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EXIT PROBLEMS FOR KOU'S PROCESS IN A MARKOVIAN ENVIRONMENT

In this paper, we consider a path-wise sum of a Brownian motion plus a compound Poisson process with exponentially distributed positive and negative jumps with parameters that depend on some finite Markov chain. Using known fluctuation identities we investigate one-sided and two-sided exit problems generalizing some results for Kou's processes to the setting of regime switching models without exploiting the fluid embedding technique. The generating function for the hitting time of the state-dependent levels is analyzed. For the case of two states, the numerical examples are given.

1. Introduction

Kou's model is a double exponential jump diffusion model introduced in [35] for the purpose of option pricing. This model represents a generalization of the Black–Scholes model which accurately fits price dynamics of different assets (see for instance [43] and [44]) and incorporates the asymmetric leptokurtic feature and the volatility smile. Moreover, the model is analytically tractable and provides exact solutions for a variety of option pricing problems (see also [37]).

Kou's process is a special case of Lévy processes and a wide range of applications can be captured within the correspondent fluctuation theory. We refer to the monographs [18] and [39] for an introduction to the recent results of the fluctuation theory and its application for different types of Lévy processes. A number of fluctuation identities are semi-explicit up to the components of the Spitzer-Rogozin factorization, which are usually called as the Wiener-Hopf factors. The closed form of these factors are known only for special classes of Lévy processes. A rich enough class contains the processes the jumps of which form a compound Poisson process with a rational characteristic function of jumps. In [4, Chapter 2, Section 2], it was shown that the Wiener-Hopf factors for such processes can be expressed in terms of roots of the cumulant equation, see also [41] and [13]. In [38], Kuznetsov et al. represented meromorphic Lévy processes for which the Wiener-Hopf factors can be expressed as rational functions of infinite degree. For Kou's process the Wiener-Hopf factors are rational functions of degree two with poles of multiplicity 1. This essentially simplifies fluctuation identities preserving its general structure. With this fact in mind, Kou's model can be considered as a handy delegate of the class of Lévy processes with rational Wiener-Hopf factors.

Parameters of Lévy processes do not depend on time which is not completely consistent with the real dynamics of different time series. To incorporate time-inhomogeneity Hamilton [20] proposed an econometric model with parameters that vary with the state of the environment. The parameters of the model take different values when the discrete-state Markov process is in different states. Such a state is usually referred to as a regime and the corresponding Markov-modulated processes are called regime switching processes.

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Regime switching Lévy processes belong to the class of Markov additive processes (MAP). For an introduction to MAP we refer to [1, Chapter III, Section 4]. One of the important advantages of MAP is that the classical results of the fluctuation theory for Lévy processes can be extended without sacrificing the analytical tractability; see [17]. Van Beek et al. [46] mentioned that this is also true for an affine process. As well as in the scalar case (that is, when the environment has only one state) we need some additional restrictions on the process structure to receive closed-form representations for the distribution of different functionals. A quite general class of MAPs for which these closed-form representations are known includes the so called phase-type MAPs. For the definition and basic properties of phase-type distributions we refer to [1, Chapter IX]. Breuer [5] showed that the Laplace transform of the first passage time for phase-type MAPs has a phase-type form and presented a numerically stable iterative procedure to determine the parameter matrices. The fluid embedding method used in the paper was based on adding new states that correspond to jumps so as to reformulate the initial problem in terms of a related continuous MAP (see [6]–[10] for more detail). A Markov modulated Kou process is a phase-type MAP and there exists a one-to-one correspondence with a Markov modulated Brownian motion but due to the simple structure we can do analysis without extending the state space and rebuilding the process. This allows us to get generalizations of results known for the scalar case in a more direct way. Moreover, working with a model with jumps rather than with a modified continuous version we can detect possible pitfalls connected to the overshoot problem. These remarks formed the motivation for our paper.

Fluctuation theory studies the properties of the supremum and the infimum functionals and their connection with one and two sided exit problems. For MAPs, the corresponding investigations appeared in 1970s and have been an active research topic over the past decades. These investigations provide us a series of different approaches for finding explicit expressions for the integral transforms of the functional distributions for Kou's processes defined on a finite Markov chain. The arguments we use here are generalizations of ideas from [18, Section 5.7] and [31]. Using the corresponding results we discuss an algorithm of obtaining the Laplace transform of the joint distribution of the first passage through state-dependent levels and the value of process at that moment. The results are of interest to applications such as the option pricing in finance and the dividend problem in risk theory. Following Jiang and Pistorius [27], if the stock price process is modeled as a geometric MAP, then the optimal exercise strategy for a perpetual American put option is the hitting time of the state-dependent levels (see also [2]), and the value function could be represented in terms of the corresponding generating function. Jiang and Pistorius [28] showed that the state-dependent levels define the optimal dividend strategy for a company whose cumulative net revenues evolve as a regime switching Brownian motion with positive drifts.

The rest of the paper is organized as follows. In Section 2, we introduce Kou's process in a Markovian environment, and then derive an algorithm for finding the densities of killed extrema and the integral transforms of one-sided and two-sided boundary functionals. Some alternatives are discussed in the remarks. In Section 3, we give a representation of the integral transform of the joint distribution for the first passage time through state-dependent levels and the overshoot at this moment in terms of those for one-sided and two-sided boundary functionals. For the case when the environment has two possible states, we represent numerical examples.

2. Model

Let $Z_t = \{X_t, J_t\}$ be the bivariate Markov process, where J_t is a finite irreducible nonperiodic Markov chain with finite state space $E = \{1, \ldots, N\}$, transition rate matrix

$\mathbf{Q} = \|q_{ij}\|_{i,j=1}^{N}$, and the stationary distribution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_N)$. Process X_t is given by

$$dX_t = a_{J_t}dt + \sigma_{J_t}dW(t) + \sum_{k=1}^{N} I_{\{J_t=k\}}dS_k(t), \quad X_0 = 0.$$

That is, conditional on $J_t = k$, the increments of X_t is determined by the increments of $X_k(t) = a_k t + \sigma_k W(t) + S_k(t)$. Here,

- constants a_k, σ_k are the drift and volatility of the diffusion part,
- $\{W(t), t \geq 0\}$ is a standard Wiener process,
- $\{S_k(t), t \geq 0\}_{k=1}^{N}$ are compound Poisson processes with exponentially distributed positive and negative jumps.

The notation $I_{\{\cdot\}}$ denotes the indicator function of $\{\cdot\}$. Processes $\{W(t), S_1(t), \ldots, S_N(t)\}$ are mutually independent. For the scalar case ($N = 1$) see also [36].

The derivation of our results relies on the factorization theory for MAPs (see [17]). This theory generalizes the Wiener-Hopf theory for Lévy processes. The factorization method turn out to provide useful tool to analyze various functionals of MAPs, such as running maximum and minimum

$$X_t^+ = \sup_{0\leq u\leq t} X_u, \quad X_t^- = \inf_{0\leq u\leq t} X_u,$$

the first passage time over a positive level $x \geq 0$ (below a negative level $x \leq 0$)

$$\tau^+(x) = \inf\{t > 0 : X_t > x\} \quad \left(\tau^-(x) = \inf\{t > 0 : X_t < x\}\right),$$

and the corresponding overshoots:

$$\gamma^+(x) = X_{\tau^+(x)} - x; \quad \gamma^-(x) = x - X_{\tau^-(x)};$$

the exit time from an interval $(x - b, x)$, $0 < x < b$:

$$\tau(x, b) = \inf\{t > 0 : X_t \notin (x - b, x)\},$$

and the overshoot at the moment of exit from the interval

$$\gamma(x, b) = \left(X_{\tau(x,b)} - x\right) I_{\left\{X_{\tau(x,b)}\geq x\right\}} + \left(x - b - X_{\tau(x,b)}\right) I_{\left\{X_{\tau(x,b)}\leq x-b\right\}}.$$

Using these functionals we can investigate the first passage time over (below) a level that depends on the current state of the environment

$$T_b^+ = \inf\{t > 0 : X_t > b_{J_t}\} \quad \left(T_b^- = \inf\{t > 0 : X_t < b_{J_t}\}\right) \tag{1}$$

and the overshoot

$$\gamma_b^+ = X_{T_b^+} - b_{J_{T_b^+}} \quad \left(\gamma_b^- = b_{J_{T_b^-}} - X_{T_b^-}\right). \tag{2}$$

Let P_i and E_i be shorthand for $\mathsf{P}\{\cdot|X_0 = 0, J_0 = i\}$ and $\mathsf{E}\{\cdot|X_0 = 0, J_0 = i\}$ respectively and $\mathsf{E}_i[\xi;\, A] = \mathsf{E}_i[\xi\, I_{\{A\}}]$. The distribution of a MAP is determined by the matrix moment generating function

$$\mathbf{E}e^{rX_t} = \left\|\mathsf{E}_i\left[e^{rX_t};\ J_t = j\right]\right\|_{i,j=1}^{N} = e^{t\mathbf{K}[r]},\ \mathrm{Re}[r] = 0, \quad \mathbf{K}[0] = \mathbf{Q}.$$

We understand $\mathbf{E}g(\tau)$ as the $N \times N$ matrix with elements $\mathsf{E}_i[g(\tau); J_\tau = j]$. For the Kou process the cumulant function (the analogue of the Laplace exponent for the scalar case) is as follows (cf. [17, p. 14])

$$\mathbf{K}[r] = r\mathbf{A} + r^2\boldsymbol{\Sigma} + \int_{-\infty}^{\infty} (e^{rx} - 1)\, \boldsymbol{\Lambda}\mathbf{f}(x)dx + \mathbf{Q}, \tag{3}$$

$$\mathbf{f}(x) = \mathbf{p}^+\mathbf{B}^+e^{-\mathbf{B}^+x}I_{\{x\geq 0\}} + \mathbf{p}^-\mathbf{B}^-e^{\mathbf{B}^-x}I_{\{x<0\}}, \tag{4}$$

where $\mathbf{A} = diag\,(a_1, \dots, a_N)$, $\mathbf{\Sigma} = diag\,(\sigma_1^2/2, \dots, \sigma_N^2/2)$, $\mathbf{\Lambda} = diag\,(\lambda_1, \dots, \lambda_N)$, λ_k are the rates of the time intervals between two neighboring jumps, $\mathbf{B}^+ = diag\,(\beta_1^+, \dots, \beta_N^+)$, $\mathbf{B}^- = diag\,(\beta_1^-, \dots, \beta_N^-)$, β_k^+ and β_k^- are the parameters of exponentially distributed positive and negative jumps, correspondingly, $\mathbf{p}^\pm = diag\,(p_1^\pm, \dots, p_N^\pm)$: $p_k^\pm > 0, p_k^+ + p_k^- = 1$, $k = 1, \dots N$. Combining (3) and (4) yields

$$\mathbf{K}[r] = r\mathbf{A} + r^2\mathbf{\Sigma} + r\mathbf{\Lambda}\mathbf{p}^+\left(\mathbf{B}^+ - r\mathbf{I}\right)^{-1} - r\mathbf{\Lambda}\mathbf{p}^-\left(\mathbf{B}^- + r\mathbf{I}\right)^{-1} + \mathbf{Q}.$$

That is, $\mathbf{K}[r]$ is the $N \times N$ matrix with elements

$$ra_k + r^2\frac{\sigma_k^2}{2} + \frac{\lambda_k p_k^+ r}{\beta_k^+ - r} - \frac{\lambda_k p_k^- r}{\beta_k^- + r} + q_{kk}$$

on the diagonal and with q_{kj} as off-diagonal elements. Hence, all entries of $\mathbf{K}[r]$ are rational functions with respect to r.

2.1. **Time-reversed process.** In order to formulate the statement about factorization decomposition, we need some more notation related to time-reversion. To the process J_t we can associate the so called reversed process $\hat{J}_t$, where the process J_t is considered in the reversed time. The reversed process $\hat{J}_t$ is a Markov process with the initial distribution $\boldsymbol{\pi}$ and its transition rates are $\hat{q}_{ij} = \frac{\pi_j q_{ji}}{\pi_i}$, or using notation $\mathbf{\Delta}_\pi = diag\,(\pi_1, \dots, \pi_N)$

$$\hat{\mathbf{Q}} = \mathbf{\Delta}_\pi^{-1}\mathbf{Q}^\top\mathbf{\Delta}_\pi.$$

In general, the reversed process is a different process from the original one. The process J_t is reversible if $\hat{\mathbf{Q}} = \mathbf{Q}$ or if $\mathbf{\Delta}_\pi\mathbf{Q}$ is a symmetric matrix (see, for example, [45, Chapter 6]).

The process $\hat{Z}_t = \{\hat{X}_t, \hat{J}_t\}$ with cumulant $\hat{\mathbf{K}}[r] = \mathbf{\Delta}_\pi^{-1}\mathbf{K}^\top[r]\mathbf{\Delta}_\pi$ is called reversed process for Z_t. The characteristics of $\hat{Z}_t$ will be indicated by using a hat over the existing notation for the characteristics of Z_t. Process Z_t is called reversible if $\mathbf{\Delta}_\pi\mathbf{K}[r]$ is a symmetric matrix. If Z_t is reversible, then it has the same law as $\hat{Z}_t$ (see, e.g., [22, Section 2.5]). For the process with cumulant function (3) the reversibility could be established from the reversibility of J_t. So, if $N = 2$, then Z_t is reversible. For a more thorough discussion on time reversal for MAP see [26].

