Theory of Stochastic Processes
Vol. 25 (41), no. 2, 2020, pp. 15–24

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LOOP-ERASED RANDOM WALKS ASSOCIATED WITH MARKOV PROCESSES

A new class of loop-erased random walks (LERW) on a finite set, defined as functionals from a Markov chain is presented. We propose a scheme in which, in contrast to the general settings of LERW, the loop-erasure is performed on a non-markovian sequence and moreover, not all loops are erased with necessity. We start with a special example of a random walk with loops, the number of which at every moment of time does not exceed a given fixed number. Further we consider loop-erased random walks, for which loops are erased at random moments of time that are hitting times for a Markov chain. The asymptotics of the normalized length of such loop-erased walks is established. We estimate also the speed of convergence of the normalized length of the loop-erased random walk on a finite group to the Rayleigh distribution.

1. Introduction

In this article we propose and discuss new constructions of loop-erased random walks on a finite set. These walks are defined as functionals from a Markov chain on an enlarged set. This Markov chain defines not only the elements of the walk, but also the rule, in accordance to which the loops that appear in the walk are erased. This makes the difference with the original loop-erased random walk defined in [1], where any loop is erased as soon as it appear.

We present two constructions. The first one is based on a multi-dimensional Ehrenfest model, which is a Markov chain $\{\nu_n = (i_1^n, \dots, i_N^n), n \geq 0\}$ on the set $\{0, 1, \dots, m\}^N$ with the transition probabilities

$$p_{\vec{i},\vec{j}} = \begin{cases} 0, & \text{if } \sum\limits_{k=1}^{N} |i_k - j_k| > 1, \\ \frac{1}{m+1}, & \text{if } \sum\limits_{k=1}^{N} |i_k - j_k| = 0, \\ \frac{1}{N}\frac{i_k}{m+1}, & \text{if } j_k = i_k - 1, \\ \frac{1}{N}\frac{m-i_k}{m+1}, & \text{if } j_k = i_k + 1. \end{cases}$$

The evolution of this Markov chain can be described as follows. At every step we uniformly choose one coordinate of the vector $\vec{i}$, say i_1. Then change it as in the lazy Ehrenfest's urn model: considering i_1 and $m - i_1$ as the number of particles in two boxes, we first randomly choose a particle and then either move it to another box with probability $m/(m+1)$, or leave it in its box with probability $1/(m+1)$.

Now we construct a random walk on the set $\{1, \dots, N\}$ according to the following algorithm: if for the multiple Ehrenfest chain the k-th coordinate is chosen and $i_k^{n+1} = i_k^n + 1$, then we add to the trajectory of the random walk the element $k \in \{1, \dots, N\}$; if $i_k^{n+1} = i_k^n - 1$, then we erase the part of the trajectory that starts from the latest occurrence of k in the trajectory; if $\vec{i}^{n+1} = \vec{i}^n$, then the trajectory of the walk is not changed.

2000 *Mathematics Subject Classification.* 60J10, 60J67.
Key words and phrases. loop-erased random walk, Ehrenfest model.

The random walk constructed in the proposed way admits loops. Note, however, that each element of the set $\{1,\dots,N\}$ can occur in the walk trajectory no more than m times. Moreover, the probability of erasing loops increases with the number of repeated occurrences of the elements in the path. For such a random walk we establish the asymptotic behavior of the length of the walk trajectory as both n and m tend to infinity.

In the second model proposed in the paper, we allow a finite but not bounded number of loops in the walk trajectory. We start from a Markov chain $\{\nu_n; n\ge 0\}$ on the set $\{0,1\}\times\{1,\dots,N\}$ and construct a random walk on $\{1,\dots,N\}$ as follows. If $\nu_{n+1}=(1,k)$, we add the element k to the trajectory of the walk; if $\nu_{n+1}=(0,k)$, we erase the part of the trajectory that starts from the first entry of k to the trajectory. The asymptotics of the normalized length of such loop-erased walk is investigated.

