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ASYMPTOTICS OF ERROR PROBABILITIES OF OPTIMAL TESTS

We consider first and second error probabilities of asymptotically optimal tests (Neyman-Pearson, minimax, Bayesian) when two simple hypotheses H_1^t and H_2^t parametrized by time $t \geq 0$ are tested under the observation X^t of arbitrary nature. The paper provides details on the conditions of asymptotic decrease of probabilities of optimal criteria errors determined by α type Hellinger integral between measures P_1^t and P_2^t, demonstrating that in the case of minimax and Bayesian criteria it is sufficient to investigate Hellinger integral, when $\alpha \in (0,1)$, and in the case of Neyman-Pearson criterion it is observed only in the environment of point $\alpha = 1$. Whereas Kullback-Leibler information distance is always larger than Chernoff distance; we discover that, in the case of Neyman-Pearson criterion, the probability of type II error decreases faster than in the cases of minimax or Bayesian criteria. This is proven by the examples of marked point processes of the i.i.d. case, non-homogeneous Poisson process and the geometric renewal process presented at the end of the paper.

1. Introduction

When testing the two simple hypotheses H_1^n and H_2^n, for observation $X^n = (X_1, X_2, \ldots, X_n)$, where $X_1, X_2, \ldots, X_n$ are i.i.d., asymptotic investigation of the asymptotic error probabilities started from the two formulas.

Lemma 1.1. [2]. *1) If $I(P_1, P_2) < \infty$, then*

$$\lim_{n\to\infty} \inf_{\delta^n \in B_n(\alpha)} n^{-1} \ln \alpha_2(\delta^n) = -I(P_1, P_2); \tag{1}$$

2) if $J(P_1, P_2) < \infty$, then

$$\lim_{n\to\infty} n^{-1} \inf_{\delta^n \in \Delta^n} \ln \max\{\alpha_1(\delta^n), \alpha_2(\delta^n)\} = -J(P_1, P_2); \tag{2}$$

where δ^n – statistical criterion-test, $B_n(\alpha) = \{\delta^n \in \Delta^n : \alpha_1(\delta^n) \leq \alpha, 0 < \alpha < 1\}$, Δ^n is a set of the criteria, $\alpha_1(\delta^n)$, $\alpha_2(\delta^n)$ denote the probabilities of 1st and 2nd type errors, respectively for test $\delta^n \in \Delta^n$; $I(P_1, P_2)$ – Kullback-Leibler information of measures P_1 and P_2, $J(P_1, P_2)$ – Chernoff information of measures P_1 and P_2.

In formula (2), the constant $J(P_1, P_2)$ is defined through α type Hellinger integral between measures P_1 and P_2:

$$J(P_1, P_2) = -\inf_{0<\alpha<1} \ln H(\alpha, P_1, P_2) = -\inf_{0<\alpha<1} \ln \int \left(\frac{dP_1}{dQ}\right)^{\alpha} \left(\frac{dP_2}{dQ}\right)^{1-\alpha} dQ.$$

Natural generalisation of Lemma 1.1.2) under natural conditions formulated for α type Hellinger integral $H_t(\alpha, P_1^t, P_2^t)$ is presented in V. Kanišauskas' paper [7] (this is also presented in Lemma 3.3 of this paper).

The proof of the formula (1) is based on the likelihood ratio $\ln \frac{dP_2^n}{dP_1^n}$ convergence in probability, when $P_2^n \ll P_1^n$. For observations X^t of arbitrary nature the proof of the

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analogical formula (1) is also based on the law of large numbers of the likelihood ratio $\ln \frac{dP_2^t}{dP_1^t}$, when $P_2^t \ll P_1^t$, $t \in \mathbb{R}_+$, [9, 20]:

$$P_1^t - \lim_{t\to\infty} \varphi_t^{-1} \ln \frac{dP_2^t}{dP_1^t}\left(X^t\right) = -I_{1,2} < \infty, \tag{3}$$

where $P_1^t - \lim_{t\to\infty}$ means the convergence in probability, and the function φ_t is such that $\varphi_t \to \infty$ as $t \to \infty$.

For the likelihood ratio $\ln \frac{dP_2^t}{dP_1^t}$, when the law of large numbers (3) is valid, large deviations can be investigated to the normalized likelihood ratio measure $\mu_t(B) = P_1^t\left(\varphi_t^{-1} \ln \frac{dP_2^t}{dP_1^t} \in B\right)$, $B \in \mathcal{B}(R)$. For this measure generalized Cramer condition has the form

$$\varphi(\lambda) = \lim_{t\to\infty} \varphi_t^{-1} \ln \mathbf{E}_1^t \exp\left\{\lambda \ln \frac{dP_2^t}{dP_1^t}\right\} < \infty, \quad \lambda \in (\lambda_-, \lambda_+) \ni 0. \tag{4}$$

When this formula is valid we can define a Legendre-Frenchel transformation $I(\gamma) = \sup\limits_{\lambda}(\gamma\lambda - \varphi(\lambda))$ and get large deviations [5], which was used in the theory for testing hypotheses.

It is easy to see that for $P_2^t \ll P_1^t$, $t \in \mathbb{R}_+$, the Hellinger integral of order α between measures P_1^t and P_2^t has the form

$$H_t(\alpha) = \mathbf{E}_Q^t\left(Z_1^t\right)^\alpha\left(Z_2^t\right)^{1-\alpha} = \mathbf{E}_1^t\left(Z_{2,1}^t\right)^{1-\alpha} = \mathbf{E}_1^t \exp\left\{(1-\alpha)\ln\frac{dP_2^t}{dP_1^t}\right\},$$

for which analogical (4) generalized Cramer conditions, under which the theory of large deviations can be applied [5], can be formulated.