2.2. **Factorization identity.** Let θ_s be an exponentially distributed random variable with parameter $s > 0$ ($\mathsf{P}\{\theta_s > t\} = e^{-st}, t \geq 0$), independent of Z_t. We can rewrite the matrix moment generating function of X_{θ_s} as follows

$$\mathbf{E}e^{rX_{\theta_s}} = s\int_0^\infty e^{-st}\mathbf{E}e^{rX_t}dt = s\left(s\mathbf{I} - \mathbf{K}[r]\right)^{-1}, \tag{5}$$

where $\mathbf{I}$ is the identity matrix. The Wiener-Hopf procedure hinges on finding a product factorization for $s\mathbf{I} - \mathbf{K}[r]$ in the form $s\mathbf{I} - \mathbf{K}[r] = \mathbf{K}_-(s, r)\mathbf{K}_+(s, r)$, where components $\mathbf{K}_\mp$ define the moment generating functions of the running minimum and the process reflected at its minimum killed at time θ_s. More precisely, we have the identity of infinitely divisible factorization [17, Theorem 2.2] (see also [33, Theorem 1]):

$$\mathbf{E}e^{rX_{\theta_s}} = \begin{cases} \mathbf{E}e^{rX_{\theta_s}^+}\mathbf{P}_s^{-1}\mathbf{E}e^{r\left(X_{\theta_s} - X_{\theta_s}^+\right)}, \\ \mathbf{E}e^{rX_{\theta_s}^-}\mathbf{P}_s^{-1}\mathbf{E}e^{r\left(X_{\theta_s} - X_{\theta_s}^-\right)}, \end{cases} \quad \mathrm{Re}[r] = 0,$$

where $\mathbf{P}_s = s\left(s\mathbf{I} - \mathbf{K}[0]\right)^{-1}$. Note that the factorization components admit the analytic continuation into the corresponding half-plane ($\mp\mathrm{Re}[r] > 0$).

Set

$$\mathbf{P}[r] = \left(\mathbf{B}^- + r\mathbf{I}\right)\left(\mathbf{I} - s^{-1}\mathbf{K}[r]\right)\left(\mathbf{B}^+ - r\mathbf{I}\right), \tag{6}$$

then the moment generating function of X_{θ_s} can be represented as

$$\mathbf{E}e^{rX(\theta_s)} = \left(\mathbf{B}^+ - r\mathbf{I}\right)\mathbf{P}^{-1}[r]\left(\mathbf{B}^- + r\mathbf{I}\right).$$

Following [40, Section 3.7], if equation $\det \mathbf{P}[r] = 0$ has all distinct real roots, then $\mathbf{P}[r]$ can be decomposed into a product of linear factors:

$$\mathbf{P}[r] = (r\mathbf{I} - \mathbf{X}_4)(r\mathbf{I} - \mathbf{X}_3)\, s^{-1}\mathbf{\Sigma}\,(r\mathbf{I} - \mathbf{X}_2)(r\mathbf{I} - \mathbf{X}_1). \tag{7}$$

This decomposition is not unique we can choose matrices $\mathbf{X}_i$ such that $\mathbf{X}_1, \mathbf{X}_2$ have only positive eigenvalues: $\rho_i^+, i = 1, \ldots, 2N$, and $\mathbf{X}_3, \mathbf{X}_4$ have only negative eigenvalues: $-\rho_j^-, j = 1, \ldots, 2N$. Hence, ρ_i^+ define the factorization component $\mathbf{E}e^{rX_{\theta_s}^+}$ and ρ_j^- define the component $\mathbf{E}e^{r\left(X_{\theta_s} - X_{\theta_s}^+\right)}$. Inverting factorization components with respect to r we can get the distribution of $X_{\theta_s}^+$ and $X_{\theta_s} - X_{\theta_s}^+$.

Proposition 2.1. *If equation* $\det \mathbf{P}[r] = 0$ *has all distinct real roots, then the densities of extrema and process reflected at its extrema could be represented as follows*

$$\mathbf{f}_\pm(x) = \left\| \frac{\partial}{\partial x} \mathsf{P}_i\left\{X_{\theta_s}^\pm < x,\ J_{\theta_s} = j\right\}\right\|_{i,j=1}^N = \sum_{k=1}^{2N} \mathbf{A}_k^\pm e^{\mp\rho_k^\pm x}, \pm x \geq 0, \tag{8}$$

$$\mathbf{f}^\pm(x) = \left\| \frac{\partial}{\partial x} \mathsf{P}_i\left\{X_{\theta_s} - X_{\theta_s}^\mp < x,\ J_{\theta_s} = j\right\}\right\|_{i,j=1}^N = \sum_{l=1}^{2N} \bar{\mathbf{A}}_l^\pm e^{\mp\rho_l^\pm x}, \pm x \geq 0, \tag{9}$$

for some matrices $\mathbf{A}_k^\pm, \bar{\mathbf{A}}_l^\pm$, $k, l = 1, \ldots, 2N$.

Proof. Since $\forall i, j \in E$: $\mathsf{E}_i[|X_t|; J_t = j] < \infty$, from [19, Th. 1] we get

$$\mathbf{E}e^{rX_{\theta_s}^+} = \left[\mathbf{I} - r\left(\bar{\mathbf{C}}_1^+ + s^{-1}\mathbf{k}(r)\right)\right]^{-1} \mathbf{P}_s,$$

where $\bar{\mathbf{C}}_1^+ = s^{-1}\mathbf{f}^-(0)\,\mathbf{\Sigma}$ and

$$\mathbf{k}(r) = \int_0^\infty e^{rx} \int_{-\infty}^0 \mathbf{f}^-(y) \int_{x-y}^\infty \mathbf{\Lambda}\mathbf{f}(z)dzdydx.$$

Since every diagonal matrix commutes with all other diagonal matrices of the same dimension, substituting (4) gives

$$\begin{aligned}
\mathbf{E}e^{rX_{\theta_s}^+} &= \left(\mathbf{I} - r\bar{\mathbf{C}}_1^+ - \frac{r}{s}\int_{-\infty}^0 \mathbf{f}^-(y)e^{\mathbf{B}^+ y}dy\mathbf{\Lambda}\mathbf{p}^+\left(\mathbf{B}^+ - r\mathbf{I}\right)^{-1}\right)^{-1}\mathbf{P}_s \\
&= \left(\left(\left(\mathbf{I} - r\bar{\mathbf{C}}_1^+\right)\left(\mathbf{B}^+ - r\mathbf{I}\right) - \frac{r}{s}\int_{-\infty}^0 \mathbf{f}^-(y)e^{\mathbf{B}^+ y}dy\mathbf{\Lambda}\mathbf{p}^+\right)\left(\mathbf{B}^+ - r\mathbf{I}\right)^{-1}\right)^{-1}\mathbf{P}_s \\
&= \left(\mathbf{B}^+ - r\mathbf{I}\right)\left(\mathbf{B}^+ + r^2\bar{\mathbf{C}}_1^+ - r\mathbf{I} - r\bar{\mathbf{C}}_1^+\mathbf{B}^+ - \frac{r}{s}\int_{-\infty}^0 \mathbf{f}^-(y)e^{\mathbf{B}^+ y}dy\mathbf{\Lambda}\mathbf{p}^+\right)^{-1}\mathbf{P}_s.
\end{aligned}$$

Write

$$\bar{\mathbf{C}}_2^+ = \mathbf{I} + \bar{\mathbf{C}}_1^+\mathbf{B}^+ + s^{-1}\int_{-\infty}^0 \mathbf{f}^-(y)e^{\mathbf{B}^+ y}dy\mathbf{\Lambda}\mathbf{p}^+,$$

then

$$\mathbf{E}e^{rX_{\theta_s}^+} = \left(\mathbf{B}^+ - r\mathbf{I}\right)\left(\mathbf{B}^+ - r\bar{\mathbf{C}}_2^+ + r^2\bar{\mathbf{C}}_1^+\right)^{-1}\mathbf{P}_s. \tag{10}$$

Considering process $\bar{X}_t = -X_t$ we get the similar representation for the moment generating function of $X_{\theta_s}^-$:

$$\mathbf{E}e^{rX_{\theta_s}^-} = \left(\mathbf{B}^- + r\mathbf{I}\right)\left(\mathbf{B}^- + r\bar{\mathbf{C}}_2^- + r^2\bar{\mathbf{C}}_1^-\right)^{-1}\mathbf{P}_s, \tag{11}$$

where

$$\bar{\mathbf{C}}_1^- = s^{-1}\mathbf{f}^+(0)\,\mathbf{\Sigma},\ \bar{\mathbf{C}}_2^- = \mathbf{I} + \bar{\mathbf{C}}_1^-\mathbf{B}^- + s^{-1}\int_0^\infty \mathbf{f}^+(y)e^{-\mathbf{B}^- y}dy\mathbf{\Lambda}\mathbf{p}^-.$$

Taking into account that $\mathbf{E}e^{r\left(X_{\theta_s}-X^{\pm}_{\theta_s}\right)} = \mathbf{\Delta}_\pi^{-1}\left(\mathbf{E}e^{r\hat{X}^{\mp}_{\theta_s}}\right)^\top \mathbf{\Delta}_\pi$ (see, for instance, [33]) we deduce

$$\mathbf{E}e^{r\left(X_{\theta_s}-X^{\pm}_{\theta_s}\right)} = \mathbf{P}_s\left(\mathbf{B}^{\mp} \pm r\mathbf{C}_2^{\mp} + r^2\mathbf{C}_1^{\mp}\right)^{-1}\left(\mathbf{B}^{\mp} \pm r\mathbf{I}\right), \tag{12}$$

$$\mathbf{C}_1^{\pm} = s^{-1}\mathbf{\Sigma}\mathbf{f}_{\mp}(0), \; \mathbf{C}_2^{\pm} = \mathbf{I} + \mathbf{B}^{\pm}\mathbf{C}_1^{\pm} + s^{-1}\mathbf{\Lambda}\mathbf{p}^{\pm}\int_0^\infty e^{-\mathbf{B}^{\pm}y}\mathbf{f}_{\mp}(\mp y)dy.$$

So, the identity of infinitely divisible factorization gives

$$\mathbf{E}e^{rX_{\theta_s}} = \left(\mathbf{B}^+ - r\mathbf{I}\right)\Big[\Big(\mathbf{B}^-\left(\mathbf{C}_1^-\right)^{-1} + r\mathbf{C}_2^-\left(\mathbf{C}_1^-\right)^{-1} + r^2\mathbf{I}\Big)\mathbf{C}_1^-\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^+ \times$$
$$\times\Big(\left(\bar{\mathbf{C}}_1^+\right)^{-1}\mathbf{B}^+ - r\left(\bar{\mathbf{C}}_1^+\right)^{-1}\bar{\mathbf{C}}_2^+ + r^2\mathbf{I}\Big)\Big]^{-1}\left(\mathbf{B}^- + r\mathbf{I}\right)$$

But

$$\mathbf{E}e^{rX_{\theta_s}} = \left(\mathbf{B}^+ - r\mathbf{I}\right)\mathbf{P}^{-1}[r]\left(\mathbf{B}^- + r\mathbf{I}\right),$$

hence

$$\mathbf{P}[r] = \Big(\mathbf{B}^-\left(\mathbf{C}_1^-\right)^{-1} + r\mathbf{C}_2^-\left(\mathbf{C}_1^-\right)^{-1} + r^2\mathbf{I}\Big)\left(\mathbf{C}_1^-\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^+\right) \times$$
$$\times\Big(\left(\bar{\mathbf{C}}_1^+\right)^{-1}\mathbf{B}^+ - r\left(\bar{\mathbf{C}}_1^+\right)^{-1}\bar{\mathbf{C}}_2^+ + r^2\mathbf{I}\Big).$$

Comparing the leading coefficient matrices on the left and right-hand sides of the equality gives $\mathbf{C}_1^-\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^+ = s^{-1}\mathbf{\Sigma}$. Thus,

$$\mathbf{P}[r] = \Big(\mathbf{B}^-\left(\mathbf{C}_1^-\right)^{-1} + r\mathbf{C}_2^-\left(\mathbf{C}_1^-\right)^{-1} + r^2\mathbf{I}\Big)\left(s^{-1}\mathbf{\Sigma}\right) \times \tag{13}$$
$$\times\Big(\left(\bar{\mathbf{C}}_1^+\right)^{-1}\mathbf{B}^+ - r\left(\bar{\mathbf{C}}_1^+\right)^{-1}\bar{\mathbf{C}}_2^+ + r^2\mathbf{I}\Big).$$

Since the equation $\det\mathbf{P}[r] = 0$ has all distinct real roots, we can write the decomposition

$$\mathbf{P}[r] = \left(r\mathbf{I} - \mathbf{X}_4\right)\left(r\mathbf{I} - \mathbf{X}_3\right)s^{-1}\mathbf{\Sigma}\left(r\mathbf{I} - \mathbf{X}_2\right)\left(r\mathbf{I} - \mathbf{X}_1\right)$$

with matrices $\mathbf{X}_i$ such that $\mathbf{X}_1, \mathbf{X}_2$ have only positive eigenvalues and $\mathbf{X}_3, \mathbf{X}_4$ have only negative eigenvalues. Moreover, substituting $r = 0$ into (6) and (7) gives

$$\mathbf{B}^-\mathbf{P}_s^{-1}\mathbf{B}^+ = \mathbf{X}_4\mathbf{X}_3\left(s^{-1}\mathbf{\Sigma}\right)\mathbf{X}_2\mathbf{X}_1.$$