In the last part of the article we consider loop-erased random walks on a finite group. It is known [3], that the normalized length of such a walk converges to the Rayleigh distribution. We estimate the speed of the convergence using coupling arguments.

2. Loop-erased random walk associated with the Ehrenfest multiple model

Let $\{\nu_n; n\ge 0\}$ be a Markov chain on the phase space $D=\{0,1,\dots,m\}^{|E|}$, where $E=\{1,2\dots,N\}$, $m\ge 1$, i.e. $D=\{(i_1,i_2,\dots,i_N), i_j=0,\dots,m, j=1,\dots,N\}$. Define the transition matrix of $\{\nu_n; n\ge 1\}$ as follows. From the state $(i_1,\dots,i_N)$ one-step transitions are allowed only to the states $(j_1,\dots,j_N)$, such that

$$\sum_{k=1}^{N}|i_k-j_k|\in\{0,1\},$$

and

$$p_{(i_1,\dots,i_N),(i_1,\dots,i_N)}=\frac{1}{m+1},$$
$$p_{(i_1,\dots,i_{k_0},\dots,i_N),(i_1,\dots,i_{k_0}+1,\dots,i_N)}=\frac{1}{N}\frac{m-i_{k_0}}{m+1},\ k_0=1,\dots,N,$$
$$p_{(i_1,\dots,i_{k_0},\dots,i_N),(i_1,\dots,i_{k_0}-1,\dots,i_N)}=\frac{1}{N}\frac{i_{k_0}}{m+1},\ k_0=1,\dots,N.$$

When $N=1$ the corresponding Markov chain is the well-known Ehrenfest chain,which is known to have the unique invariant distribution

$$\pi_i=C_m^i\frac{1}{2^m},\ i=0,\dots,m.$$

The multi-dimensional chain $\{\nu_n; n\ge 1\}$ on D has the unique invariant distribution of the form

$$\pi_{(i_1,\dots,i_N)}=\prod_{k=1}^{N}\pi_{i_k}=\frac{1}{2^{mN}}\cdot\prod_{k=1}^{N}C_m^{i_k},\ \ (i_1,\dots,i_N)\in D.$$

Now let us construct a loop-erased random walk $\{\Gamma_n^m; n\ge 0\}$, using multi-dimensional Ehrenfest chain $\{\nu_n; n\ge 1\}$. We will start from $\nu_0=(0,\dots,0)$ and put $\Gamma_0^m=\emptyset$, $l_0^m=|\Gamma_0^m|=0$. The next elements of $\{\Gamma_n^m; n\ge 1\}$ are constructed inductively as follows. Suppose $\Gamma_n^m=(x_1,\dots,x_r)$ has been already defined. If $\nu_{n+1}=\nu_n$, then set $\Gamma_{n+1}^m=\Gamma_n^m$. If

$$\nu_n=(i_1,\dots,i_{k_0},\dots,i_N)\text{ and }\ \nu_{n+1}=(i_1,\dots,i_{k_0}+1,\dots,i_N),$$

then set

$$x_{r+1}=k_0,\ \Gamma_{n+1}^m=(x_1,\dots,x_{r+1}),\ l_{n+1}^m=l_n^m+1.$$

Otherwise, if

$$\nu_n=(i_1,\dots,i_{k_0},\dots,i_N),\ \ \nu_{n+1}=(i_1,\dots,i_{k_0}-1,\dots,i_N)\text{ and }\ k_0\notin(x_1,\dots,x_r),$$

then put

$$\Gamma^m_{n+1} = \Gamma^m_n,\ l^m_{n+1} = l^m_n.$$

In the case

$$\nu_n = (i_1, \dots, i_{k_0}, \dots, i_N),\ \ \nu_{n+1} = (i_1, \dots, i_{k_0} - 1, \dots, i_N) \text{ and } \ k_0 \in (x_1, \dots, x_r),$$

define

$$\Gamma^m_{n+1} = (x_1, \dots, x_{i-1}),\ l^m_{n+1} = i - 1,$$

where $i = \max\{j : k_0 = x_j\}$.