H. Chernoff was the first who applied large deviations in the theory for testing hypotheses for observations of i.i.d. random variables [3, 4]. He applied the large deviation theorems to investigate the rate of decrease of Bayesian risk. The main result of this work was traditionally called Chernoff theorem. In 1981, L. Birgé [1] continued application of large deviations in problems on testing hypotheses and obtained the exponent of 2nd type error probabilities for Neyman-Pearson tests. To a large extent, Yu. N. Lin'kov specifically applied the theory of large deviations in problems of testing hypotheses ([13]–[18], [21]–[26]). In 1995, he formulated the general scheme of their application [19].

Similar problems in testing hypotheses have been dealt with by I. Vajda and F. Liese who gave priority to applications of convex functions [11, 31, 32]. In Igor Vajda's paper [32], one can find generalization of Stein and Chernoff theorems in a classical (i.i.d.) case, when no results from large deviation theory are used. The investigations started in these papers were continued in other papers [12, 30].

The paper consists of the introduction and 3 chapters (Sections 2–4).

Section 2 defines the concepts and presents additional results. Other two sections present new or corrected results.

Section 3 comprises 3 subsections.

Even though the result of Subsection 3.1 is similar to already available results, still it must be compared within a complex containing other results of the paper. To be precise, asymptotics of error probabilities of Neyman-Pearson and minimax as well as Bayesian criteria are expressed through $\alpha \in [0, 1]$ type Hellinger integral which is normed by the same increasing function in all cases.

Formulas of the first and the second type error probabilities of the minimax and Bayesian criteria presented in Subsections 3.2 and 3.3 are a natural conclusion from V. Kanišauskas' [7] result and are more precise than analogical results provided by V. Linkov [20] on the question under discussion.

The results of the investigation are demonstrated in examples of marked point processes of the i.i.d. case, non-homogeneous Poisson process and the geometric renewal process provided in Section 4.

2. Major concepts and auxiliary results

Let $(\mathcal{X}^t, \mathcal{F}^t, \{P_1^t, P_2^t\})$, $t \geq 0$, be a family of statistical experiments with observations $X^t \in \mathcal{X}^t$ and let H_1^t and H_2^t be two simple hypotheses according to which the distributions of observations X^t are defined by measures P_1^t and P_2^t respectively. Let δ^t be a measurable mapping from $(\mathcal{X}^t, \mathcal{F}^t)$ into $([0,1], \mathcal{B}[0,1])$, which is called a test for testing the hypotheses H_1^t and H_2^t under observation X^t. Let Δ^t be a collection of all tests δ^t, $\alpha_1(\delta^t)$ and $\alpha_2(\delta^t)$ denote the probabilities of the 1st and 2nd type errors, respectively for test $\delta^t \in \Delta^t$, namely

$$\alpha_1\left(\delta^t\right) = \mathbb{E}_1^t \delta^t\left(X^t\right), \quad \alpha_2\left(\delta^t\right) = \mathbb{E}_2^t\left(1 - \delta^t\left(X^t\right)\right),$$

where $\mathbb{E}_i^t$ is an expectation with respect to P_i^t.

The criterion $\delta_0^{t,\alpha}$ is called the most powerful of α level, if

$$\alpha_2\left(\delta_0^{t,\alpha}\right) = \min_{\delta^t \in K_\alpha^t} \alpha_2\left(\delta^t\right),$$

where $K_\alpha^t = \{\delta^t : \alpha_1(\delta^t) \leq \alpha\}$, $\alpha \in (0,1)$. According to I. Vajda (see [32]), such criterion is called optimum α-tests.

In the case of Bayesian principle, initial hypotheses H_1^t and H_2^t are random events whose probabilities $\pi^t = \pi_1^t = P_1^t(H_1^t)$ and $\pi_2^t = P_2^t(H_2^t) = 1 - \pi^t$ are known and called a priori probabilities of the hypotheses H_1^t and H_2^t. In the Bayesian case, the property of the criterion is reflected by the average error probability

$$e_{\pi^t}\left(\delta^t\right) = \pi^t \alpha_1\left(\delta^t\right) + \left(1 - \pi^t\right)\alpha_2\left(\delta^t\right).$$

The criterion δ_π^t is called Bayesian criterion with respect to a priori distribution $(\pi^t, 1-\pi^t)$, if

$$e_{\pi^t}\left(\delta_\pi^t\right) = \min_{\delta^t} e_{\pi^t}\left(\delta^t\right). \tag{5}$$

It is known (see [20]) that Bayesian criterion δ_π^t is the most powerful in the type $K_{\alpha_0}^t = \{\delta^t : \alpha_2(\delta^t) \leq \alpha_0\}$, when $\alpha_0 = \alpha_1(\delta_\pi^t)$. In the case of minimax principle, the quality of the criterion δ^t is indicated by the measure

$$e\left(\delta^t\right) = \max_j \alpha_j\left(\delta^t\right) = \max_{\pi^t} e_{\pi^t}\left(\delta^t\right).$$

A criterion δ_0^t is called minimax if

$$e\left(\delta_0^t\right) = \min_{\delta^t} e\left(\delta^t\right).$$

Lemma 2.1. *(See [20], Theorem 1.2.3.) If there exists Bayesian criterion δ_π^t with a priori distribution $(\pi^t, 1-\pi^t)$ for which*

$$\alpha_1\left(\delta_\pi^t\right) = \alpha_2\left(\delta_\pi^t\right),$$

then δ_π^t is the minimax criterion.