So, we obtain the next decomposition

$$\mathbf{E}e^{rX_{\theta_s}} = \left(\mathbf{B}^+ - r\mathbf{I}\right)\left[\left(r\mathbf{I} - \mathbf{X}_2\right)\left(r\mathbf{I} - \mathbf{X}_1\right)\right]^{-1}\Big(\mathbf{X}_2\mathbf{X}_1\left(\mathbf{B}^+\right)^{-1}\Big)\mathbf{P}_s\mathbf{P}_s^{-1} \times \tag{14}$$
$$\times\,\mathbf{P}_s\Big(\left(\mathbf{B}^-\right)^{-1}\mathbf{X}_4\mathbf{X}_3\Big)\left[\left(r\mathbf{I} - \mathbf{X}_4\right)\left(r\mathbf{I} - \mathbf{X}_3\right)\right]^{-1}\left(\mathbf{B}^- + r\mathbf{I}\right)$$

The factor $\left(\mathbf{B}^+ - r\mathbf{I}\right)\left[\left(r\mathbf{I} - \mathbf{X}_2\right)\left(r\mathbf{I} - \mathbf{X}_1\right)\right]^{-1}\Big(\mathbf{X}_2\mathbf{X}_1\left(\mathbf{B}^+\right)^{-1}\Big)\mathbf{P}_s$ admits the analytic continuation into the half-plane $\mathrm{Re}[r] < 0$ and the second factor $\mathbf{P}_s\Big(\left(\mathbf{B}^-\right)^{-1}\mathbf{X}_4\mathbf{X}_3\Big) \times$ $\times\left[\left(r\mathbf{I} - \mathbf{X}_4\right)\left(r\mathbf{I} - \mathbf{X}_3\right)\right]^{-1}\left(\mathbf{B}^- + r\mathbf{I}\right)$ into $\mathrm{Re}[r] > 0$. Hence, these factors define the components of the factorization (for details see [17, Chapter I]) and

$$\bar{\mathbf{C}}_1^+ = \mathbf{B}^+\left(\mathbf{X}_2\mathbf{X}_1\right)^{-1}, \; \mathbf{C}_1^- = \left(\mathbf{X}_4\mathbf{X}_3\right)^{-1}\mathbf{B}^-, \tag{15}$$

$$\bar{\mathbf{C}}_2^+ = \mathbf{B}^+\left(\mathbf{X}_2\mathbf{X}_1\right)^{-1}\left(\mathbf{X}_2 + \mathbf{X}_1\right), \; \mathbf{C}_2^- = -\left(\mathbf{X}_4 + \mathbf{X}_3\right)\left(\mathbf{X}_4\mathbf{X}_3\right)^{-1}\mathbf{B}^-. \tag{16}$$

Substituting these formulas in relations (10) and (12) and inverting with respect of r gives the representations for $\mathbf{f}_+$ and $\mathbf{f}^-$. Considering process $\bar{X}_t = -X_t$ completes the proof. □

Remark 2.1. From (10) it follows that the generating function for the distribution of $X_{\theta_s}^{+}$ is a matrix with entries that are rational functions of degree two. To get the explicit expression we need to specify matrices $\bar{\mathbf{C}}_1^{+}$ and $\bar{\mathbf{C}}_2^{+}$ in terms of the process parameters. For the scalar case we have a simple algorithm which is based on finding roots of the cumulant equation; see [18, Section 5.7.1]. Asmussen et al. [3] offers a recent treatment for a jump diffusion with phase-type jumps. In general, we need a factorization of the matrix polynomial $\mathbf{P}[r] = \mathbf{P}_{+}[r]\left(s^{-1}\mathbf{\Sigma}\right)\mathbf{P}_{-}[r]$, where $\mathbf{P}_{+}[r]$ and $\mathbf{P}_{-}[r]$ are monic polynomials with disjoint spectra (see [15, Chapter 4] for more detail). The condition on roots of $\det \mathbf{P}[r]$ allows us to obtain the factorization in terms of matrices $\mathbf{X}_1, \ldots, \mathbf{X}_4$ which are (right) solvents for $\mathbf{P}[\mathbf{X}]$. Moreover, this condition provides the exponential form of the densities. So, we have the following algorithm:

(1) Solve the equation $\det \mathbf{P}[r] = 0$ and verify that all solutions are different.
(2) Find solvents (diagonalizable) for $\mathbf{P}[\mathbf{X}]$, such that
   - $\mathbf{X}_1, \mathbf{X}_2$ have only positive eigenvalues, and
   - $\mathbf{X}_3, \mathbf{X}_4$ have only negative eigenvalues.
(3) Express $\mathbf{E}e^{rX_{\theta_s}^{+}}$ in terms of $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{E}e^{r\left(X_{\theta_s}-X_{\theta_s}^{+}\right)}$ in terms of $\mathbf{X}_3, \mathbf{X}_4$ using (10), (15) and (12), (16) respectively.
(4) Invert the factorization components with respect to r to get the matrices $\mathbf{A}_k^{+}$ and $\bar{\mathbf{A}}_k^{-}$, $k = 1, \ldots, 2N$.

Breuer [5] used fixed-point equations to obtain the Laplace transform of the first passage time for phase-type MAPs without the restriction on roots of $\det \mathbf{P}[r]$. For the purpose of numerical inversion, Kim et al. [32] provided an algorithm for obtaining the Laplace transform with complex parameters. Since $\mathbf{E}\left[e^{-s\tau^{+}(x)}, \tau^{+} < \infty\right]\mathbf{P}_s = \mathbf{P}\left\{X_{\theta_s}^{+} > x\right\}$ we can adapt the iterative procedures to get a representation of the density of $X_{\theta_s}^{+}$ in a matrix exponential form. Using the theory of generalized Jordan chains D'Aura et al. [11] (see also [22]) obtained the Jordan normal form of the corresponding matrix exponent for a spectrally negative MAP which can be applied for Kou's processes in a Markov environment via the fluid embedding technique.

2.3. **One-sided exit problems.** The Kou process can cross a level x by hitting the level exactly (by creeping) or by a jump which causes an overshoot over (below) the level. That is, we have two different cases $\gamma^{\pm}(x) = 0$ and $\gamma^{\pm}(x) > 0$ (to stress the cases we use the corresponding left subscripts in our notation). Following [36], we can show that the first passage and the overshoot at first passage are conditionally independent given the state of J and that the overshoot is more than 0. Moreover, the overshoot is conditionally memoryless; see also [2].

Set

$$
{}_0\mathbf{T}_r^{\pm} = \mathbf{A}_r^{\pm}\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^{\pm}, \quad {}_{>}\mathbf{T}_r^{\pm} = \mathbf{A}_r^{\pm}\mathbf{P}_s^{-1}\left(\left(\rho_r^{\pm}\right)^{-1}\mathbf{I} - \bar{\mathbf{C}}_1^{\pm}\right),
$$

$$
v_{ij}^{\pm}(x, dy) = \mathsf{E}_i\left[e^{-s\tau^{\pm}(x)};\ \tau^{\pm}(x) < \infty, \gamma^{\pm}(x) \in dy, J_{\tau^{\pm}(x)} = j\right].
$$

Proposition 2.2. *Under the assumptions of Proposition 2.1, the integral transform of the joint distribution of* $\{\tau^{\pm}(x), \gamma^{\pm}(x)\}$ *has the next representation*

$$
v_{ij}^{\pm}(x, dy) = {}_0v_{ij}^{\pm}(x)\,\delta(dy) + {}_{>}v_{ij}^{\pm}(x)\,\beta_j^{\pm}e^{-\beta_j^{\pm}y}dy, \ \pm x \geq 0, y \geq 0,
$$

where passages by creeping and by jump are as follows

$$
{}_0v_{ij}^{\pm}(x) = \sum_{r=1}^{2N}\left({}_0\mathbf{T}_r^{\pm}\right)_{ij}e^{\mp\rho_r^{\pm}x}, \quad {}_{>}v_{ij}^{\pm}(x) = \sum_{r=1}^{2N}\left({}_{>}\mathbf{T}_r^{\pm}\right)_{ij}e^{\mp\rho_r^{\pm}x},
$$

$\delta(\cdot)$ *is the Dirac measure at 0.*

Proof. From [17, Corollary I.3.4], we can deduce

$$\begin{aligned}(17)\quad &\mathbf{E}\left[e^{-s\tau^{+}(x)};\ \tau^{+}(x)<\infty, X_{\tau^{+}(x)}-x\in dy, x-X_{\tau^{+}(x)-0}\in dz\right]\\ &=\left\|\mathsf{E}_i\left[e^{-s\tau^{+}(x)};\ \tau^{+}(x)<\infty,\gamma^{+}(x)\in dy, x-X_{\tau^{+}(x)-0}\in dz, J_{\tau^{+}(x)}=j\right]\right\|_{i,j=1}^{N}\\ &=\frac{\partial}{\partial x}\mathbf{P}\left\{X^{+}_{\theta_s}<x\right\}\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^{+}\delta(dy)\,\delta(dz)\\ &+s^{-1}\int_{u=0}^{\min\{x,z\}}\mathbf{P}\left\{X^{+}_{\theta_s}\in x-du\right\}\mathbf{P}_s^{-1}\mathbf{P}\left\{X^{+}_{\theta_s}-X_{\theta_s}\in dz-u\right\}\mathbf{\Lambda f}(y+z)\,dy.\end{aligned}$$

Substituting the representation of $\mathbf{f}$ and integrating in y and z yields

$$\begin{aligned}\mathbf{E}\left[e^{-s\tau^{+}(x)};\ \tau^{+}(x)<\infty\right]&=\frac{\partial}{\partial x}\mathbf{P}\left\{X^{+}_{\theta_s}<x\right\}\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^{+}+\\ &+s^{-1}\int_0^x\mathbf{f}_{+}(u)\,\mathbf{P}_s^{-1}\int_{-\infty}^{0}\mathbf{f}^{-}(z)\,\mathbf{\Lambda p}^{+}e^{-\mathbf{B}^{+}(x-z-u)}dzdu.\end{aligned}$$

Hence,

$$\begin{aligned}s^{-1}\int_0^x\mathbf{f}_{+}(u)\,\mathbf{P}_s^{-1}\int_{-\infty}^{0}\mathbf{f}^{-}(z)\,\mathbf{\Lambda p}^{+}e^{-\mathbf{B}^{+}(x-z-u)}dzdu\\ =\mathbf{P}\left\{X^{+}_{\theta_s}>x\right\}\mathbf{P}_s^{-1}-\frac{\partial}{\partial x}\mathbf{P}\left\{X^{+}_{\theta_s}<x\right\}\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^{+}\end{aligned}$$

and from (17) we can deduce

$$\begin{aligned}\mathbf{E}\left[e^{-s\tau^{+}(x)};\ \tau^{+}(x)<\infty,\gamma^{+}(x)\in dy\right]&=\frac{\partial}{\partial x}\mathbf{P}\left\{X^{+}_{\theta_s}<x\right\}\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^{+}\delta(dy)\\ &+\left(\mathbf{P}\left\{X^{+}_{\theta_s}>x\right\}\mathbf{P}_s^{-1}-\frac{\partial}{\partial x}\mathbf{P}\left\{X^{+}_{\theta_s}<x\right\}\mathbf{P}_s^{-1}\bar{\mathbf{C}}_1^{+}\right)\mathbf{B}^{+}e^{-\mathbf{B}^{+}y}dy.\end{aligned}$$

Taking into account the results from Proposition 2.1 and considering process $\bar{X}_t=-X_t$ completes the proof. □

Remark 2.2. Fluctuation identity (17) for the scalar case can be considered as a marginal of the quintuple law for overshoots and undershoots from [12] and represents the Gerber-Shiu measure in risk theory. Breuer [6] obtained an explicit matrix exponential representation of the quintuple law for phase-type MAPs. Using [6, Theorem 1] we can get an alternative representation for $\boldsymbol{\pi}\mathbf{E}\left[e^{-s\tau^{+}(x)};\ \tau^{+}(x)<\infty,\gamma^{+}(x)\in dy\right]\mathbf{e}$, where $\mathbf{e}$ is a column vector of 1's of appropriate dimension.

For passage by jump we have

$$\begin{aligned}&\mathsf{E}\left[e^{-s\tau^{+}(x)};\ \gamma^{+}(x)\in dy|\tau^{+}(x)<\infty,\gamma^{+}(x)>0,J_{\tau^{+}(x)}=j\right]\\ &\qquad=\mathsf{E}\left[e^{-s\tau^{+}(x)}|\tau^{+}(x)<\infty,\gamma^{+}(x)>0,J_{\tau^{+}(x)}=j\right]\beta_j^{+}e^{-\beta_j^{+}y}dy,\end{aligned}$$

which justify the conditional independence of the first passage and the overshoot at the first passage.

2.4. **Two-sided exit problems.** Set

$$u_{ij}^{+}(x,dy) = \mathsf{E}_i\left[e^{-s\tau(x,b)};\ \gamma(x,b)\in dy, J(\tau(x,b))=j, X_{\tau(x,b)}\geq x\right],$$

$$u_{ij}^{-}(x,dy) = \mathsf{E}_i\left[e^{-s\tau(x,b)};\ \gamma(x,b)\in dy, J(\tau(x,b))=j, X_{\tau(x,b)}\leq x-b\right],$$

$$\mathbf{U}^{\pm}(x,v) = \left\|\int_0^{\infty} e^{-vy}u_{ij}^{\pm}(x,dy)\right\|_{i,j=1}^{N}.$$

Here v is such that $\int_0^{\infty} e^{-vy}u_{ij}^{\pm}(x,dy)$ are finite.

To derive the main result in this section we employ an approximation method generalizing ideas given in [31]. For a non-step Lévy process ξ_t we can construct a sequence of step processes $\xi_n(t)$ such that $\mathsf{E}e^{r\xi_n(t)}\to\mathsf{E}e^{r\xi_t}$, $n\to\infty$. Since $\{\tau^{+}(x),\gamma^{+}(x)\}$ are functionals almost everywhere continuous in the Skorokhod topology on cádlág functions, the joint Laplace transform for $\{\tau_n^{+}(x),\gamma_n^{+}(x)\}$ converges to the Laplace transform of $\{\tau^{+}(x),\gamma^{+}(x)\}$, see [14, Theorem IV.3]; for two-sided boundary functionals see also [4, Chapter 1, Section 3].