In this way we obtain a random walk $\{\Gamma^m_n; n \geq 0\}$ on E, such that for every $n \geq 1$ the sequence Γ^m_n contains no more than m entries of every element of the set E. Moreover, at every moment of time, the erasure of a loop associated with the element $i \in E$, occurs with probability that is proportional to the current number of entries of i in Γ^m.

Clearly, the maximal length of Γ^m_n does not exceed $m \cdot N$

$$\forall n \geq 1: \quad l^m_n = |\Gamma^m_n| \leq m \cdot N.$$

It is easy to check, that the sequence $\{\Gamma^m_n; n \geq 0\}$ is a Markov chain in the space of finite sequences with elements from E of length at most mN.

We will study the asymptotics of the length l^m_n as m, n tend to infinity. The following statement holds.

Theorem 1. *For any $\alpha > 0$ there exist $m_0 > 0$, $C > 0$, such that for all $m > m_0$*

$$P\{l^m_{m^{1+\alpha}} < \frac{m}{2} - \frac{1}{2}\sqrt{(1+\alpha)m \ln m}\} \leq C\, m^{-(1+\alpha)/2}.$$

Proof. For every $n \geq 1$ the state ν_n of the N-dimensional Ehrenfest chain can be written as $\nu_n = (\nu^1_n. \dots, \nu^N_n)$. It can be easily checked by induction, that for every $n \geq 1$

$$l^m_n \geq \min_{j=\overline{1,N}} \nu^j_n.$$

Consequently, in order to prove the theorem, it is enough to proove the statement of the theorem for $\nu^j_{m^{1+\alpha}}$ as m tends to infinity.

Note, that for a fixed j the sequence $\{\nu^j_n; n \geq 1\}$ is the lazy Ehrenfest chain with the transition probability matrix

$$\frac{1}{N}Q + (1 - \frac{1}{N})I,$$

where Q is the transition matrix of the one-dimensional Ehrenfest chain and I is the identity matrix. It is known, that for probability distributions $Q^n\delta_0$ the following relations hold.

Theorem 2. [2] *For an arbitrary $c > 0$*

$$\|Q^{m \ln m + cm}\delta_0 - \pi_m\| \leq e^{-c}.$$

Here π_m is the invariant distribution of the Ehrenfest chain - the binomial distribution with parameters m and $\frac{1}{2}$, and $\| \cdot \|$ is the variation distance.

Now take $\alpha > 0$. Consider m_0 such that
$\forall\ m \geq m_0$:

$$m \ln m \leq \frac{1}{3N} m^{1+\alpha}.$$

Then for $m \geq m_0$

$$\|Q^{m^{1+\alpha}}\delta_0 - \pi_m\| \leq e^{-(1-\frac{2}{3N})m^\alpha}.$$

Consequently,

$$\|(\frac{1}{N}Q + (1 - \frac{1}{N})I)^{m^{1+\alpha}}\delta_0 - \pi_m\| \leq$$

$$\leq 2 \sum_{k=0}^{m\ln m+\frac{1}{2N}m^{1+\alpha}} C^k_{m^{1+\alpha}}(\frac{1}{N})^k \cdot (1-\frac{1}{N})^{m^{1+\alpha}-k} + e^{-(1-\frac{2}{3N})m^{\alpha}}.$$

In the last inequality the upper bound of summation is chosen in such a way that Theorem 2 can be applied for other summands.