The distribution $(\pi^t, 1-\pi^t)$ which corresponds to Bayesian criterion δ_π^t with $\alpha_1(\delta_\pi^t) = \alpha_2(\delta_\pi^t)$ is treated at the worst because its average criterion probability $e_{\pi^t}(\delta_\pi^t)$ is the highest:

$$\max_{\pi^t} e_{\pi^t}\left(\delta_\pi^t\right) = \max_{\pi^t}\min_{\delta^t} e\left(\delta^t\right).$$

The criterion $\delta_{c,\varepsilon}^t$ with parameters $c \in [0,\infty)$ and $\varepsilon \in [0,1]$ is called the likelihood ratio criterion if

$$\delta_{c,\varepsilon}^t = \mathbb{I}\left(Z_t > c\right) + \varepsilon \mathbb{I}\left(Z_t = c\right), \quad Z_t = \frac{dP_2^t}{dP_1^t}, P_2^t \ll P_1^t, t \in \mathbb{R}_+.$$

According to the fundamental Neyman-Pearson lemma (see [20], Theorem 1.1.1), the likelihood ratio criterion of order $\alpha \in (0, \alpha_0)$ is the most powerful criterion of order α, here $\alpha_0 = P_1^t(Z_t > 0)$.

Minimax criterion δ_0^t can be found in such a way.

Lemma 2.2. *(See [20], Theorem 1.2.4.) There exists the likelihood ratio criterion $\delta_{c,q}^t$ with specific parameters c and $q = const.$, which is minimax; moreover, parameters c and q are found from equality $\alpha_1\left(\delta_{c,q}^t\right) = \alpha_2\left(\delta_{c,q}^t\right)$, which is*

$$P_1^t\left(Z_{2,1}^t > c\right) + P_2^t\left(Z_{2,1}^t > c\right) + q\left[P_1^t\left(Z_{2,1}^t = c\right) + P_2^t\left(Z_{2,1}^t = c\right)\right] = 1,$$

where $Z_{2,1}^t = \frac{dP_2^t}{dP_1^t}$.

Let a measure Q^t be defined on the measurable space $(\mathcal{X}^t, \mathcal{F}^t)$ such that $P_i^t \ll Q^t$, $i = 1, 2$, for all $t \in R_+$ and let $Z_i^t = \frac{dP_i^t}{dQ^t}$, $i = 1, 2$, be versions of Radon-Nikodym derivatives.

The Hellinger integral of order α between measures P_1^t and P_2^t is defined by

$$H_t(\alpha) = \mathbb{E}_Q^t\left(Z_1^t\right)^\alpha\left(Z_2^t\right)^{1-\alpha} = \begin{cases} \mathbb{E}_Q^t Z_2^t \mathbb{I}\left(Z_1^t > 0\right), & \alpha = 0; \\ \mathbb{E}_Q^t\left(Z_1^t\right)^\alpha\left(Z_2^t\right)^{1-\alpha}, & \alpha \in (0,1); \\ \mathbb{E}_Q^t Z_1^t \mathbb{I}\left(Z_2^t > 0\right), & \alpha = 1. \end{cases}$$

If $P_i^t \sim Q^t$, $t \in R_+$, then $Z_i^t > 0$, Q^t – a.s., $Z_i^t = \frac{dP_i^t}{dQ^t}$, $i = 1, 2$. Therefore

$$H_t(\alpha) = \mathbb{E}_Q^t\left(Z_1^t\right)^\alpha\left(Z_2^t\right)^{1-\alpha}, \ \alpha \in [0,1], \text{ and } H_t(0) = H_t(1) = 1.$$

For $P_1^t \sim P_2^t$, $t \in R_+$, we have

$$H_t(\alpha) = \mathbb{E}_2^t\left(Z_{1,2}^t\right)^\alpha = \mathbb{E}_1^t\left(Z_{2,1}^t\right)^{1-\alpha}, \ \alpha \in [0,1],$$

where $Z_{1,2}^t = \frac{dP_1^t}{dP_2^t}$, $Z_{2,1}^t = \frac{dP_2^t}{dP_1^t}$.

The Chernoff information between measures P_1^t and P_2^t is denoted by [29]:

$$J\left(P_1^t, P_2^t\right) = -\ln \inf_{0 \le \alpha \le 1} H_t(\alpha) = -\ln \inf_{0 \le \alpha \le 1} \mathbb{E}_Q^t\left(Z_1^t\right)^\alpha\left(Z_2^t\right)^{1-\alpha}.$$

The divergence

$$I\left(P_1^t, P_2^t\right) = \begin{cases} \int \ln\left(\frac{dP_1^t}{dP_2^t}\right) dP_1^t, & \text{if } P_1^t \ll P_2^t; \\ \infty & \text{otherwise} \end{cases}$$

of arbitrary distributions P_1^t, P_2^t called Kullback-Leibler divergence-information [12]. It is known that Kullback-Leibler information is positive because $I(P_1, P_2) = 0 \iff P_1 = P_2$ and because of the inequality $\ln(1 + V) - V \le 0$,

$$I(P_1, P_2) = \int p_1(x) \ln \frac{p_1(x)}{p_2(x)} dx = -\int\left[\ln \frac{p_1(x)}{p_2(x)} - \left(\frac{p_1(x)}{p_2(x)} - 1\right)\right] p_1(x) dx \ge 0.$$

The following is known of the property for concave function $f(x)$:

$$f\left(\int \alpha(x) y(x) dx\right) \ge \int \alpha(x) f(y(x)) dx,$$

where $\int \alpha(x) dx = 1$, $\alpha(x) \ge 0$ with each x.