For the Markov modulated Kou process we construct an approximating process $Z_t^{(n)}=\left\{X^{(n)}(t),J_t\right\}$ as follows. If $J(t)=k$, $X^{(n)}(t)$ evolves like a compound Poisson process with drift $a_k^{(n)}=a_k+\frac{3n}{2}\sigma_k^2$, with intensity of jumps $\lambda_k^{(n)}=\lambda_k p_k^{+}+3n^2\sigma_k^2+\lambda_k p_k^{-}e^{-\frac{\beta_k^{-}}{n}}$, and the density of jumps is

$$f_k^{(n)}(x)=\begin{cases} p_1(k,n)\,\beta_k^{+}e^{-\beta_k^{+}x}, & x\geq 0;\\ p_2(k,n)\,n, & -\frac{1}{n}\leq x<0;\\ p_3(k,n)\,\beta_k^{-}e^{\beta_k^{-}(x+1/n)}, & x\leq -\frac{1}{n},\end{cases}$$

where $p_1(k,n)=\frac{\lambda_k p_k^{+}}{\lambda_k^{(n)}}, p_2(k,n)=\frac{3n^2\sigma_k^2}{\lambda_k^{(n)}}, p_3(k,n)=\frac{\lambda_k p_k^{-}}{\lambda_k^{(n)}}e^{-\beta_k^{-}/n}$. Such a construction of $X_t^{(n)}$ grants both possibilities for passage over a level by creeping (for large enough n) and by exponential jump. In this section, we add right subscript (n) to our notation to distinguish the corresponding characteristics of $Z^{(n)}$ from those of Z. To justify the name approximating process we consider the cumulant function for $Z^{(n)}$:

$$\mathbf{K}_{(n)}[r]=r\mathbf{A}_{(n)}+\int_{-\infty}^{\infty}(e^{rx}-1)\,\mathbf{\Lambda}_{(n)}\mathbf{f}_{(n)}(x)dx+\mathbf{Q}.$$

Using the representations of $\mathbf{A}_{(n)}=diag\left(a_1^{(n)},\dots,a_N^{(n)}\right)$, $\mathbf{\Lambda}_{(n)}=diag\left(\lambda_1^{(n)},\dots,\lambda_N^{(n)}\right)$, and $\mathbf{f}_{(n)}(x)=diag\left(f_1^{(n)}(x),\dots,f_N^{(n)}(x)\right)$ given above after some manipulations yields (for the scalar case see [31])

$$\mathbf{K}_{(n)}[r]=r\mathbf{A}+r^2\mathbf{\Sigma}(1+o(1))+r\mathbf{\Lambda}\mathbf{p}^{+}\left(\mathbf{B}^{+}-r\mathbf{I}\right)^{-1} - r\mathbf{\Lambda}\mathbf{p}^{-}\left(\mathbf{B}^{-}+r\mathbf{I}\right)^{-1}e^{-\frac{1}{n}\mathbf{B}^{-}}+\mathbf{Q},$$

where $o(1)\to 0$, as $n\to\infty$. Hence,

$$\mathbf{E}e^{rX_t^{(n)}}=e^{t\mathbf{K}_{(n)}[r]}\underset{n\to\infty}{\longrightarrow}e^{t\mathbf{K}[r]}=\mathbf{E}e^{rX_t}.$$

Following [17, pp. 17–18], we can get the Laplace transform for $\{\tau(x,b),\gamma(x,b)\}$ as the limit of corresponding Laplace transform for the approximating process: $\mathbf{U}^{\pm}(x,v)=\lim_{n\to\infty}\mathbf{U}_{(n)}^{\pm}(x,v)$.

Write

$$\left\|{}_0L_{ij}^{\pm}(x)\right\|_{i,j=1}^{N}=\begin{pmatrix}\mathbf{J}_1^{\pm}(x) & \mathbf{J}_2^{\pm}(x)\end{pmatrix}\begin{pmatrix}\mathbf{J}_1^{\pm}(b) & \mathbf{J}_2^{\pm}(b)\\ \tilde{\mathbf{J}}_1^{\pm} & \mathbf{I}+\tilde{\mathbf{J}}_2^{\pm}\end{pmatrix}^{-1}\begin{pmatrix}{}_0\mathbf{V}^{\pm}\\ {}_0\tilde{\mathbf{V}}^{\pm}\end{pmatrix},$$

$$\left\|{}_{>}L_{ij}^{\pm}(x)\right\|_{i,j=1}^{N} = \begin{pmatrix} \mathbf{J}_1^{\pm}(x) & \mathbf{J}_2^{\pm}(x) \end{pmatrix} \begin{pmatrix} \mathbf{J}_1^{\pm}(b) & \mathbf{J}_2^{\pm}(b) \\ \tilde{\mathbf{J}}_1^{\pm} & \mathbf{I}+\tilde{\mathbf{J}}_2^{\pm} \end{pmatrix}^{-1} \begin{pmatrix} {}_{>}\mathbf{V}^{\pm} \\ \mathbf{I}+{}_{>}\tilde{\mathbf{V}}^{\pm} \end{pmatrix},$$

where

$$\mathbf{J}_1^{\pm}(x) = \sum_{k,l=1}^{2N} \mathbf{A}_k^{\pm}\mathbf{P}_s^{-1}\bar{\mathbf{A}}_l^{\mp} \frac{e^{-\rho_l^{\mp} b}}{\rho_k^{\pm}+\rho_l^{\mp}} \left(e^{\rho_l^{\mp} x} - e^{-\rho_k^{\pm} x}\right);$$

$$\mathbf{J}_2^{\pm}(x) = \sum_{k,l=1}^{2N} \mathbf{A}_k^{\pm}\mathbf{P}_s^{-1}\bar{\mathbf{A}}_l^{\mp} \frac{e^{-\rho_l^{\mp} b}}{\rho_k^{\pm}+\rho_l^{\mp}} \left(e^{\rho_l^{\mp} x} - e^{-\rho_k^{\pm} x}\right) s^{-1}\mathbf{\Lambda}\mathbf{p}^{\pm}\left(\rho_l^{\mp}\mathbf{I}+\mathbf{B}^{\pm}\right)^{-1} e^{-\mathbf{B}^{\pm} b};$$

$${}_{0}\mathbf{V}^{\pm} = \left\|{}_{0}v_{ij}^{\pm}(\pm b)\right\|, \ {}_{0}\tilde{\mathbf{V}}^{\pm} = \int_0^b \mathbf{B}^{\pm} e^{\mathbf{B}^{\pm} z} \left\|{}_{0}v_{ij}^{\pm}(\mp z)\right\| dz,$$

$${}_{>}\mathbf{V}^{\pm} = \left\|{}_{>}v_{ij}^{\pm}(\pm b)\right\|, \ {}_{>}\tilde{\mathbf{V}}^{\pm} = \int_0^b \mathbf{B}^{\pm} e^{\mathbf{B}^{\pm} z} \left\|{}_{>}v_{ij}^{\pm}(\mp z)\right\| dz,$$

$$\tilde{\mathbf{J}}_l^{\pm} = \int_0^b \mathbf{B}^{\pm} e^{\mathbf{B}^{\pm} z} \mathbf{J}_l^{\pm}(z)\, dz, l = 1, 2.$$

As for one-sided boundary functionals, we can state that the first exit time from an interval and the corresponding overshoot are conditionally independent given that $\{\gamma(x,b) > 0\}$.

Proposition 2.3. *Under the assumptions of Proposition 2.1, if the corresponding inverse matrices exist, then the integral transform of the joint distribution of* $\{\tau(x,b), \gamma(x,b)\}$ *has the next representation*

$$u_{ij}^{\pm}(x, dy) = {}_{0}u_{ij}^{\pm}(x)\,\delta(dy) + {}_{>}u_{ij}^{\pm}(x)\,\beta_j^{\pm} e^{-\beta_j^{\pm} y} dy, \ \pm x \geq 0, y \geq 0, \tag{18}$$

where passage over upper level x *is determined by*

$${}_{0}u_{ij}^{+}(x) = {}_{0}v_{ij}^{+}(x) - {}_{0}L_{ij}^{+}(x), \ {}_{>}u_{ij}^{+}(x) = {}_{>}v_{ij}^{+}(x) - {}_{>}L_{ij}^{+}(x),$$

and passage below lower level $x-b$ *by*

$${}_{0}u_{ij}^{-}(x) = {}_{0}v_{ij}^{-}(x-b) - {}_{0}L_{ij}^{-}(b-x), \ {}_{>}u_{ij}^{-}(x) = {}_{>}v_{ij}^{-}(x-b) - {}_{>}L_{ij}^{-}(b-x).$$

Proof. Consider the approximating process $Z_t^{(n)} = \left\{X^{(n)}(t), J_t\right\}$. According to the total probability law and the strong Markov property the following integro-differential equation holds $(0 < x < b)$

$$\mathbf{A}_{(n)} \frac{\partial}{\partial x}\mathbf{U}_{(n)}^{+}(x,v) = -\left(s\mathbf{I} - \mathbf{Q} + \mathbf{\Lambda}_{(n)}\right)\mathbf{U}_{(n)}^{+}(x,v) + \int_{-\infty}^{\infty} \mathbf{\Lambda}_{(n)}\mathbf{f}_{(n)}(z)\,\mathbf{U}_{(n)}^{+}(x-z,v)\,dz,$$

with boundary conditions

$$\mathbf{U}_{(n)}^{+}(x,v) = \begin{cases} 0, & x \geq b; \\ e^{vx}\mathbf{I}, & x \leq 0. \end{cases}$$

Extending the equation for $x \geq b$ gives

$$\mathbf{A}_{(n)} \frac{\partial}{\partial x}\mathbf{U}_{(n)}^{+}(x,v) = -\left(s\mathbf{I} - \mathbf{Q} + \mathbf{\Lambda}_{(n)}\right)\mathbf{U}_{(n)}^{+}(x,v) + \int_{-\infty}^{\infty} \mathbf{\Lambda}_{(n)}\mathbf{f}_{(n)}(z)\,\mathbf{U}_{(n)}^{+}(x-z,v)\,dz$$
$$-\mathbf{\Lambda}_{(n)}\mathbf{p}_{(n)}^{+} e^{-\mathbf{B}^{+}x}\left(\mathbf{B}^{+}\left(v\mathbf{I}+\mathbf{B}^{+}\right)^{-1} + \int_0^b \mathbf{B}^{+}e^{\mathbf{B}^{+}z}\mathbf{U}_{(n)}^{+}(z,v)\,dz\right) I_{\{x\geq b\}}, \ 0 < x < \infty.$$

For the scalar case, this equation and the one we get after the passage to the limit as $n \to \infty$ were studied in [31]. Applying similar considerations as in the scalar case and paying extra attention to non commutativity of matrix product yields

$$\mathbf{U}^{+}(x,v)=\mathbf{E}\left[e^{-s\tau^{+}(x)-v\gamma^{+}(x)};\ \tau^{+}(x)<\infty\right]$$
$$-\int_{-b}^{x-b}\mathbf{f}_{+}(x-y-b)\,\mathbf{P}_s^{-1}\mathbf{f}^{-}(y)\,dy\mathbf{C}_1^{+}(v,b)$$
$$-\int_0^x\mathbf{f}_{+}(y)\,\mathbf{P}_s^{-1}\int_{-\infty}^{x-y-b}\mathbf{f}^{-}(z)\,s^{-1}\mathbf{\Lambda}\mathbf{p}^{+}e^{-\mathbf{B}^{+}(x-y-z)}dzdy\mathbf{C}_2^{+}(v,b),\tag{19}$$

where $\mathbf{C}_1^{+}(v,b)$ and $\mathbf{C}_2^{+}(v,b)$ satisfy the following system of equations

$$\begin{cases}\mathbf{U}^{+}(b,v)=0,\\ \mathbf{B}^{+}\left(v\mathbf{I}+\mathbf{B}^{+}\right)^{-1}+\int_0^b\mathbf{B}^{+}e^{\mathbf{B}^{+}z}\mathbf{U}^{+}(z,v)\,dz=\mathbf{C}_2^{+}(v,b).\end{cases}$$

Under the conditions of the proposition we find

$$\begin{pmatrix}\mathbf{C}_1^{+}(v,b)\\ \mathbf{C}_2^{+}(v,b)\end{pmatrix}=\begin{pmatrix}\mathbf{J}_1^{+}(b) & \mathbf{J}_2^{+}(b)\\ \tilde{\mathbf{J}}_1^{+} & \mathbf{I}+\tilde{\mathbf{J}}_2^{+}\end{pmatrix}^{-1}\left(\begin{pmatrix}{}_0\mathbf{V}^{+}\\ {}_0\tilde{\mathbf{V}}^{+}\end{pmatrix}+\begin{pmatrix}{}_>\mathbf{V}^{+}\\ \mathbf{I}+{}_>\tilde{\mathbf{V}}^{+}\end{pmatrix}\mathbf{B}^{+}\left(v\mathbf{I}+\mathbf{B}^{+}\right)^{-1}\right)$$

and combining with (19) gives (18).

