that in this constructions sequences $\{\hat{S}_\xi^{(n)}(k)\}$ depend on n but $\{S_\eta(k)\}$ does not. Often we will leave out indexing by n, that is $\hat{S}_\xi^{(n)}$ becomes $\hat{S}_\xi$.

It follows from inequality (8) that

$$S_\eta(\hat{T}(nt)-1) \le \hat{m}(nt) \text{ a.s.} \tag{11}$$

for all $t \ge 0$.

Divide both sides by $\sqrt{n}$. By (10) we have

$$\forall \varepsilon_1 > 0\ \exists n_1\ \forall n > n_1 \quad \frac{1}{\sqrt{n}} S_\eta(\hat{T}(nt)-1) \le -\min_{s\le t} \hat{W}(s) + \varepsilon_1 \text{ a.s.} \tag{12}$$

Note that $\lim_{k\to\infty} \hat{T}(k) = +\infty$ a.s. because $\liminf_{k\to\infty} S_\xi(k) = -\infty$ a.s. Thus

$$\forall \varepsilon_1 > 0\ \exists n_1'\ \forall n > n_1' \quad \frac{\hat{T}(nt)-1}{\sqrt{n}} \le \frac{\hat{T}(nt)-1}{S_\eta(\hat{T}(nt)-1)}(-\min_{s\le t} \hat{W}(s) + \varepsilon_1) \text{ a.s.} \tag{13}$$

The strong law of the large numbers for S_η implies

$$\limsup_{n\to\infty} \frac{\hat{T}(nt)}{\sqrt{n}} \le \frac{-\min_{s\le t} \hat{W}(s)}{\mathbb{E}\eta} \text{ a.s.} \tag{14}$$

When $\mathbb{E}\eta = \infty$, the right hand side is 0.

Corollary 1.

$$\sup_{0\le t\le 1} \frac{T(nt)}{n} \xrightarrow{\mathbb{P}} 0,\ n\to\infty.$$

Proof. Since $\hat{T}$ is non-decreasing we have

$$\sup_{0\le t\le 1} \frac{\hat{T}(nt)}{n} \le \frac{\hat{T}(n)}{n}.$$

Recall that $\hat{T} \overset{d}{=} T$, hence the corollary follows from (14). ■

Lemma 2. *The uniform convergence on* $[0,1]$ *holds as* $n\to\infty$

$$\frac{\hat{m}(nt-\hat{T}(nt))}{\sqrt{n}} \rightrightarrows -\min_{s\le t} \hat{W}(t) \ \textit{a.s.}$$

Proof. Equations (10) and (14) yield the uniform convergences on $[0,1]$

$$\frac{1}{\sqrt{n}}\hat{m}(nt) = \frac{1}{\sqrt{n}}\hat{m}^{(n)}(nt) \rightrightarrows -\min_{s\le t} \hat{W}(t), \quad n\to\infty,$$

and

$$t - \frac{\hat{T}(nt)}{n} = t - \frac{\hat{T}^{(n)}(nt)}{n} \rightrightarrows t, \quad n\to\infty,$$

with probability 1.

Since $t - \frac{\hat{T}(nt)}{n} \in [0,1]$ and the limit functions are continuous, their compositions a.s. uniformly converge to the composition of limits. ■

Remark 3. In Lemma 2 we may use both linear and piecewise constant approximation of the corresponding processes.

3. Proof of Theorem 1

It suffices to prove uniform convergence over $t \in [0,1]$ as $n \to \infty$

$$\frac{1}{\sqrt{n}} S_\eta(\hat{T}(nt)) \rightrightarrows -\min_{s\le t} \hat{W}(s) \text{ a.s.} \tag{15}$$

The other part of the statement of Theorem 1 is provided by (10).

From Lemma 1 and Lemma 2 we infer that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} S_\eta(\hat{T}(nt)-1) \le -\min_{s\le t} \hat{W}(s) \text{ a.s.} \tag{16}$$

$$\liminf_{n\to\infty} \left(\frac{1}{\sqrt{n}} S_\eta(\hat{T}(nt)) + \frac{\max_{i\le n} |\hat{\xi}_i|}{\sqrt{n}} \right) \ge -\min_{s\le t} \hat{W}(s) \text{ a.s.} \tag{17}$$

To prove (15) it suffices to show

$$\frac{\max_{i\le n} |\hat{\xi}_i|}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0, \ n\to\infty. \tag{18}$$

and

$$\frac{1}{\sqrt{n}} |S_\eta(\hat{T}(nt)) - S_\eta(\hat{T}(nt)-1)| = \frac{1}{\sqrt{n}} \eta_{\hat{T}(nt)} \xrightarrow{\mathbb{P}} 0, \ n\to\infty, \tag{19}$$

Observe that for a given $\delta > 0$ due to (14) it is always possible to choose $K > 0$ such that

$$\mathbb{P}(\hat{T}(n) \le K\sqrt{n}) > 1-\delta.$$

Hence both (18) and (19) follow from Theorem 6.2.1 in [3].