Then we use it for concave function $f(x) = \ln x$, we get

$$\begin{aligned} \ln H(\alpha) &= \ln\left(\int q(x)\left(\frac{p(x)}{q(x)}\right)^\alpha dx\right) \ge \int q(x) \ln\left(\frac{p(x)}{q(x)}\right)^\alpha dx = \\ &= \alpha \int q(x) \ln \frac{p(x)}{q(x)} dx = -\alpha \int q(x) \ln \frac{q(x)}{p(x)} dx. \end{aligned}$$

From here

$$-\ln H(\alpha) \leq \alpha \int q(x) \ln \frac{q(x)}{p(x)} dx = \alpha I\left(P_1, P_2\right) < I\left(P_1, P_2\right)$$

with each $\alpha \in (0,1)$. Thus,

$$J\left(P_1, P_2\right) = -\inf_{0<\alpha<1} \ln H(\alpha) < I\left(P_1, P_2\right).$$

3. Testing of two simple hypotheses. Main results

3.1. **Neyman-Pearson test.** Let δ_0^t is $\alpha \in (0,1)$ Neyman-Pearson test. We study the asymptotic decrease of error probabilities $\alpha_2\left(\delta_0^t\right)$.

We will formulate now the conditions A.

A0. $P_2^t \ll P_1^t$, $t \in \mathbb{R}_+$.

A1. $P_1^t \sim P_2^t$, $t \in \mathbb{R}_+$.

A2. There is a function φ_t, $\varphi_t \to \infty$ as $t \to \infty$, such that

$$P_1^t - \lim_{t\to\infty} \varphi_t^{-1} \ln \frac{dP_2^t}{dP_1^t}\left(X^t\right) = -I_{1,2} < \infty.$$

Lemma 3.1. *Let conditions A0 and A2 be satisfied. Then*

$$\lim_{t\to\infty} \inf_{\delta^t \in B_t(\alpha)} \varphi_t^{-1} \ln \alpha_2\left(\delta^t\right) = -I_{1,2},$$

where $B_t(\alpha) = \left\{\delta^t \in \Delta^t : \alpha_1\left(\delta^t\right) \leq \alpha, 0 < \alpha < 1\right\}$.

Proof. By the condition A0 exists

$$\frac{dP_2^t}{dP_1^t}\left(X^t\right).$$

Further, we repeat the proof of the theorem from [9]. ☐

By definition of Neyman-Pearson test δ_0^t from Lemma 3.1 we get

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_2\left(\delta_0^t\right) = -I_{1,2}.$$

We will find out that

$$I_{1,2} = \lim_{t\to\infty} \varphi_t^{-1} I\left(P_1^t, P_2^t\right),$$

$$I\left(P_1^t, P_2^t\right) = \int \ln\left(\frac{dP_1^t}{dP_2^t}\right) dP_1^t.$$

For this purpose we will use Lemma 3.2.

Lemma 3.2. *(See* [5], *Lemma IV.6.3.) Let* $\{f_n; n = 1, 2, \ldots\}$ *be sequence of convex functions on an open interval A of R, such that* $f(t) = \lim_{n\to\infty} f_n(t)$ *exists for each* $t \in A$. *If each* f_n *and* f *are differentiable at some point* $t_0 \in A$, *then* $\lim_{n\to\infty} f_n'\left(t_0\right)$ *exists and equals* $f'\left(t_0\right)$.

Let Y_t, $t \geq 0$, be a family of random variables which are defined on a probability space $(\Omega, \mathcal{F}, P)$. A normalizing sequence φ_t, $\varphi_t \to \infty$, as $t \to \infty$, defines

$$\psi_t(\lambda) = \varphi_t^{-1} \ln \mathbb{E}\left[\exp\left(\lambda Y_t\right)\right].$$

We introduce the following conditions C.

C1. Each function $\psi_t(\lambda)$ is finite for all $\lambda \in (\lambda_-, \lambda_+) \ni 0$ and $t \in \mathbb{R}_+$.

C2. $\psi(\lambda) = \lim_{t\to\infty} \psi_t(\lambda)$ exists for all $\lambda \in (\lambda_-, \lambda_+)$ and is finite.

C3. The function $\psi(\lambda)$ is differentiable at point $\lambda = 0$ and $\mu = \psi'(0)$.

Theorem 3.1. *Let conditions C be satisfied. Then*

$$P - \lim_{t\to\infty} \varphi_t^{-1} Y_t = \mu = \psi'(0).$$

If $\lim_{t\to\infty} \varphi_t (\ln t)^{-1} = \infty$, *then the convergence is with the probability* 1.

The proof is analogous to the proof of Theorem II.6.3 in [5].

It is easy to see that for $P_1^t \sim P_2^t$, $t \in \mathbb{R}_+$, the Hellinger integral of order α between measures P_1^t and P_2^t has the form

$$H_t(\alpha) = \mathbb{E}_Q^t \left(Z_1^t\right)^\alpha \left(Z_2^t\right)^{1-\alpha} = \mathbb{E}_1^t \left(Z_{2,1}^t\right)^{1-\alpha} = \mathbb{E}_1^t \exp\left\{(1-\alpha)\ln \frac{dP_2^t}{dP_1^t}\right\}.$$

For $H_t(\alpha)$ we can formulate generalized Cramer conditions B, analogical to conditions C.