To get the expression for $\mathbf{U}^{-}(x,v)$ we can use the fact that $\mathbf{U}^{-}(x,v)=\bar{\mathbf{U}}^{+}(b-x,v)$, where $\bar{\mathbf{U}}^{+}(x,u)$ is the integral transform of the joint distribution of $\{\tau(x,b),\gamma(x,b)\}$ for the process $\bar{Z}_t=\{-X_t,J_t\}$. ☐

Corollary 2.1. *Under the assumptions of Proposition 2.1, the density of X killed before exit from $(x-b,x)$ can be expressed as follows $(x-b<z<x)$*

$$\frac{\partial}{\partial z}\mathbf{P}\left\{X_{\theta_s}<z,\ \tau(x,b)>\theta_s\right\}=\left\|\frac{\partial}{\partial z}\mathsf{P}_i\left\{X_{\theta_s}<z,\ \tau(x,b)>\theta_s,J_{\theta_s}=j\right\}\right\|_{i,j=1}^{m}$$
$$=\sum_{k,l=1}^{2N}\mathbf{A}_k^{+}\mathbf{P}_s^{-1}\bar{\mathbf{A}}_l^{-}\frac{1}{\rho_k^{+}+\rho_l^{-}}\left(e^{-\rho_k^{+}z}\left(e^{\left(\rho_k^{+}+\rho_l^{-}\right)(z\wedge 0)}-e^{\left(\rho_k^{+}+\rho_l^{-}\right)(x-b)}\right)\right.$$
$$\left.-e^{-\rho_l^{-}b}\left(e^{\rho_l^{-}(b-x+z)}-e^{-\rho_k^{+}(b-x+z)}\right)\left({}_0\mathbf{U}^{+}(x)+{}_>\mathbf{U}^{+}(x)\,\mathbf{B}^{+}\left(\rho_l^{-}\mathbf{I}-\mathbf{B}^{+}\right)^{-1}\right)\right),$$

where ${}_0\mathbf{U}^{+}(x)=\left\|{}_0u_{ij}^{+}(x)\right\|$, ${}_>\mathbf{U}^{+}(x)=\left\|{}_>u_{ij}^{+}(x)\right\|$.

Proof. Inverting the fluctuation identity [30, (12)] with respect to α gives

$$\mathbf{P}\left\{X_{\theta_s}\in dz,\ \tau(x,b)>\theta_s\right\}=\int_{x-b}^{z}\mathbf{P}\left\{X_{\theta_s}^{+}\in dz-y\right\}\mathbf{P}_s^{-1}$$
$$\times\left(\mathbf{P}\left\{\left(X_{\theta_s}-X_{\theta_s}^{+}\right)\in dy\right\}I_{\{y\le 0\}}\right.$$
$$\left.-\int_0^{\infty}\mathbf{P}\left\{\left(X_{\theta_s}-X_{\theta_s}^{+}\right)\in dy-x-v\right\}\mathbf{E}\left[e^{-s\tau(x,b)};\ \gamma(x,b)\in dv,X_{\tau(x,b)}\ge x\right]\right)\tag{20}$$

and Proposition 2.3 leads to

$$\frac{\partial}{\partial z}\mathbf{P}\left\{X_{\theta_s}<z,\ \tau(x,b)>\theta_s\right\}=\int_{x-b}^{z\wedge 0}\mathbf{f}_{+}(z-y)\,\mathbf{P}_s^{-1}\mathbf{f}^{-}(y)\,dy$$
$$-\int_{x-b}^{z}\mathbf{f}_{+}(z-y)\,\mathbf{P}_s^{-1}\mathbf{f}^{-}(y-x)\,\mathbf{U}_0^{+}(x)\,dy$$
$$-\int_{x-b}^{z}\mathbf{f}_{+}(z-y)\,\mathbf{P}_s^{-1}\int_0^{\infty}\mathbf{f}^{-}(y-x-v)\,\mathbf{U}_>^{+}(x)\,\mathbf{B}^{+}e^{-\mathbf{B}^{+}v}dvdy,$$

which gives the assertion of the corollary when combined with (8) and (9). □

Remark 2.3. For scalar MAPs Kadankov and Kadankova obtained representations of the integral transforms of the joint distribution for $\{\tau(x,b), \gamma(x,b)\}$ in terms of series of integral transforms for one-boundary functionals (see [29, Theorem 1]); for a series form of linear operators we refer to [38, Theorem 4]. We can extend these representations for general MAPs and then convert them to the specific setting of Markov modulated Kou's processes. Moreover, for a Markov modulated Brownian motion Gusak in [16] (see also [17]) showed that $\mathbf{U}^{\pm}(x,0)$ can be represented as a "fraction" of two matrix exponential expressions. We can use this fact to justify that the nonsingularity assumption for $\begin{pmatrix} \mathbf{J}_1^{\pm}(b) & \mathbf{J}_2^{\pm}(b) \\ \tilde{\mathbf{J}}_1^{\pm} & \mathbf{I}+\tilde{\mathbf{J}}_2^{\pm} \end{pmatrix}$ is a mild restriction. For MAPs with phase-type jumps the two-sided exit problem was solved in [27, Proposition 1]; see also [9, Section 3.3]. Ivanovs [21] and Ivanovs et al. [23] considered two-sided exit problems for a Markov-modulated Brownian motion with a two-sided reflection.

To solve one-sided and two-sided exit problems for spectrally negative MAPs, Korolyuk and Shurenkov [34] proposed the so called potential method. They showed that $\mathbf{U}^{+}(x,0)$ can be represented as a "fraction" of the values of the resolvent function (the scale matrix) at x and b. For the probabilistic construction of the scale matrix see [24]. The reader is referred to [25] for the densities of potential measures for spectrally negative MAPs with reflecting or terminating upper and lower barriers.

For the scalar case the fluctuation identity (20) is a modification of the Kemperman-Pecherskii identity given in [42] (see also [4, Section 3.7]). A series form for the joint distribution of the supremum, infimum and the values of the process

$$\mathsf{P}\left\{X_t \in (\alpha,\beta), X_t^- \geq x-b, X_t^+ \leq x\right\} = \mathsf{P}\left\{X_t \in (\alpha,\beta), \tau(x,b) \geq t\right\},$$

$x-b \leq \alpha < \beta \leq x$, in terms of "convolutions" of $\mathsf{P}\{\tau^{\pm}(\cdot) \in dt, \gamma^{\pm}(\cdot) \in dy\}$ may be found in [14, (IV.2.57)] and [4, Lemma 6.1]. It provides an alternative way of reasoning.

2.5. **Example.** To demonstrate, we consider the two state space, i.e., $N=2$. Let $s=5$,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \mathbf{\Sigma} = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}, \mathbf{\Lambda} = \begin{pmatrix} 5/2 & 0 \\ 0 & 4 \end{pmatrix},$$

$$\mathbf{p}^{+} = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/4 \end{pmatrix}, \mathbf{B}^{+} = \begin{pmatrix} 2 & 0 \\ 0 & 3/2 \end{pmatrix}, \mathbf{B}^{-} = \begin{pmatrix} 3 & 0 \\ 0 & 5/2 \end{pmatrix},$$

and $\mathbf{Q} = \begin{pmatrix} -0.1 & 0.1 \\ 0.05 & -0.05 \end{pmatrix}$.

Process J_t is time-reversible with stationary distribution $\boldsymbol{\pi} = (1/3, 2/3)$ and $\mathsf{E}X_1 = -1.49\bar{4}$. The roots of $\det \mathbf{P}[r] = 0$ (to 3 s.f.) are (see Figure 1)

$$\rho_4^+ = 2.11, \rho_3^+ = 1.64, \rho_2^+ = 1.13, \rho_1^+ = 0.878,$$

$$-\rho_1^- = -0.623, -\rho_2^- = -1.30, -\rho_3^- = -2.64, -\rho_4^- = -3.28.$$

The solvents of $\mathbf{P}[\mathbf{X}]$ are diagonazible and can be found by the method from [47] (let $\mathbf{v}$ be an eigenvector for $\mathbf{X}$ with corresponding eigenvalue λ which is a root of the equation $\det \mathbf{P}[r] = 0$. Then $\mathbf{v}$ can be taken from the nonzero elements of $Null(\mathbf{P}[\lambda])$):

$$\mathbf{X}_1 = \begin{pmatrix} 2.11 & -0.00292 \\ -0.000199 & 1.64 \end{pmatrix}, \mathbf{X}_2 = \begin{pmatrix} 1.13 & -0.0107 \\ -0.00309 & 0.878 \end{pmatrix},$$

$$\mathbf{X}_3 = \begin{pmatrix} -1.30 & 0.00481 \\ 0.00830 & -0.623 \end{pmatrix}, \mathbf{X}_4 = \begin{pmatrix} -3.28 & 0.000306 \\ 0.000459 & -2.64 \end{pmatrix}.$$

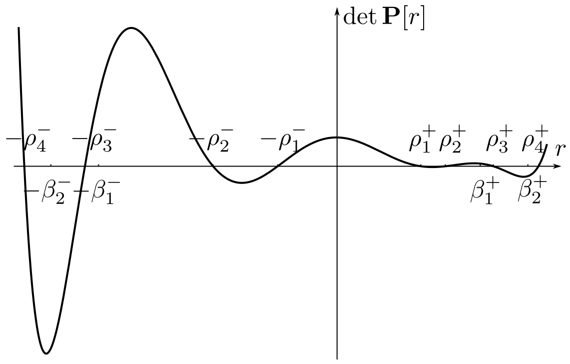

FIGURE 1. Roots of $\det \mathbf{P}[r] = 0$

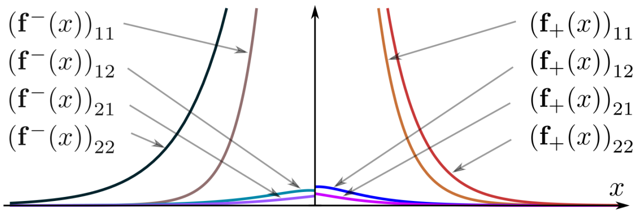

FIGURE 2. Defective densities of $X_{\theta_s}^+$ and $X_{\theta_s} - X_{\theta_s}^+$

The coefficient matrices for the density of killed running maximum in (8) are as follows

$$\mathbf{A}_4^+ = \begin{pmatrix} 0.133 & -0.000387 \\ -0.000308 & 9.00 \times 10^{-7} \end{pmatrix}, \mathbf{A}_3^+ = \begin{pmatrix} 1.27 \times 10^{-6} & -0.00280 \\ -0.0000778 & 0.172 \end{pmatrix},$$
$$\mathbf{A}_2^+ = \begin{pmatrix} 1.04 & -0.0268 \\ -0.0101 & 0.000261 \end{pmatrix}, \mathbf{A}_1^+ = \begin{pmatrix} 0.000839 & 0.0395 \\ 0.0165 & 0.777 \end{pmatrix}.$$

Similarly, matrices in (9) are

$$\bar{\mathbf{A}}_4^- = \begin{pmatrix} 0.000195 & 0.0162 \\ 0.00729 & 0.606 \end{pmatrix}, \bar{\mathbf{A}}_3^- = \begin{pmatrix} 1.20 & -0.00864 \\ -0.00270 & 0.0000195 \end{pmatrix},$$
$$\bar{\mathbf{A}}_2^- = \begin{pmatrix} -4.59 \times 10^{-7} & 0.000255 \\ -0.0000832 & 0.0461 \end{pmatrix}, \bar{\mathbf{A}}_1^- = \begin{pmatrix} 0.196 & -0.000264 \\ 0.000347 & -4.68 \times 10^{-7} \end{pmatrix}.$$

Figure 2 illustrates graphs of defective densities $(\mathbf{f}_+(x))_{ij} = \frac{\partial}{\partial x} \mathsf{P}_i \left\{ X_{\theta_s}^+ < x,\ J_{\theta_s} = j \right\}$ and $(\mathbf{f}^-(x))_{ij} = \frac{\partial}{\partial x} \mathsf{P}_i \left\{ X_{\theta_s} - X_{\theta_s}^+ < x,\ J_{\theta_s} = j \right\}$, $i, j \in \{1, 2\}$.

Using the results from [5] we can also deduce the following representation

$$\mathbf{f}^+(x) = -\begin{pmatrix} 0 & \mathbf{I} \end{pmatrix} \left(\mathbf{R} e^{\mathbf{R}x} \right) \begin{pmatrix} \mathbf{I} \\ \mathbf{I} \end{pmatrix} \mathbf{P}_s,$$

where matrix $\mathbf{R}$ satisfies the fixed point equations given in [5, Theorem 3]. After 5000 iterations we get

$$\mathbf{R} = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -1.5 & 0 & 1.5 \\ 0.048 & 0.000 & -1.245 & 0.014 \\ 0.000 & 0.057 & 0.003 & -1.016 \end{pmatrix}$$

and the representation for $\mathbf{f}^+$ is consistent with the results above.