4. Proof of Theorem 2

It suffices to prove (2). Set

$$N_\eta(x) = \inf\{k : S_\eta(k) > x\}, \ x \ge 0.$$

Let $c > 0$, $K > 0$, $0 < \varepsilon < K$ and $0 < t \le 1$ be arbitrary real numbers, then

$$\begin{aligned} \left\{ \max_{i\le nt} \frac{\tilde{S}(i)}{\sqrt{n}} \ge c \right\} &= \left\{ \max_{i\le nt} \Big(S_\xi(i - T(i)) + S_\eta(T(i)) \Big) \ge c\sqrt{n} \right\} \\ &\supset \left\{ \min_{i\le n} S_\xi(i) \ge -K\sqrt{n} \right\} \cap \left\{ S_\eta(T(nt)) > (K+c)\sqrt{n} \right\} \\ &= \left\{ \min_{i\le n} S_\xi(i) \ge -K\sqrt{n} \right\} \cap \left\{ N_\eta((K+c)\sqrt{n}) \le T(nt) \right\} \\ &\supset \left\{ \min_{i\le n} S_\xi(i) \ge -K\sqrt{n} \right\} \cap \left\{ N_\eta(\varepsilon\sqrt{n}) \le T(nt) \right\} \\ &\qquad \cap \left\{ \eta_{N_\eta(\varepsilon\sqrt{n})} > (K+c)\sqrt{n} \right\}. \end{aligned} \tag{20}$$

For the second inclusion observe that if

$$\eta_{N_\eta(\varepsilon\sqrt{n})} > (K+c)\sqrt{n},$$

then automatically

$$N_\eta((K+c)\sqrt{n}) = N_\eta(\varepsilon\sqrt{n}).$$

Lemma 3. *Let $j, l, a > 0$ be real numbers such that $j + l < n$, then*

$$\left\{ \min_{i\le j} S_\xi(i) \le -a \right\} \cap \left\{ T(n) < l \right\} \subset \left\{ N_\eta(a) \le T(j+l) \right\}.$$

Proof. Suppose $\min_{i\le j} S_\xi(i) = -A \le -a$ and it is firstly attained at $i = i^* \le j$. Denote by u the smallest solution of

$$u - T(u) = i^*.$$

Let k be such that

$$\tilde S(u) \le \tilde S(u+1) \le \cdots \le \tilde S(u+k-1) \le 0 < \tilde S(u+k),$$

meaning that $\tilde S$ needs k jumps to become positive. Observe that $T(u) + k = T(u+k)$.

Assume $T(n) < l$. Then $k < l$. Thus

$$u + k < j + l < n.$$

As k is such that $\tilde S(u+k) > 0$, then from definition of $\tilde S$ we deduce that $S_\eta(T(u+k)) > A$ and so

$$N_\eta(a) \le N_\eta(A) \le T(u+k) = T(i^* + T(u) + k) = T(i^* + T(u+k)) \le T(j+l).$$

■

Apply Lemma 3 with $j = l = \frac{nt}{2}$ and $a = \varepsilon\sqrt{n}$ to further inclusion (20)

$$\begin{aligned}
&\Big\{\max_{i\le nt} \frac{\tilde S(i)}{\sqrt{n}} \ge c\Big\}\\
&\supset \Big\{\min_{i\le n} S_\xi(i) \ge -K\sqrt{n}\Big\} \cap \Big\{N_\eta(\varepsilon\sqrt{n}) \ge T(nt)\Big\} \cap \Big\{\eta_{N_\eta(\varepsilon\sqrt{n})} > (K+c)\sqrt{n}\Big\}\\
&\supset \Big\{\min_{i\le n} S_\xi(i) \ge -K\sqrt{n}\Big\} \cap \Big\{\min_{i\le nt/2} S_\xi(i) \le -\varepsilon\sqrt{n}\Big\} \cap \Big\{T(n) < \frac{nt}{2}\Big\}\\
&\qquad \cap \Big\{\eta_{N_\eta(\varepsilon\sqrt{n})} > (K+c)\sqrt{n}\Big\}.
\end{aligned} \tag{21}$$

Hence

$$\begin{aligned}
\mathbb{P}\Big(\max_{i\le nt} \frac{\tilde S(i)}{\sqrt{n}} \ge c\Big) &\ge \mathbb{P}\Big(\text{r.h.s. of (21)}\Big)\\
&= 1 - \mathbb{P}\Big(\text{complement to the r.h.s. of (21)}\Big)\\
&\ge 1 - \mathbb{P}\Big(\min_{i\le n} S_\xi(i) \le -K\sqrt{n}\Big) - \mathbb{P}\Big(\min_{i\le nt/2} S_\xi(i) \ge -\varepsilon\sqrt{n}\Big)\\
&\quad - \mathbb{P}\Big(T(n) > \frac{nt}{2}\Big) - \mathbb{P}\Big(\eta_{N_\eta(\varepsilon\sqrt{n})} < (K+c)\sqrt{n}\Big).
\end{aligned} \tag{22}$$

Let $\delta > 0$ be an arbitrary small number. We proceed by proving that every negative term in the r.h.s. of the last inequality is greater than $-\delta$ for n big enough.

Choose $K > 0$ and $0 < \varepsilon < K$ so that

$$\mathbb{P}\Big(|N_\eta(0,1)| > K\Big) < \delta, \quad \mathbb{P}\Big(|N_\eta(0,1)| < \varepsilon\sqrt{\frac{2}{t}}\Big) < \delta.$$

Since

$$-\min_{i\le nt} \frac{S_\xi(i)}{\sqrt{n}} \xrightarrow{w} |N_\eta(0,t)|, \ n \to \infty,$$

we get

$$\limsup_{n\to\infty} \Bigg(\mathbb{P}\Big(\min_{i\le n} S_\xi(i) \le -K\sqrt{n}\Big) + \mathbb{P}\Big(\min_{i\le nt/2} S_\xi(i) \le -\varepsilon\sqrt{n}\Big)\Bigg) \le 2\delta.$$

It follows from Theorem 8.8.2 in [1] that

$$\frac{\eta_{N_\eta(\varepsilon\sqrt{n})}}{\sqrt{n}} \xrightarrow{\mathbb{P}} \infty, \ n \to \infty. \tag{23}$$

So the last term in the right hand side of (22) converges to 0.

Corollary 1 concludes the proof.

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