B1. Let $\varepsilon > 0$ exists such that $H_t(\alpha)$ defined as $\alpha \in U_\varepsilon(1) = (1-\varepsilon, 1+\varepsilon)$ and for all $\alpha \in U_\varepsilon(1)$ the limit

$$\lim_{t\to\infty} \varphi_t^{-1} \ln H_t(\alpha) = c(\alpha)$$

exists, where $\varphi_t \to \infty$ as $t \to \infty$, the function $c(\alpha)$ is a differentiable at the point 1.

B2. There is a function φ_t, $\varphi_t \to \infty$ as $t \to \infty$, and a strictly convex, differentiable function $c(\alpha)$, such that for each $\alpha \in (0,1)$

$$\lim_{t\to\infty} \varphi_t^{-1} \ln H_t(\alpha) = c(\alpha).$$

B3. Let $(\pi^t, 1-\pi^t)$ are a priori probabilities of the hypotheses H_1^t and H_2^t under observation X^t, such that

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \pi^t = 0, \tag{6}$$

where a function φ_t is such that $\varphi_t \to \infty$ as $t \to \infty$.

Theorem 3.2. *Let the conditions A0 and B1 be satisfied. Then*

1) there exists $\frac{dP_2^t}{dP_1^t}$ *and*

$$P_1^t - \lim_{t\to\infty} \varphi_t^{-1} \ln \frac{dP_2^t}{dP_1^t}\left(X^t\right) = -I_{1,2} < \infty,$$

where $I_{1,2} = c'(1) = \lim_{t\to\infty} \varphi_t^{-1} I\left(P_1^t, P_2^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \mathbb{E}_1^t \ln \frac{dP_2^t}{dP_1^t}$.

If $\lim_{t\to\infty} \varphi_t (\ln t)^{-1} = \infty$, *then the convergence is with the probability* 1.

2) If δ_0^t *is* $\alpha \in (0,1)$ *Neyman-Pearson test, then*

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_2\left(\delta_0^t\right) = -I_{1,2},$$

where $I_{1,2} = c'(1) = \lim_{t\to\infty} \varphi_t^{-1} I\left(P_1^t, P_2^t\right)$.

3.2. **Minimax test.** Let δ_0^t is a minimax test for testing the hypotheses H_1^t and H_2^t under observation X^t.

Lemma 3.3. [7] *Let the conditions A1 and B2 be satisfied. Then*

$$\lim_{t\to\infty} \varphi_t^{-1} \inf_{\delta^t \in \Delta^t} \ln \max\left\{\alpha_1\left(\delta^t\right), \alpha_2\left(\delta^t\right)\right\} = -J_{1,2} = \inf_{0<\alpha<1} c(\alpha) < \infty. \tag{7}$$

By definition for a minimax risk $e_0\left(\delta_0^t\right)$ for the minimax test δ_0^t the (7) formula has a form

$$\lim_{t\to\infty} \varphi_t^{-1} \ln e_0\left(\delta_0^t\right) = -J_{1,2}.$$

From this connection and the fact that (see Lemma 2.1)

$$\alpha_1\left(\delta_0^t\right) = \alpha_2\left(\delta_0^t\right) = e_0\left(\delta_0^t\right)$$

we obtain that

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_1\left(\delta_0^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_2\left(\delta_0^t\right) = -J_{1,2}.$$

We get the second main result:

Theorem 3.3. *Let the conditions A1 and B2 be satisfied. Then*

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_1 \left(\delta_0^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_2 \left(\delta_0^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \ln e_0 \left(\delta_0^t\right) = -J_{1,2},$$

where $0 > -J_{1,2} = c\left(\alpha_0\right) = \inf\limits_{0<\alpha<1} c\left(\alpha\right)$, $c\left(\alpha\right) = \lim\limits_{t\to\infty} \varphi_t^{-1} \ln H_t\left(\alpha\right)$; δ_0^t *is a minimax test for testing the hypotheses* H_1^t *and* H_2^t *under observation* X^t*, and* $e_0\left(\delta_0^t\right)$ *is minimax risk for the test* δ_0^t.

3.3. **Bayesian test.** Let δ_π^t is a Bayesian test for testing the hypotheses H_1^t and H_2^t, $(\pi^t, 1-\pi^t)$ are a priori probabilities of the hypotheses H_1^t and H_2^t.

We will formulate the third main result.

Theorem 3.4. *Let the conditions A1 and B2, B3 be satisfied. Then*

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_1 \left(\delta_\pi^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_2 \left(\delta_\pi^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \ln e_\pi \left(\delta_\pi^t\right) = -J_{1,2},$$

where δ_π^t *is Bayesian test for testing the hypotheses* H_1^t *and* H_2^t, $(\pi^t, 1-\pi^t)$ *are a priori probabilities of the hypotheses* H_1^t *and* H_2^t; $e_\pi\left(\delta_\pi^t\right)$ *is the risk of the test* δ_π^t.