In order to estimate the first summand, consider a sequence $\{\xi_n; n \geq 1\}$ of independent random variables taking values zero and one with probabilities $1-\frac{1}{N}$ and $\frac{1}{N}$ respectively. Define

$$S_n = \sum_{k=1}^{n} \xi_k, \ n \geq 1.$$

Then

$$\sum_{k=0}^{m\ln m+\frac{1}{2N}m^{1+\alpha}} C^k_{m^{1+\alpha}}(\frac{1}{N})^k(1-\frac{1}{N})^{m^{1+\alpha}-k} =$$

$$= P\{S_{m^{1+\alpha}} \leq m\ln m + \frac{1}{3}m^{1+\alpha}\} =$$

$$= P\left\{ \frac{1}{m^{\frac{1+\alpha}{2}}} \cdot \frac{1}{\sqrt{\frac{1}{N}(1-\frac{1}{N})}}(S_{m^{1+\alpha}} - \frac{1}{N}m^{1+\alpha}) \leq \right.$$

$$\left. \leq \frac{1}{m^{\frac{1+\alpha}{2}}} \cdot \frac{1}{\sqrt{\frac{1}{N}(1-\frac{1}{N})}}(m\ln m - \frac{1}{2N}m^{1+\alpha}) \right\} \leq$$

$$\leq \Phi\left(\frac{1}{\sqrt{\frac{1}{N}(1-\frac{1}{N})}} \cdot \frac{m\ln m - \frac{1}{2N}m^{1+\alpha}}{m^{\frac{1+\alpha}{2}}} \right) + C\frac{1}{m^{\frac{1+\alpha}{2}}}$$

due to the Berry-Esseen inequality. Here Φ is the distribution function of the standard normal law and the constant C depends on N. Finally,

$$\|(\frac{1}{N}Q + (1-\frac{1}{N})I)^{m^{1+\alpha}}\delta_0 - \pi_m\| \leq C_1 m^{-\frac{1+\alpha}{2}}.$$

Recall that π_m is binomial distribution with parameters m and $\frac{1}{2}$. In order to make the estimation for ν^j_n, it is enough to make the estimation for the Gaussian distribution and use the Berry-Esseen inequality for π_m. Hence, we get

$$P\{l^m_{m^{1+\alpha}} < \frac{m}{2} - \frac{1}{2}\sqrt{(1+\alpha)m\ln m}\} \leq C_2\, m^{-(1+\alpha)/2}.$$

The theorem is proved. □

3. Loop-erased random walks associated with a renewal process

Let $\{\nu_n = (z_n, \sigma_n); n \geq 1\}$ be a Markov chain on the phase space $F = \{1, 2, \ldots, N\} \times \{0, 1\}$. We define a loop-erased sequence $\{\Gamma_n; n \geq 1\}$ inductively as follows. Let $\Gamma_0 = \emptyset$. If $\sigma_1 = 1$, set $x_1 = z_1$ and $\Gamma_1 = (x_1)$. Otherwise, if $\sigma_1 = 0$, set $\Gamma_1 = \Gamma_0 = \emptyset$. Inductively, if $\Gamma_n = (x_1, \ldots, x_k)$ has been defined, then:

if $\sigma_{n+1} = 1$, set

$$x_{k+1} = z_{n+1},\ \Gamma_{n+1} = (x_1, \ldots, x_k, x_{k+1}),$$

otherwise, if $\sigma_{n+1} = 0$, then start the loop-erasing procedure:

if $z_{n+1} \notin (x_1, \ldots, x_k)$, then set

$$\Gamma_{n+1} = \Gamma_n,$$

otherwise, if $z_{n+1} \in (x_1, \ldots, x_k)$, define

$$\Gamma_{n+1} = (x_1, \ldots, x_{i-1}),$$

where $i = \min\{m : z_{n+1} = x_m\}$. By this procedure loops are erased at random moments, at which the second component of the chain $\{\nu_n; n \geq 1\}$ equals zero.

The asymptotics of the normalized length $|\Gamma_n|$ as $n \to \infty$ is described in the following theorem.