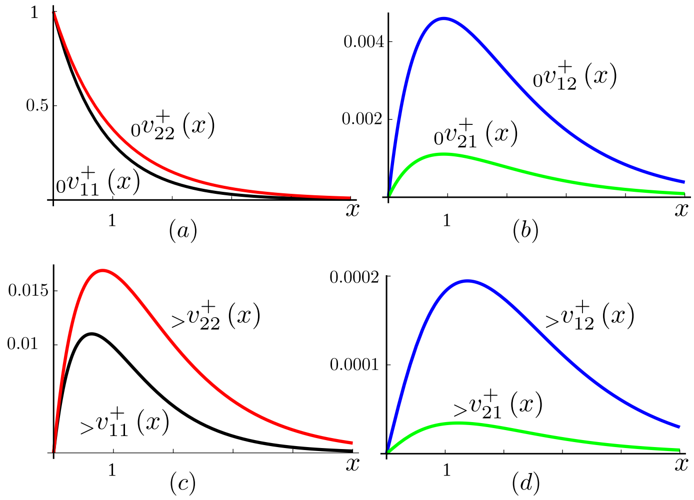

FIGURE 3. Passage by creeping $((a),(b))$ and by jump $((c),(d))$. *Take attention to different scales for y axis.*

For passage by creeping (to 3 s.f.) we have

$$\begin{aligned}
{}_0v_{11}^+(x) &= 0.113e^{-2.11x} + 0.0000162e^{-1.64x} + 0.886e^{-1.13x} + 0.000502e^{-0.878x},\\
{}_0v_{12}^+(x) &= -0.00154e^{-2.11x} - 0.00295e^{-1.64x} - 0.0371e^{-1.13x} + 0.0416e^{-0.878x},\\
{}_0v_{21}^+(x) &= -0.000263e^{-2.11x} - 0.000995e^{-1.64x} - 0.00862e^{-1.13x} + 0.00988e^{-0.878x},\\
{}_0v_{22}^+(x) &= 3.57\times10^{-6}e^{-2.11x} + 0.181e^{-1.64x} + 0.000361e^{-1.13x} + 0.819e^{-0.878x},
\end{aligned}$$

and for passage by jump

$$\begin{aligned}
{}_>v_{11}^+(x) &= -0.0490e^{-2.11x} + 1.66\times10^{-6}e^{-1.64x} + 0.0490e^{-1.13x} + 0.0000236e^{-0.878x},\\
{}_>v_{12}^+(x) &= 0.0000936e^{-2.11x} + 0.00122e^{-1.64x} - 0.00512e^{-1.13x} + 0.00380e^{-0.878x},\\
{}_>v_{21}^+(x) &= 0.0001142e^{-2.11x} - 0.000102e^{-1.64x} - 0.000476e^{-1.13x} + 0.000464e^{-0.878x},\\
{}_>v_{22}^+(x) &= -2.17\times10^{-7}e^{-2.11x} - 0.0749e^{-1.64x} + 0.0000498e^{-1.13x} + 0.0749e^{-0.878x}.
\end{aligned}$$

Figure 3 shows behavior of ${}_0v_{km}^+(x)$ and ${}_>v_{km}^+(x)$.

Particularly, for passage by creeping we get

$$\begin{aligned}
\mathsf{E}&\left[e^{-s\tau^+(x)}; \tau^+(x) < \infty; \gamma^+(x) = 0\right]\\
&= 0.0371e^{-2.11x} + 0.119e^{-1.64x} + 0.278e^{-1.13x} + 0.566e^{-0.878x},
\end{aligned}$$

which is consistent with the result we can get from [6, Corollary 2].

With $b = 3$, for passages by creeping and by jump through upper and lower bounds we obtain correspondingly

$$\begin{aligned}
{}_0u_{11}^+(x) &= 0.113e^{-2.11x} + 0.0000161e^{-1.64x} + 0.887e^{-1.13x} + 0.000505e^{-0.878x}\\
&\quad - 1.87\times10^{-6}e^{0.623x} - 0.00055e^{1.30x} + 1.15\times10^{-11}e^{2.64x} - 1.34\times10^{-7}e^{3.28x},
\end{aligned}$$

$$\begin{aligned}
{}_>u_{11}^+(x) &= -0.0490e^{-2.11x} + 1.66\times10^{-6}e^{-1.64x} + 0.0490e^{-1.13x} + 0.0000237e^{-0.878x}\\
&\quad - 8.15\times10^{-8}e^{0.623x} - 0.0000286e^{1.30x} + 4.17\times10^{-13}e^{2.64x} - 7.01\times10^{-9}e^{3.28x},
\end{aligned}$$

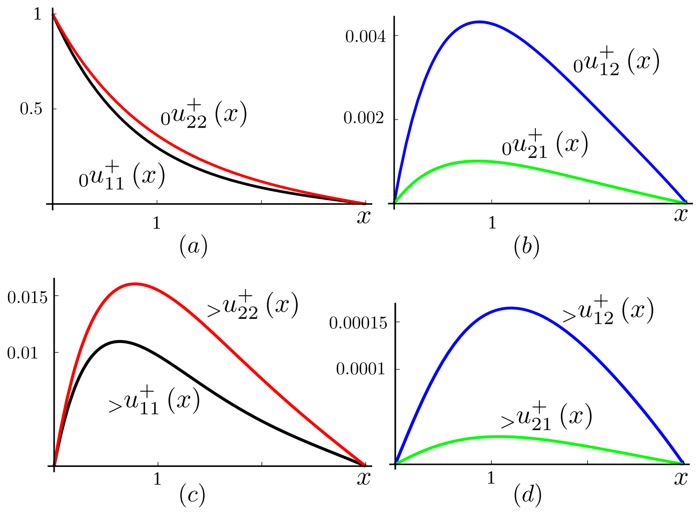

FIGURE 4. Passage by creeping $((a),(b))$ and by jump $((c),(d))$ over upper level x. *Take attention to different scales for y axis.*

$$\begin{aligned}{}_0u_{11}^{-}(x) &= -0.00148e^{-2.11x} + 6.15\times 10^{-7}e^{-1.64x} - 0.0159e^{-1.13x} - 0.0000285e^{-0.878x} \\ &\quad + 0.0000124e^{0.623x} + 0.0174e^{1.30x} + 1.30\times 10^{-10}e^{2.64x} + 7.53\times 10^{-6}e^{3.28x},\end{aligned}$$

$$\begin{aligned}{}_>u_{11}^{-}(x) &= -0.000137e^{-2.11x} + 4.22\times 10^{-8}e^{-1.64x} - 0.00147e^{-1.13x} - 2.33\times 10^{-6}e^{-0.878x} \\ &\quad + 9.72\times 10^{-7}e^{0.623x} + 0.00162e^{1.30x} + 3.17\times 10^{-11}e^{2.64x} - 4.27\times 10^{-6}e^{3.28x}.\end{aligned}$$

Figures 4 and 5 illustrate graphs of ${}_0u_{km}^{\pm}(x)$ and ${}_>u_{km}^{\pm}(x)$.

Using Corollary 2.1 we get the density

$$\frac{\partial}{\partial z}\mathsf{P}\{X_{\theta_s} < z,\ \tau(x,b) > \theta_s\} = \frac{\partial}{\partial z}\boldsymbol{\pi}\mathbf{P}\{X_{\theta_s} < z,\ \tau(x,b) > \theta_s\}\mathbf{e}$$

with $b = 3$ (see Figure 6). E.g., for $x = \frac{1}{2}$ we have

$$\begin{aligned}&\frac{\partial}{\partial z}\mathsf{P}\{X_{\theta_s} < z,\ \tau(0.5,3) > \theta_s\} \\ &= \begin{cases} 0.0173e^{-2.11z} + 0.0316e^{-1.64z} + 0.182e^{-1.13z} \\ +0.226e^{-0.878z} - 0.111e^{0.623z} - 0.0539e^{1.30z} & -2.5 < z \le 0, \\ -0.00131e^{2.64z} - 0.00185e^{3.28z}, \\ -2.24\times 10^{-6}e^{-2.11z} - 0.0000602e^{-1.64z} - 0.000259e^{-1.13z} \\ -0.00285e^{-0.878z} + 0.137e^{0.623z} + 0.133e^{1.30z} & 0 < z < 0.5. \\ +0.00677e^{2.64z} + 0.0155e^{3.28z}, \end{cases}\end{aligned}$$

3. FIRST PASSAGE THROUGH STATE-DEPENDENT LEVELS

The main objects of research in this section are the integral transforms

$$w_{ij}^{\pm}(b) = w_{ij}^{\pm}(b,s,v) = \mathsf{E}_i\left[e^{-sT_b^{\pm}+v\gamma_b^{\pm}};\ T_b^{\pm} < \infty, J_{T_b^{\pm}} = j\right].$$

For definition of $T_b^{\pm}$ and $\gamma_b^{\pm}$ see (1) and (2). Since $w_{ij}^{-}(b) = \bar{w}_{ij}^{+}(-b)$, where $\bar{w}_{ij}^{+}$ is the corresponding integral transform for $\bar{Z}_t = \{-X_t, J_t\}$, we concentrate attention on w_{ij}^{+}. To determine an integral equation for $w_{ij}^{+}(b)$ we use probabilistic reasoning (for definiteness, we assume that the levels are ordered as $b_1 < \ldots < b_N$).

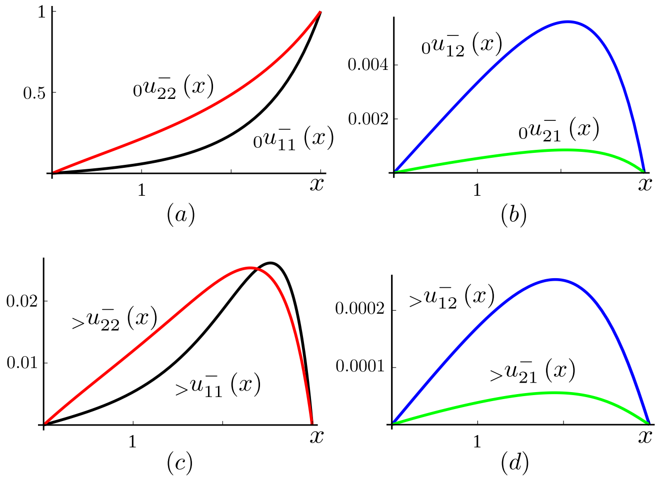

FIGURE 5. Passage by creeping $((a), (b))$ and by jump $((c), (d))$ below lower level $x - b$. *Take attention to different scales for y axis.*

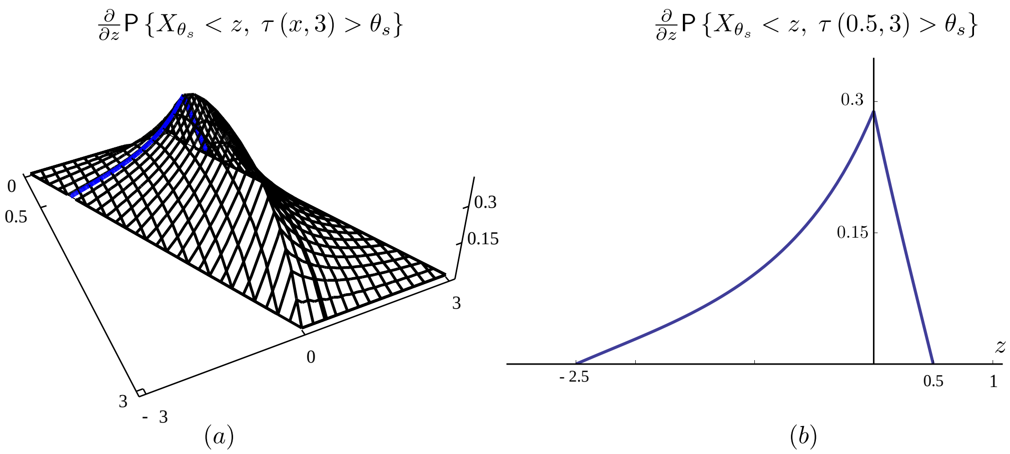

FIGURE 6. The density of X killed before exit from $(x - b, x)$ with $b = 3$. Behavior of the density for different values of $x \in (0, b)$ is illustrated in (a) and for the fixed value $x = \frac{1}{2}$ in (b).

To shorten notation, we write $\tau_1 = \tau^+(b_1)$ and $\gamma_1 = \gamma^+(b_1)$ for the first passage time over the level b_1 and the corresponding overshoot, $\tilde{\tau}_k = \tau(b_{k+1}, b_{k+1} - b_k)$ and $\tilde{\xi}_k = X_{\tilde{\tau}_k}$ for the first exit time from the interval (b_k, b_{k+1}) and the value of the process at that time. Denote by ζ the epoch of the first regime switch and by η the value of X at this switch.

Theorem 3.1. *If $0 < b_1$, then*

$$w_{ij}^+(b) = \int_{b_j}^{\infty} v_{ij}^+(b_1, dy - b_1)\, e^{v(y-b_j)} + \sum_{l=2}^{N} \int_{b_1}^{b_l} v_{il}^+(b_1, dy - b_1)\, w_{lj}^+(b - y),$$

where $v_{ij}^+(b_1, dy) = \mathsf{E}_i\left[e^{-s\tau_1};\ \tau_1 < \infty, \gamma_1 \in dy, J_{\tau_1} = j\right]$, $i, j = 1, \ldots, N$, *and* $b - y = (b_1 - y, \ldots, b_N - y)$.