Proof. Let $(\pi^t, 1-\pi^t)$ are a priori probabilities of the hypotheses H_1^t and H_2^t. Then by the definition of Bayesian and minimax tests we have (see proof of Theorem 2.3.7 in [20])

$$e_\pi\left(\delta_\pi^t\right) = \min_{\delta^t} e_\pi\left(\delta^t\right) \le \min_{\delta^t} \max\left\{\alpha_1\left(\delta^t\right), \alpha_2\left(\delta^t\right)\right\} = e\left(\delta_0^t\right),$$

$$e_\pi\left(\delta_\pi^t\right) \ge \left(\min\left\{\pi^t, 1-\pi^t\right\}\right) \max\left\{\alpha_1\left(\delta^t\right), \alpha_2\left(\delta^t\right)\right\} \ge \left(\min\left\{\pi^t, 1-\pi^t\right\}\right) e\left(\delta_0^t\right).$$

Therefore, for arbitrary $(\pi^t, 1-\pi^t)$

$$\min\left\{\pi^t, 1-\pi^t\right\} e\left(\delta_0^t\right) \le e_\pi\left(\delta_\pi^t\right) \le e\left(\delta_0^t\right).$$

Then

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \min\left\{\pi^t, 1-\pi^t\right\} + \lim_{t\to\infty} \varphi_t^{-1} \ln e\left(\delta_0^t\right) \le \lim_{t\to\infty} \varphi_t^{-1} \ln e_\pi\left(\delta_\pi^t\right) \le \lim_{t\to\infty} \varphi_t^{-1} \ln e\left(\delta_0^t\right).$$

By condition B3 and Theorem 3.3, it follows that

$$\lim_{t\to\infty} \varphi_t^{-1} \ln e_\pi\left(\delta_\pi^t\right) = -J_{1,2}. \tag{8}$$

The formula (8) is valid for each a priori probability π^t with (6), i.e. it does not depend on $(\pi^t, 1-\pi^t)$.

Then by Lemma 2.1 and definition of Bayesian criterion (5)

$$\begin{aligned}\lim_{t\to\infty} \varphi_t^{-1} \ln e_\pi\left(\delta_\pi^t\right) &= \lim_{t\to\infty} \varphi_t^{-1} \ln \min_{\delta^t} e_{\pi^t}\left(\delta^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \ln \max_{\pi^t} \min_{\delta^t} e\left(\delta^t\right) = \\ &= \lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_1\left(\delta_\pi^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_2\left(\delta_\pi^t\right) = -J_{1,2}.\end{aligned}$$

□

We will formulated the condition O.

O. There exists an interval (α_-, α_+) containing the interval $[0,1]$ and such that for all $\alpha \in (\alpha_-, \alpha_+)$ there exists a finite limit

$$\lim_{t\to\infty} \varphi_t^{-1} \ln H_t\left(\alpha\right) = c(\alpha),$$

where $\varphi_t \to \infty$ as $t \to \infty$ and $c(\alpha)$ is a strictly convex and differentiable function on (α_-, α_+).

Corollary 3.1. *Let the condition O be satisfied. Then*

1) if δ^t is Neyman-Pearson criterion, then

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_2 \left(\delta^t\right) = -I_{1,2}, \tag{9}$$

where $I_{1,2} = c'(1) = \lim_{t\to\infty} \varphi_t^{-1} I\left(P_1^t, P_2^t\right) = \lim_{t\to\infty} \varphi_t^{-1} \mathbb{E}_1^t \ln \frac{dP_1^t}{dP_2^t}$;

2) if δ_0^t is a minimax or Bayesian (with condition B3) criterion, then

$$\lim_{t\to\infty} \varphi_t^{-1} \ln \alpha_2 \left(\delta_0^t\right) = -J_{1,2}, \tag{10}$$

where $J_{1,2} = \lim\limits_{t\to\infty} \varphi_t^{-1} J\left(P_1^t, P_2^t\right) = -c\left(\alpha_0\right) = -\inf\limits_{0<\alpha<1} c\left(\alpha\right)$, α_0 get from $c'\left(\alpha_0\right) = 0$, $c\left(\alpha\right) = \lim\limits_{t\to\infty} \varphi_t^{-1} \ln H_t\left(\alpha\right)$. The normalizing function φ_t is the same in formulas (9) and (10).

4. Examples

4.1. **Marked point processes of the i.i.d. case.** Suppose we observe $(X_1, X_2, \ldots, X_n)$, $n \in \mathbb{N}$, where $X_1, \ldots, X_n$ are independent identically distributed random variables taking values in the measurable space $(E, \mathcal{E})$ with distribution P_{θ_i}, where $\theta_i \in \Theta$, $i = 1, 2$. The marked point process associated with the sequence of observations is

$$\mu\left([0,t], B\right) = \sum_{j\geq 1} \mathbb{I}(j \leq t)\mathbb{I}\left(X_j \in B\right), \quad B \in \mathcal{E}.$$

Its (P_{θ_i}, F) compensator is deterministic:

$$\nu\left([0,t], B\right) = \sum_{j\geq 1} \mathbb{I}(j \leq t) P_{\theta_i}(B),\ i = 1, 2;\ B \in \mathcal{E}.$$

Let

$$P_{\theta_i}(B) = \int_B p_{\theta_i}(x)dx,\ i = 1, 2;\ B \in \mathcal{E}.$$

Then (see [6])

$$H_t(\alpha) = H_t\left(\alpha, P_{\theta_1}^t, P_{\theta_2}^t\right) = \left(\int\limits_{-\infty}^{+\infty} p_{\theta_1}^{\alpha}(x) p_{\theta_2}^{1-\alpha}(x)dx\right)^{[t]}.$$

Therefore,

$$\lim_{[t]\to\infty} [t]^{-1} \ln H_t(\alpha) = \ln \int\limits_{-\infty}^{+\infty} p_{\theta_1}^{\alpha}(x) p_{\theta_2}^{1-\alpha}(x)dx = \ln H\left(\alpha, P_{\theta_1}, P_{\theta_2}\right) = c(\alpha).$$