Theorem 3. *Suppose, that the matrix of transition probabilities of the Markov chain $\{\nu_n; n \geq 1\}$ consists of positive numbers. Then there exists a positive constant c_0 such, that*

$$P\{\overline{\lim_{n\to\infty}} \frac{|\Gamma_n|}{\ln n} = c_0\} = 1.$$

Proof. Let us define two sequences of moments of time $\{m_n; n \geq 1\}$ and $\{\tau_n; n \geq 1\}$ as follows:

$$m_1 = \min\{k \geq 1 : \sigma_k = 1\};\ \tau_1 = \min\{k > m_1 : \nu_k = (z_{m_1}, 0)\}$$

and for $n \geq 2$

$$m_n = \min\{k \geq \tau_{n-1} + 1 : \sigma_k = 1\};\ \tau_n = \min\{k > m_n : \nu_k = (z_{m_n}, 0)\}.$$

It follows from this definition and the construction of $\{\Gamma_n; n \geq 1\}$ that m_1 is the first moment for which $\Gamma_{m_1} \neq \emptyset$, while τ_1 is the first moment after m_1 such that $\Gamma_{\tau_1} = \emptyset$; m_2 is the first moment after τ_1 for which $\Gamma_{m_2} \neq \emptyset$, while τ_2 is the first moment after m_2 such that $\Gamma_{\tau_2} = \emptyset$ and so on.

Define a probability kernel $K(x, y)$ on E as follows. For each starting point $(x, 0)$ of the chain $\{\nu_n; n \geq 1\}$ let $\varkappa'$ be the first moment when $\nu_{\varkappa'} \in E \times \{1\}$. If $\nu_{\varkappa'} = (y, 1)$ then let $\varkappa'' > \varkappa'$ be the first moment for which $\nu_{\varkappa''} = (y, 0)$. Define

$$K(x, y) = P(\nu_{\varkappa''} = (y, 0)).$$

Assuming that the transition probabilities of the Markov chain $\{\nu_n; n \geq 1\}$ are positive, the transition matrix K also has positive elements. Hence, the sequence of the blocks $\{\Gamma_1, \ldots, \Gamma_{\tau_1}\}$, $\{\Gamma_{\tau_1+1}, \ldots, \Gamma_{\tau_2}\}, \ldots$ forms a Markov chain which satisfies the exponential mixing condition.

Note, that in the k-th block $\{\Gamma_{\tau_{k-1}+1}, \ldots, \Gamma_{\tau_k}\}$ the maximal length of Γ-s is at most $\tau_k - 1 - m_k$, and is at least $\tau_k^* - 1 - m_k$, where $\tau_k^* = \min\{m_k < r \leq \tau_k : \sigma_r = 0\}$ is the first moment in $[\tau_{k-1} + 1, \ldots, \tau_k]$, when the loop-erasing procedure starts.

In view of the ergodic theorem

$$\tau_n \sim nC,\ n \to \infty \text{ a.s.},$$

where C is the expectation of τ_1 under the stationary distribution of the chain $\{\{\Gamma_{\tau_n+1}, \ldots, \Gamma_{\tau_{n+1}}\}; n \geq 0\}$. Consequently, in order to obtain an upper estimation of $\frac{|\Gamma_n|}{\ln n}$, it is enough to consider $\overline{\lim_{n\to\infty}} \frac{\alpha_n}{\ln n}$, where $\alpha_n = \max\{|\Gamma_r| : \tau_n + 1 \leq r < \tau_{n+1}\}$-is the maximal length of Γ-s in the block $\{\Gamma_{\tau_n+1}, \ldots, \Gamma_{\tau_{n+1}}\}$. Since, as has been already noted, $\alpha_n \leq \tau_{n+1} - m_{n+1} - 1$ and the transition probabilities of $\{\nu_n; n \geq 1\}$ are assumed to be positive, we get

$$P\{\alpha_n > N\} \leq q^N$$

for some $q \in (0; 1)$. From this inequality and the Borel-Cantelli lemma it follows that with probability one