If $b_k \le 0 < b_{k+1}$, $1 \le k < N$, *or* $k = N$ *and* $b_N \le 0$, *then*

$$\begin{aligned} w_{ij}^{+}(b) = e^{-vb_j} I_{\{i\le k, i=j\}} &+ \int_{b_i}^{\infty} \mathsf{E}_i \left[e^{-s\tilde{\tau}_k}; J_{\tilde{\tau}_k} = i, \zeta \ge \tilde{\tau}_k, \tilde{\xi}_k \in dy\right] e^{-v(b_j - y)} I_{\{i=j, i>k\}} \\ &+ \int_{b_k}^{b_{k+1}} \mathsf{E}_i \left[e^{-s\zeta};\ \zeta < \tilde{\tau}_k, J_\zeta = j, \eta \in dy\right] e^{-v(b_j-y)} I_{\{j\le k, i>k\}} \\ &+ \int_{b_{k+1}}^{b_i} \mathsf{E}_i \left[e^{-s\tilde{\tau}_k};\ J_{\tilde{\tau}_k} = i, \zeta \ge \tilde{\tau}_k, \tilde{\xi}_k \in dy\right] w_{ij}^{+}(b-y) I_{\{i>k+1\}} \\ &+ \int_{b_k}^{b_{k+1}} \sum_{l=k+1}^{N} \mathsf{E}_i \left[e^{-s\zeta};\ \zeta < \tilde{\tau}_k, J_\zeta = l, \eta \in dy\right] w_{lj}^{+}(b-y) I_{\{i>k\}} \\ &+ \int_{-\infty}^{b_k} \mathsf{E}_i \left[e^{-s\tilde{\tau}_k};\ J_{\tilde{\tau}_k} = i, \zeta \ge \tilde{\tau}_k, \tilde{\xi}_k \in dy\right] w_{ij}^{+}(b-y) I_{\{i>k\}}. \end{aligned}$$

Remark 3.1. By definition, X_t evolves as $X_i(t)$ for $t < \zeta$, given $J_0 = i$. Hence,

$$\begin{aligned} &\mathsf{E}_i \left[e^{-s\tilde{\tau}_k};\ J_{\tilde{\tau}_k} = i, \zeta \ge \tilde{\tau}_k, \tilde{\xi}_k \in dy\right] \\ &\qquad = \begin{cases} \mathsf{E}\left[e^{-(s-q_{ii})\tau_i(b_{k+1}, b_{k+1}-b_k)}; \gamma_i(b_{k+1}, b_{k+1}-b_k) \in dy - b_{k+1}\right], & y \ge b_{k+1}, \\ \mathsf{E}\left[e^{-(s-q_{ii})\tau_i(b_{k+1}, b_{k+1}-b_k)}; \gamma_i(b_{k+1}, b_{k+1}-b_k) \in b_k - dy\right], & y \le b_k, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\mathsf{E}_i \left[e^{-s\zeta};\ \zeta < \tilde{\tau}_k, J_\zeta = l, \eta \in dy\right] \\ &\qquad\qquad = \frac{q_{il}}{s-q_{ii}} \frac{\partial}{\partial y} \mathsf{P}\left\{X_i(\theta_{s-q_{ii}}) < y, \tau_i(b_{k+1}, b_{k+1}-b_k) > \theta_{s-q_{ii}}\right\} dy, \end{aligned}$$

where $\tau_i(b_{k+1}, b_{k+1}-b_k)$ and $\gamma_i(b_{k+1}, b_{k+1}-b_k)$ are the exit time from the interval (b_k, b_{k+1}) for $X_i(t)$ and the corresponding overshoot. Applying Propositions 2.2 and 2.3 gives that $\mathsf{E}_i[e^{-s\tau_1}; \tau_1 < \infty, J_{\tau_1} = l, \gamma_1 \in dy]$ and $\mathsf{E}_i\left[e^{-s\tilde{\tau}_k}; J_{\tilde{\tau}_k} = i, \zeta \ge \tilde{\tau}_k, \tilde{\xi}_k \in dy\right]$ are degenerate integral kernels, whereas Corollary 2.1 shows that this is not the case for $\mathsf{E}_i\left[e^{-s\zeta};\ \zeta < \tilde{\tau}_k, J_\zeta = l, \eta \in dy\right]$. Nevertheless, if $N = 2$, then we can reduce the equations for w_{ij}^{+} in Theorem 3.1 to a system of linear algebraic equations.

Proof. Let us consider the first case $0 < b_1$. The first passage through state-dependent levels can happen at the first passage over b_1 or after that moment whether or not the overshoot exceeds $b_{J_{\tau^+(b_1)}} - b_1$ (for the case $N = 2$, see Figure 7). So, by conditioning on τ_1 and on γ_1 we obtain

$$\begin{aligned} &\mathsf{E}_i \left[e^{-sT_b^+ + v\gamma_b^+};\ T_b^+ < \infty, J_{T_b^+} = j\right] \\ &\qquad = \int_{b_j - b_1}^{\infty} \mathsf{E}_i \left[e^{-s\tau_1};\ \tau_1 < \infty, J_{\tau_1} = j, \gamma_1 \in dy\right] e^{v(y-(b_j-b_1))} \\ &\qquad\quad + \sum_{l=2}^{N} \int_0^{b_l - b_1} \mathsf{E}_i \left[e^{-s\tau_1};\ \tau_1 < \infty, J_{\tau_1} = l, \gamma_1 \in dy\right] \\ &\qquad\qquad \times \mathsf{E}_l \left[e^{-sT^+_{b-(y+b_1)} + v\gamma^+_{b-(y+b_1)}};\ T^+_{b-(y+b_1)} < \infty, J_{T^+_{b-(y+b_1)}} = j\right]. \end{aligned}$$

For the second case $b_k \le 0 < b_{k+1}$, $1 \le k < N$, or $k = N$ and $b_N \le 0$, we use the following reasoning. By definition, for $i \le k$

$$\mathsf{E}_i \left[e^{-sT_b^+ + v\gamma_b^+};\ T_b^+ < \infty, J_{T_b^+} = j\right] = e^{-vb_j} I_{\{i=j\}}. \tag{21}$$

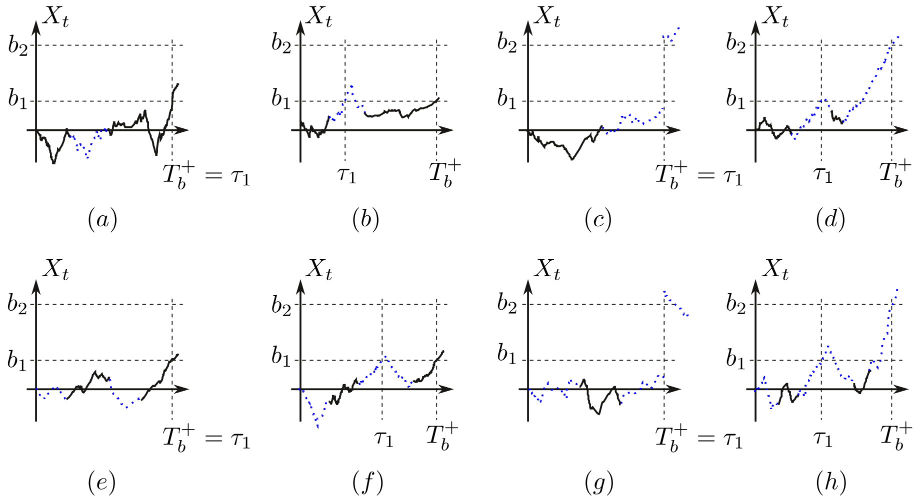

FIGURE 7. Case $0 < b_1$. The passage through levels with $J_0 = 1$ is illustrated in $(a) - (d)$ and with $J_0 = 2$ is illustrated in $(e) - (h)$. The solid lines correspond to the state 1 of J and the dotted lines to the state 2.

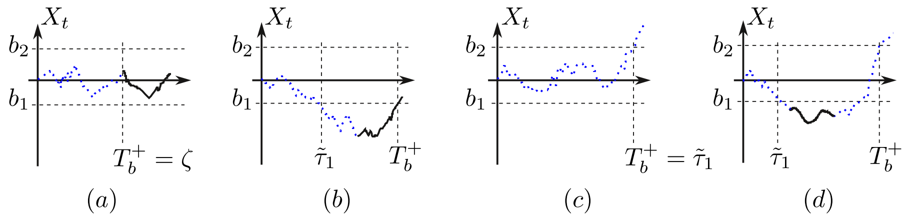

FIGURE 8. If $b_1 \leq 0 < b_2$, then only the case $J_0 = 2$ is nontrivial. The passage through levels with $J_{T_b^+} = 1$ is illustrated in (a) and (b), while with $J_{T_b^+} = 2$ in (c) and (d). The solid lines correspond to the state 1 of J and the dotted lines to the state 2.

If $i > k$, then we condition on whether or not X leaves the interval (b_k, b_{k+1}) before the first regime switch (see Figure 8 for the case $N = 2$):

$$
\begin{aligned}
&\mathsf{E}_i\left[e^{-sT_b^+ + v\gamma_b^+};\, T_b^+ < \infty, J_{T_b^+} = j\right] \\
&= \sum_{l=1}^{N} \int_{b_k}^{b_{k+1}} \mathsf{E}_i\left[e^{-s\zeta}; \zeta < \tilde{\tau}_k, J_\zeta = l, \eta \in dy\right] \mathsf{E}_l\left[e^{-sT_{b-y}^+ + v\gamma_{b-y}^+}; T_{b-y}^+ < \infty, J_{T_{b-y}^+} = j\right] \\
&\qquad + \int_{(-\infty, b_k)\cup(b_{k+1},\infty)} \mathsf{E}_i\left[e^{-s\tilde{\tau}_k};\, J_{\tilde{\tau}_k} = i, \zeta \geq \tilde{\tau}_k, \tilde{\xi}_k \in dy\right] \\
&\qquad\qquad \times \mathsf{E}_i\left[e^{-sT_{b-y}^+ + v\gamma_{b-y}^+};\, T_{b-y}^+ < \infty, J_{T_{b-y}^+} = j\right].
\end{aligned}
$$

Taking into account (21) completes the proof. □

3.1. **Two state case.** Let the environment has two states ($N = 2$). Set

$$\int_{b_1}^{b_2} \mathsf{E}_2\left[e^{-s\zeta};\ \zeta < \tilde{\tau}_1, J_\zeta = 1, \eta \in dy\right] e^{-v(b_1-y)} = h\left(b_1, b_2\right),$$

$$\lim_{y \downarrow b_1} \frac{\partial}{\partial y} h\left(b_1 - y, b_2 - y\right) = h^0,$$

$$\int_{b_1}^{b_2} \beta_2^+ e^{-\beta_2^+(y-b_1)} h\left(b_1 - y, b_2 - y\right) dy = h^+.$$

From Proposition 2.3 we have the next representation

$$\mathsf{E}_2\left[e^{-s\tilde{\tau}_1};\ J_{\tilde{\tau}_1} = 2, \zeta \geq \tilde{\tau}_1, \tilde{\xi}_1 \in dy\right]$$
$$= \begin{cases} \mathsf{E}\left[e^{-(s-q_{22})\tau_2(b_2,b_2-b_1)}; \gamma_2\left(b_2, b_2 - b_1\right) \in dy - b_2\right], & y \geq b_2, \\ \mathsf{E}\left[e^{-(s-q_{22})\tau_2(b_2,b_2-b_1)}; \gamma_2\left(b_2, b_2 - b_1\right) \in b_1 - dy\right], & y \leq b_1. \end{cases}$$
$$= \begin{cases} {}_0u_2^+\left(b_2\right)\delta\left(dy - b_2\right) + {}_>u_2^+\left(b_2\right)\beta_2^+ e^{-\beta_2^+(y-b_2)}dy, & y \geq b_2, \\ {}_0u_2^-\left(b_2\right)\delta\left(b_1 - dy\right) + {}_>u_2^-\left(b_2\right)\beta_2^- e^{-\beta_2^-(b_1-y)}dy, & y \leq b_1, \end{cases}$$

where ${}_{\substack{> \\ 0}}u_2^\pm(b_2)$ determine correspondingly passage by creeping and by jump through bounds of the interval (b_1, b_2) for X_2 killed at the rate $s - q_{22}$. Write

$$\lim_{y \downarrow b_1} \frac{\partial}{\partial y}\left({}_{\substack{> \\ 0}}u_2^\pm\left(b_2 - y\right)\right) = f^\pm_{\substack{> \\ 0}},$$

$$\int_{b_1}^{b_2} \beta_2^+ e^{-\beta_2^+(y-b_1)} {}_{\substack{> \\ 0}}u_2^\pm\left(b_2 - y\right) dy = {}_+f^\pm_{\substack{> \\ 0}},$$

and

$$\lim_{y \uparrow b_1} \frac{\partial}{\partial y}\left({}_{\substack{> \\ 0}}v_{2j}^+\left(b_1 - y\right)\right) = g_j^{\substack{> \\ 0}},$$

$$\int_{-\infty}^{b_1} \beta_2^- e^{-\beta_2^-(b_1-y)} {}_{\substack{> \\ 0}}v_{2j}^+\left(b_1 - y\right) dy = {}_-g_j^{\substack{> \\ 0}}, j = 1, 2.$$

Then from Theorem 3.1 we can deduce.