Let $c(\alpha)$, $\alpha \in [0, 1]$, is strictly convex and differentiable. Then

$$J_{1,2} = -c\left(\alpha_0\right) = -\inf_{0<\alpha<1} c(\alpha) < 0, \quad I_{1,2} = c'(1) = I\left(P_{\theta_1}, P_{\theta_2}\right) = \mathbb{E}_{\theta_1} \ln \frac{p_{\theta_1}\left(X_1\right)}{p_{\theta_2}\left(X_1\right)}.$$

Example 4.1. Let X_i has exponential distribution:

$$p_\theta(x) = \theta e^{-\theta x} - \text{density},\ x \geq 0, \theta > 0.$$

Then

$$H\left(\alpha, P_{\theta_1}, P_{\theta_2}\right) = \int\limits_0^{\infty} \left(\theta_1 e^{-\theta_1 t}\right)^{\alpha} \left(\theta_2 e^{-\theta_2 t}\right)^{1-\alpha} dt = \frac{\theta_1^{\alpha}\theta_2^{1-\alpha}}{\theta_1\alpha + (1-\alpha)\theta_2},$$

$$c(\alpha) = \alpha \ln \frac{\theta_1}{\theta_2} + \ln \theta_2 - \ln\left[\alpha\left(\theta_1 - \theta_2\right) + \theta_2\right],$$

$$c'(\alpha) = \ln\frac{\theta_1}{\theta_2} - \frac{\theta_1-\theta_2}{\alpha(\theta_1-\theta_2)+\theta_2},$$

$$I_{1,2} = c'(1) = \frac{\theta_2}{\theta_1} - 1 - \ln\frac{\theta_2}{\theta_1},$$

$c'(\alpha_0) = 0$, when $\alpha_0 = \frac{1}{\ln\frac{\theta_1}{\theta_2}} - \frac{1}{\frac{\theta_1}{\theta_2}-1}$. Then

$$J_{1,2} = -c(\alpha_0) = \frac{1}{y-1}\ln y + \ln\frac{\theta_2(y-1)}{\ln y} - \ln\theta_2 - 1,$$

where $y = \frac{\theta_1}{\theta_2}$. From here we see that $I_{1,2} > J_{1,2}$ (see Table 1).

TABLE 1. $I > J$ in the case of marked point processes of the i.i.d. case

θ_1	0.1	0.1	0.1	1	1	1	10	10	10	100	100	100
θ_2	1	10	100	0.1	10	100	0.1	1	100	0.1	1	10
$I_{1,2}$	6.697	94.395	992.092	1.403	6.697	94.395	3.615	1.403	6.697	5.909	3.615	1.403
$J_{1,2}$	0.619	2.114	3.981	0.619	0.619	2.114	2.114	0.619	0.619	3.981	2.114	0.619

4.2. **Non-homogeneous Poisson process.** Let $(\Omega, \mathcal{F}, \mathbb{F}, P_\theta, \theta \in \Theta)$ be stochastic basis on which a non-homogeneous Poisson process N_t, $t \ge 0$, with intensity $k_t(\theta)$, $t \ge 0$, $\theta \in \Theta$, is given and $\mathcal{F}_t^N = \sigma(N_s, s \le t)$, $P_{\theta_i}^t = P_{\theta_i}|\mathcal{F}_t^N$, $i = 1, 2$, $\theta_1, \theta_2 \in \Theta$, $t \ge 0$, $\theta_1 \ne \theta_2$.

Let the following condition be satisfied.

E. There exists a function ψ_t such that $\int_0^t \psi_s ds \to \infty$ as $t \to \infty$ and

$$\lim_{t\to\infty} \psi_t^{-1} k_t(\theta_i) = k(\theta_i), \; i = 1, 2, \; k(\theta_1) \ne k(\theta_2).$$

Lemma 4.1. *(See* [7, 8]*) Let the condition E be satisfied. Then*

$$c(\alpha) = -\left(\alpha k(\theta_1) + (1-\alpha)k(\theta_2) - k^\alpha(\theta_1)\, k^{1-\alpha}(\theta_2)\right)$$

for $\varphi_t = \int_0^t \psi_s ds$, *and*

$$\begin{aligned} I_{1,2} = I(P_{\theta_1}, P_{\theta_2}) = c'(1) &= k(\theta_2) - k(\theta_1) - k(\theta_1)\ln\frac{k(\theta_2)}{k(\theta_1)} = \\ &= k(\theta_1)\left(\frac{k(\theta_2)}{k(\theta_1)} - 1 - \ln\frac{k(\theta_2)}{k(\theta_1)}\right), \end{aligned}$$

$$J_{1,2} = J(\theta_1,\theta_2) = k(\theta_2) + \frac{k(\theta_1)-k(\theta_2)}{\ln(k(\theta_1)/k(\theta_2))}\left[\ln\frac{k(\theta_1)-k(\theta_2)}{k(\theta_2)\ln(k(\theta_1)/k(\theta_2))} - 1\right].$$

Example 4.2. Let $k_s(\theta) = \theta\psi(s)$, $\theta > 0$, $s \ge 0$, $\varphi_t = \int_0^t \psi_s ds$. Then

$$I_{1,2} = \theta_1(x - 1 - \ln x), \text{ where } x = \frac{\theta_2}{\theta_1};$$

$$J_{1,2} = \theta_2\left(1 + \frac{y-1}{\ln y}\left[\ln\frac{y-1}{\ln y} - 1\right]\right), \text{ where } y = \frac{\theta_1}{\theta_2} \text{ and } \theta_1 > \theta_2 > 0.$$

We find that in this case $I_{1,2} > J_{1,2}$, too (see Table 2).