$$\overline{\lim_{n\to\infty}} \frac{\alpha_n}{\ln n} < +\infty.$$

From the other side, $\alpha_n \geq \beta_n = \tau_{n+1}^* - 1 - m_{n+1}$, where

$$\tau_{n+1}^* = \min\{m_{n+1} < r \leq \tau_{n+1} : \sigma_r = 0\}$$

is the first moment in $[\tau_n + 1, \ldots, \tau_{n+1}]$, at which the loop-erasing procedure is performed. Since the transition probabilities of $\{\nu_n; n \geq 1\}$ are positive, we have

$$P\{\beta_n \geq N\} \geq p^N$$

for some $p > 0$. Note, that $\{\beta_n; n \geq 1\}$ satisfy the exponential mixing condition. Hence, the Borel-Cantelli lemma for weakly dependent random events can be applied [4], and we get that with probability one

$$\overline{\lim_{n\to\infty}} \frac{\beta_n}{\ln n} > 0.$$

Now the statement of the theorem follows from the zero-one law for stationary mixing sequences. □

4. Simple loop-erased random walk on a finite group

Consider a finite group G of order $m = |G|$ with multiplication $f \cdot g$ for elements $f, g \in G$ and the identity element e. Construct a loop-erased random walk $\{\Gamma_n, n \geq 0\}$ on G as follows. Let $\{\xi_n, n \geq 1\}$ be independent uniformly distributed random elements in G. Put $x_0 = e$ and $\Gamma_0 = (x_0)$. By induction, if $\Gamma_n = (x_0, x_1, \ldots, x_k)$ and $x_k \cdot \xi_{n+1} \notin (x_0, x_1, \ldots, x_k)$, then put $x_{k+1} = x_k \cdot \xi_{n+1}$ and

$$\Gamma_{n+1} = (x_0, x_1, \ldots, x_k, x_{k+1}).$$

Otherwise, if $x_k \cdot \xi_{n+1} = x_i$ for some $i = 0, \ldots, k$, then put

$$\Gamma_{n+1} = (x_0, x_1, \ldots, x_i).$$

By this procedure loops are erased in the order they appear and we get a trajectory of random walk on G without loops.

We assume that the length $|\Gamma|$ of the path $\Gamma = (x_0, x_1, \ldots, x_k)$ equals k, and the length of $\Gamma = (e)$ equals zero.

The following two lemmas can easily be proved.

Lemma 1. *The sequence $\{\Gamma_n; n \geq 0\}$ is a Markov chain in $G^{|G|}$ which has a unique invariant distribution.*

Denote by $\widetilde{\pi}$ the invariant distribution of $\{\Gamma_n; n \geq 0\}$. By symmetry, due to the uniform distribution of $\{\xi_n, n \geq 1\}$, the following statement holds.

Lemma 2. *For arbitrary $\Delta \subset G\backslash\{e\}$ such that $|\Delta| = l \leq m - 1$,*

$$P_{\widetilde{\pi}}(\,\Gamma_0 = (e, \Delta)\,|\,|\Gamma_0| = l) = \frac{(m-1-l)!}{(m-1)!}.$$

Here $P_{\widetilde{\pi}}$ denotes probability with respect to the invariant distribution $\widetilde{\pi}$.

To find $\widetilde{\pi}$ exactly it remains to find

$$P_{\widetilde{\pi}}\{|\Gamma_0| = k\},\ k = 0, \ldots, m-1.$$

Note, that $\{|\Gamma_n|, n \geq 1\}$ is a homogeneous Markov chain with the state space $\{0, \ldots, m-1\}$ and transition probabilities

$$p_{ij} = \begin{cases} \frac{1}{m},\ 0 \leq j \leq i \\ 1 - \frac{i+1}{m},\ j = i+1 \\ 0,\ \text{otherwise.} \end{cases}$$

It can be checked straightforwardly that the unique invariant distribution of $\{|\Gamma_n|, n \geq 1\}$ is given by

$$\begin{gathered}\pi_0 = \tfrac{1}{m},\\ \pi_1 = (1-\tfrac{1}{m})\tfrac{2}{m},\\ \vdots\\ \pi_k = (1-\tfrac{1}{m})\ldots(1-\tfrac{k}{m})\tfrac{k+1}{m},\\ \vdots\\ \pi_{m-1} = (1-\tfrac{1}{m})\ldots(1-\tfrac{m-1}{m}).\end{gathered}$$