Corollary 3.1. *For* $v < \min\left\{\beta_1^+, \beta_2^+\right\}$,

$$w_{11}^+\left(b\right) = \begin{cases} {}_0v_{11}^+\left(b_1\right) + {}_>v_{11}^+\left(b_1\right)\frac{\beta_1^+}{\beta_1^+ - v} + {}_0v_{12}^+\left(b_1\right)G_{21}^0 + {}_>v_{12}^+\left(b_1\right)G_{21}^+, & 0 < b_1, \\ e^{-vb_1}, & b_1 \leq 0. \end{cases}$$

$$w_{21}^+\left(b\right) = \begin{cases} {}_0v_{21}^+\left(b_1\right) + {}_>v_{21}^+\left(b_1\right)\frac{\beta_1^+}{\beta_1^+ - v} + {}_0v_{22}^+\left(b_1\right)G_{21}^0 + {}_>v_{22}^+\left(b_1\right)G_{21}^+, & 0 < b_1, \\ h\left(b_1, b_2\right) + {}_0u_2^-\left(b_2\right)G_{21}^0 + {}_>u_2^-\left(b_2\right)G_{21}^-, & b_1 \leq 0 < b_2, \\ 0, & b_2 \leq 0. \end{cases}$$

$$w_{12}^+\left(b\right) = \begin{cases} {}_>v_{12}^+\left(b_1\right)\frac{\beta_2^+}{\beta_2^+ - v}e^{-\beta_2^+(b_2-b_1)} + {}_0v_{12}^+\left(b_1\right)G_{22}^0 + {}_>v_{12}^+\left(b_1\right)G_{22}^+, & 0 < b_1, \\ 0 & b_1 \leq 0. \end{cases}$$

$$w_{22}^+\left(b\right) = \begin{cases} {}_>v_{22}^+\left(b_1\right)\frac{\beta_2^+}{\beta_2^+ - v}e^{-\beta_2^+(b_2-b_1)} + {}_0v_{22}^+\left(b_1\right)G_{22}^0 + {}_>v_{22}^+\left(b_1\right)G_{22}^+, & 0 < b_1, \\ {}_0u_2^+\left(b_2\right) + {}_>u_2^+\left(b_2\right)\frac{\beta_2^+}{\beta_2^+ - v} + {}_0u_2^-\left(b_2\right)G_{22}^0 + {}_>u_2^-\left(b_2\right)G_{22}^-, & b_1 \leq 0 < b_2, \\ e^{-vb_2} & b_2 \leq 0. \end{cases}$$

where G_{2j} satisfy the systems of equations

$$\begin{cases} g_1^0 + g_1^> \frac{\beta_1^+}{\beta_1^+ - v} - h^0 = \left(f_0^- - g_2^0\right) G_{21}^0 - g_2^> G_{21}^+ + f_>^- G_{21}^-, \\ h^+ = -{}_+f_0^- G_{21}^0 - {}_+f_>^- G_{21}^- + G_{21}^+, \\ {}_-g_1^0 + {}_-g_1^> \frac{\beta_1^+}{\beta_1^+ - v} = -{}_-g_2^0 G_{21}^0 - {}_-g_2^> G_{21}^+ + G_{21}^-, \end{cases}$$

and

$$\begin{cases} g_2^> \frac{\beta_2^+}{\beta_2^+ - v} e^{-\beta_2^+ (b_2 - b_1)} - \left(f_0^+ + f_>^+ \frac{\beta_2^+}{\beta_2^+ - v}\right) = \left(f_0^- - g_2^0\right) G_{22}^0 - g_2^> G_{22}^+ + f_>^- G_{22}^-, \\ {}_+f_0^+ + {}_+f_>^+ \frac{\beta_2^+}{\beta_2^+ - v} = -{}_+f_0^- G_{22}^0 - {}_+f_>^- G_{22}^- + G_{22}^+, \\ {}_-g_2^> \frac{\beta_2^+}{\beta_2^+ - v} e^{-\beta_2^+ (b_2 - b_1)} = -{}_-g_2^0 G_{22}^0 - {}_-g_2^> G_{22}^+ + G_{22}^-. \end{cases}$$

Remark 3.2. Corollary 3.1 provide us an algorithm for calculation of the integral transform for the joint distribution of $\left\{T_b^+, \gamma_b^+\right\}$. We need to solve the first passage problem for Z_t (for instance, using the results of Subsection 2.3) and two-sided exit problems for $X_2(t)$ (since X_2 is an ordinary Kou process, we can use the results of [31]). Then we solve the corresponding system of linear equations and get the representations for $w_{ij}^+(b)$, $b_1 < b_2$. If $b_1 = b_2$, then

$$w_{ij}^+(b) = \begin{cases} {}_0v_{ij}^+(b_1) + {}_>v_{ij}^+(b_1) \frac{\beta_1^+}{\beta_1^+ - v}, & 0 < b_1, \\ e^{-vb_1} I_{\{i=j\}}, & b_1 \leq 0. \end{cases}$$

For the case $b_1 > b_2$ we can consider a process Z_t^* with relabeled states: $\{J_t^* = 1\} = \{J_t = 2\}$ and $\{J_t^* = 2\} = \{J_t = 1\}$.

3.2. **Example.** As an illustrative example, we suppose that $s = 0.02$. The transition rate matrix for J_t is $\mathbf{Q} = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. Given the state of J_t, drifts are $a_1 = -1.35\bar{3}$, $a_2 = 2.4925$ and volatilities are $\sigma_1 = \sqrt{2}$, $\sigma_2 = \sqrt{5}$. The jumps follow compound Poisson processes with jump rates $\lambda_1 = 6$, $\lambda_2 = 0.225$. Upward jump sizes are exponential with rates $\beta_1^+ = 0.5$, $\beta_2^+ = 5$ and downward jump sizes are exponential with rates $\beta_1^- = 2$, $\beta_2^- = 4$. Probabilities of having upward jumps are $p_1^+ = \frac{5}{6}$, $p_2^+ = \frac{5}{9}$. Using $v = 0$ and $v = -1$, from Corollary 3.1 we can obtain representations for $\mathsf{E}_i\left[e^{-sT_b^+}; T_b^+ < \infty, J_{T_b^+} = j\right]$ and $\mathsf{E}_i\left[e^{-sT_b^+ - \gamma_b^+}; T_b^+ < \infty, J_{T_b^+} = j\right]$, $i, j = \overline{1,2}$, (see Figure 9 and Figure 10), which can be used for an option valuation problem.

Consider a market model (B, S) with the price processes $B_t = e^{-st}$, $S_t = e^{x - X_t}$, $t \geq 0$ (see Figure 11). Since $\mathsf{E}S_1 < \infty$, the market is arbitrage-free. The market is incomplete, but since we choose drifts as:

$$a_i = -s + \frac{\sigma_i^2}{2} + \lambda_i \left(\frac{p_i^-}{\beta_i^- - 1} - \frac{p_i^+}{\beta_i^+ + 1}\right),$$

we have

$$\mathsf{E}e^{-st} S_t = S_0$$

and the model is risk-neutral (see, for instance, [27, Section A.1]). Following [27, Theorem 1], the arbitrage-free price for a perpetual American put option with strike K is

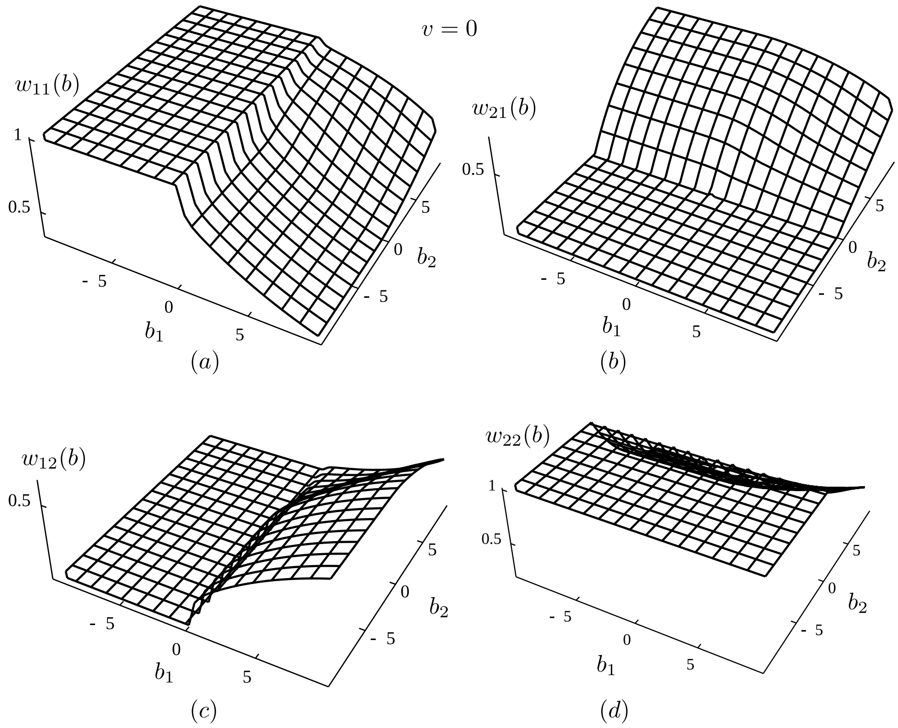

FIGURE 9. The integral transform for the distribution of T_b^+, given $J_0 = 1$ $((a),(c))$ and given $J_0 = 2$ $((b),(d))$.

given by

$$\begin{aligned} V_i(x,b) &= K\sum_{j=1}^{2}\mathsf{E}_i\left[e^{-sT^+_{x-b}};\ T^+_{x-b}<\infty, J_{T^+_{x-b}}=j\right] \\ &\quad -\sum_{j=1}^{2}e^{b_j}\,\mathsf{E}_i\left[e^{-sT^+_{x-b}-\gamma^+_{x-b}};\ T^+_{x-b}<\infty, J_{T^+_{x-b}}=j\right] \\ &= K\sum_{j=1}^{2}w^+_{ij}(x-b,s,0)-\sum_{j=1}^{2}e^{b_j}w^+_{ij}(x-b,s,-1), \end{aligned}$$

where $b=(b_1,b_2)$ satisfies

$$\lim_{x\downarrow b_1}\frac{\partial}{\partial x}V_1(x,b)=-e^{b_1},\ \lim_{x\downarrow b_2}\frac{\partial}{\partial x}V_2(x,b)=-e^{b_2}.$$

If $K=70$, then $b_1\approx -0.781$, $b_2\approx -0.669$. Fixing $x_0=3$ yields $V_1(x_0,b)\approx 68.504$ and $V_2(x_0,b)\approx 67.855$ (see Figure 12). Note that, Jiang and Pistorius [27] provided the example for the case when jumps in one state are (one-sided) exponential and no jumps occur in the other state.

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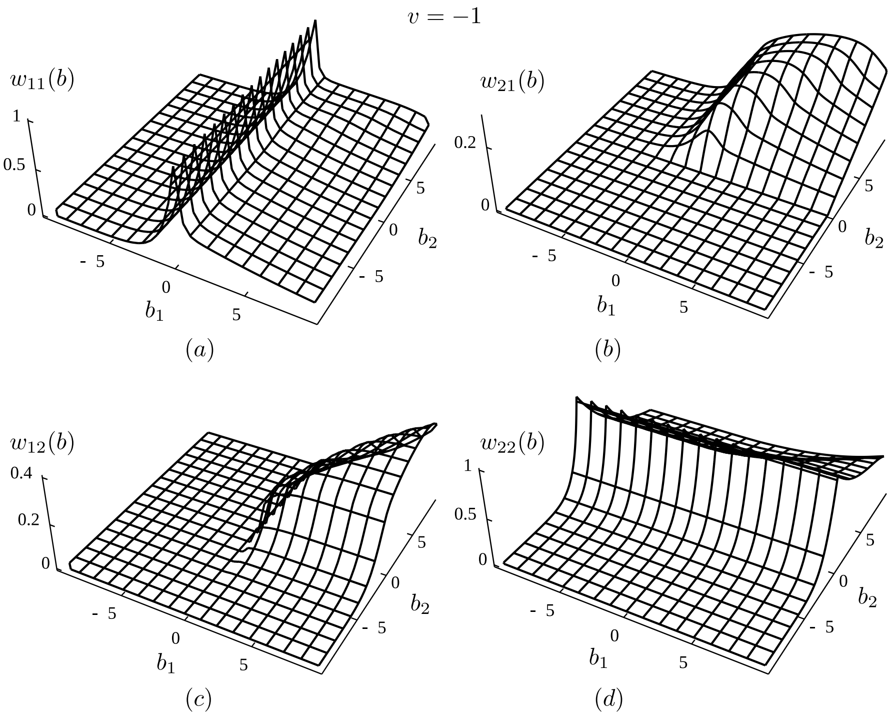

FIGURE 10. The integral transform $\mathsf{E}_i\left[e^{-sT_b^+-\gamma_b^+};\ T_b^+<\infty, J_{T_b^+}=j\right]$, $i,j=1,2$.

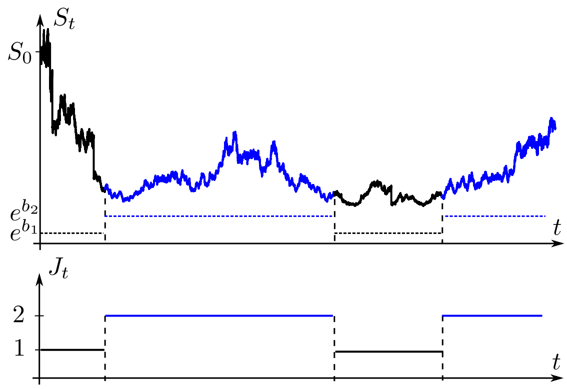

FIGURE 11. Simulated sample paths of S_t and J_t.

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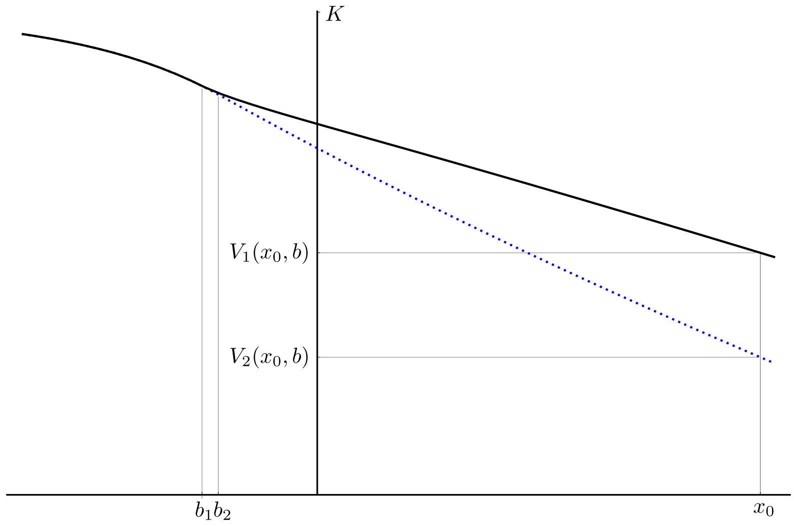

FIGURE 12. The arbitrage-free price.

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