Moreover, in this case one can notice that when there is a stable meaning of θ_1 and the meaning θ_2 increases, meanings of $I_{1,2}$ and $J_{1,2}$ decrease. When there is a stable meaning of θ_2 and the meaning θ_1 increases, meanings of $I_{1,2}$ and $J_{1,2}$ increase.

Table 2. $I > J$ in the case of non-homogeneous Poisson process

θ_1	1	1	10	10	10	50	50	50	100	100	100	100
θ_2	0.1	0.9	0.1	1	9	0.1	1	10	0.1	1	10	99
$I_{1,2}$	1.403	0.005	36.152	14.026	0.054	260.83	146.601	40.472	590.876	361.517	140.259	0.005
$J_{1,2}$	0.242	0.001	4.546	2.42	0.013	27.285	20.136	7.773	57.574	45.456	24.196	0.001

4.3. **The geometric renewal process.** Consider a geometric renewal process $N_t = \sum_{n=1}^{\infty} \mathbb{I}(T_n \le t)$, $t \ge 0$, where random variables $X_n = T_n - T_{n-1}$ have geometric distribution:

$$P_\theta\left(X_n(\omega) = k\right) = \theta(1-\theta)^{k-1},\ \theta \in \Theta = (0,1),\ k = 1, 2, \dots .$$

Then, according to [10, 27], when $\theta_1 \ne \theta_2$ and $\theta_1, \theta_2 \in \Theta = (0,1)$,

$$P\left(N_t = k\right) = C_{[t]}^{k} \theta^k (1-\theta)^{[t]-k},\ k = 0, 1, 2, \dots,$$

$$\frac{dP_{\theta_2}^t}{dP_{\theta_1}^t} = \left(\frac{\theta_2}{\theta_1}\right)^{N_t} \left(\frac{1-\theta_2}{1-\theta_1}\right)^{[t]-N_t}.$$

$$H_t(\alpha) = H_t\left(\alpha, P_{\theta_1}^t, P_{\theta_2}^t\right) = \left[\theta_1^\alpha \theta_2^{1-\alpha} + (1-\theta_1)^\alpha (1-\theta_2)^{1-\alpha}\right]^{[t]}.$$

Then

$$c(\alpha) = \lim_{t\to\infty} \frac{1}{[t]} \ln H_t(\alpha) = \ln\left[\theta_1^\alpha \theta_2^{1-\alpha} + (1-\theta_1)^\alpha (1-\theta_2)^{1-\alpha}\right].$$

In the paper [28] it has been proven that

$$0 < \theta_1^\alpha \theta_2^{1-\alpha} + (1-\theta_1)^\alpha (1-\theta_2)^{1-\alpha} < 1$$

with each $\theta_1, \theta_2 \in (0,1)$, $\theta_1 \ne \theta_2$.

$$c(\alpha) = \ln\left(\left(\frac{\theta_1}{\theta_2}\right)^\alpha \theta_2 + \left(\frac{1-\theta_1}{1-\theta_2}\right)^\alpha (1-\theta_2)\right),$$

$$c'(\alpha) = \frac{\left(\frac{\theta_1}{\theta_2}\right)^\alpha \theta_2 \ln\frac{\theta_1}{\theta_2} + \left(\frac{1-\theta_1}{1-\theta_2}\right)^\alpha (1-\theta_2) \ln\frac{1-\theta_1}{1-\theta_2}}{\left(\frac{\theta_1}{\theta_2}\right)^\alpha \theta_2 + \left(\frac{1-\theta_1}{1-\theta_2}\right)^\alpha (1-\theta_2)}.$$

$$I_{1,2} = I\left(P_{\theta_1}, P_{\theta_2}\right) = c'(1) = \theta_1 \ln\frac{\theta_1}{\theta_2} + (1-\theta_1)\ln\frac{1-\theta_1}{1-\theta_2}.$$

$$c'(\alpha) = 0$$

$$\Longrightarrow \left(\frac{\theta_1}{\theta_2}\right)^\alpha \theta_2 \ln\frac{\theta_1}{\theta_2} = -\left(\frac{1-\theta_1}{1-\theta_2}\right)^\alpha (1-\theta_2)\ln\frac{1-\theta_1}{1-\theta_2};$$

$$a_0 = \frac{\ln\left[\left(1 - \frac{1}{\theta_2}\right) \frac{\ln(1-\theta_1) - \ln(1-\theta_2)}{\ln\theta_1 - \ln\theta_2}\right]}{\ln\frac{\theta_1(1-\theta_2)}{\theta_2(1-\theta_1)}}.$$

From here $J_{1,2} = -c\left(\alpha_0\right)$.

Here, $I_{1,2} > J_{1,2}$, too (see Table 3).

Table 3. $I > J$ in the case of geometric renewal process

θ_1	0.1	0.1	0.1	0.2	0.2	0.2	0.5	0.5	0.5	0.9	0.9	0.9
θ_2	0.2	0.5	0.9	0.1	0.5	0.9	0.1	0.2	0.9	0.1	0.5	0.8
$I_{1,2}$	0.037	0.368	1.758	0.044	0.193	1.363	0.511	0.223	0.511	1.758	0.368	0.037
$J_{1,2}$	0.01	0.112	0.511	0.01	0.053	0.347	0.112	0.053	0.112	0.511	0.112	0.01

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