Hence, the invariant distribution $\widetilde{\pi}$ for a trajectory of LERW is

$$\widetilde{\pi}(e, x_1, \ldots, x_k) = (1-\frac{1}{m})\ldots(1-\frac{k}{m})\frac{k+1}{m}\cdot\frac{(m-1-k)!}{(m-1)!} = \frac{k+1}{m^{k+1}}. \tag{1}$$

From now on we will denote by Γ^m a random element in G with the distribution $\widetilde{\pi}$.

The next theorem about the limiting distribution of $|\Gamma^m|$ as $m = |G| \to \infty$ is an analogue of the result stated in [3] for LERW on finite graphs.

Theorem 4.

$$\frac{|\Gamma^m|}{\sqrt{m}} \Rightarrow \eta,\ m \to +\infty$$

where η has the Rayleigh distribution with the density $xe^{-x^2/2}, x \geq 0$.

Proof. Using the expression for the distribution of $|\Gamma^m|$ one can check, that for $x > 0$

$$P\{\frac{|\Gamma^m|}{\sqrt{m}} > x\} = \prod_{k=1}^{[x\sqrt{m}]+1}(1-\frac{k}{m})$$

and then

$$\ln P\{\frac{|\Gamma^m|}{\sqrt{m}} > x\} = \sum_{k=1}^{[x\sqrt{m}]+1}\ln(1-\frac{k}{m}).$$

Using the inequality $\ln(1-y) \leq -y,\ y \in (-\infty; 1)$, we obtain

$$\varlimsup_{m\to\infty}\ln P\{\frac{|\Gamma^m|}{\sqrt{m}} > x\} \leq -\varlimsup_{m\to\infty}\sum_{k=1}^{[x\sqrt{m}]+1}\frac{k}{m} = -\frac{x^2}{2}.$$

To estimate the lower limit, let us note, that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that
$\forall\ y \in [0; \delta]$:

$$\ln(1-y) \geq -(1+\varepsilon)y.$$

Hence

$$\varliminf_{m\to\infty}\ln P\{\frac{|\Gamma^m|}{\sqrt{m}} > x\} \geq$$

$$\geq \varliminf_{m\to\infty} -(1+\varepsilon)\sum_{k=1}^{[x\sqrt{m}]+1}\frac{k}{m} = -(1+\varepsilon)\frac{x^2}{2}.$$

Since $\varepsilon > 0$ is arbitrary, the theorem is proved.

□

In the next theorem we present the estimation on the Lévy distance
$d(F_\pi^m, F_\eta) = \inf\{\varepsilon > 0 :\ \forall\ x \in \mathbb{R} : F_\pi^m(x) \leq F_\eta(x+\varepsilon)+\varepsilon, F_\eta(x) \leq F_\pi^m(x+\varepsilon)+\varepsilon\}$
between the distribution functions F_π^m and F_η of $\frac{1}{\sqrt{m}}|\Gamma^m|$ and η respectively.

Theorem 5. *For every $\varepsilon \in (0; \frac{1}{4})$ there exists $C > 0$ such, that*

$$d(F_\pi^m, F_\eta) \leq C \cdot \frac{1}{m^{\frac{1}{4}-\varepsilon}}.$$

Proof. In order to estimate the Lévy distance between the distribution F_π^m of the normalized length of LERW trajectory and the Rayleigh distribution function F_η, we will introduce a Markov chain $\{\Gamma'_n, n \geq 0\}$, constructed using the same sequence of independent random elements $\{\xi_n; n \geq 1\}$ in G, which was used to construct $\{\Gamma_n, n \geq 0\}$, as follows. Fix $\lambda \in (0; 1)$ such, that λm is integer. Put $x'_0 = e$ and $\Gamma'_0 = x'_0$. By induction, if $\Gamma'_n = (x'_0, x'_1, \dots, x'_k)$ and $x'_k \cdot \xi_{n+1} \notin (x'_0, x'_1, \dots, x'_k)$, then put $x'_{k+1} = x'_k \cdot \xi_{n+1}$ and

$$\Gamma'_{n+1} = \begin{cases} (x'_0, x'_1, \dots, x'_k, x'_{k+1}), \ k < \lambda m \\ (x'_0, x'_1, \dots, x'_k), \ k = \lambda m. \end{cases}$$

Otherwise, if and $x'_k \cdot \xi_{n+1} = x'_i$ for some $i = 0, \dots, k$, then put

$$\Gamma'_{n+1} = (x'_0, x'_1, \dots, x'_i).$$

It can be checked, that $\{\Gamma'_n, n \geq 0\}$ is a Markov chain. Moreover $\{|\Gamma'_n|, n \geq 0\}$ is also a Markov chain with the transition matrix

$$p_{ij} = \begin{cases} \frac{1}{m}, \ 0 \leq j \leq i \\ 1 - \frac{i+1}{m}, \ j = i + 1 \\ 0, \ \text{otherwise}, \end{cases} \tag{2}$$

for $0 \leq i \leq j < \lambda m$, and

$$p_{\lambda m, j} = \begin{cases} \frac{1}{m}, j < \lambda m \\ 1 - \frac{\lambda m}{m}, \ j = \lambda m. \end{cases} \quad .$$

We can construct $\{\Gamma_n, n \geq 0\}$ and $\{\Gamma'_n, n \geq 0\}$ simultaneously on the same probability space by letting $x'_{k+1} = (x'_k)^{-1} x_k \xi_{n+1}$ on every step $k \geq 0$, as it follows from the lemma below. Define

$$\mathcal{F}_n^\xi = \sigma(\xi_1, \dots, \xi_n), \ n \geq 1.$$

Lemma 3. *Suppose, that a random element z in G is measurable with respect to $\mathcal{F}_n^\xi$. Then the sequence $\xi_1, \dots, \xi_n, z\xi_{n+1}$ is equidistributed with $\xi_1, \dots, \xi_{n+1}$.*

Proof. For any $g_1, \dots, g_n \in G$ we have

$$P\{\xi_1 = g_1, \dots, \xi_n = g_n, z\xi_{n+1} = g_{n+1}\} =$$
$$= E \prod_{k=1}^n 1\!\!1_{\xi_k = g_k} \cdot 1\!\!1_{\xi_{n+1} = z^{-1} g_{n+1}} =$$
$$= E\{\prod_{k=1}^n 1\!\!1_{\xi_k = g_k} E(1\!\!1_{\xi_{n+1} = z^{-1} g_{n+1}} / \mathcal{F}_n^\xi)\} = \frac{1}{m^{n+1}} =$$
$$= P\{\xi_1 = g_1, \dots, \xi_n = g_n, \xi_{n+1} = g_{n+1}\}.$$

□

Such coupling of x and x' helps us to estimate the Lévy distance between the invariant distributions of the lengths of Γ and Γ'. Due to (2) the invariant distribution π'_m of $|\Gamma'|$ is such, that
$\forall \ k = 0, \dots, \lambda m - 1$:

$$P_{\pi'_m}(|\Gamma'| > k) = \prod_{j=1}^{k+1} (1 - \frac{j}{m}),$$

$\forall\ k = \lambda m, \dots, m-1:$

$$P_{\pi'_m}(|\Gamma'| > k) = 0.$$

Note that

$$\sup_{x \geq 0} |P_{\pi'_m}\{\frac{|\Gamma'|}{\sqrt{m}} > x\} - e^{-\frac{x^2}{2}}| \leq \max\{e^{-\lambda^2 m/2}, \sup_{0 \leq x < \lambda\sqrt{m}} |e^{-\frac{x^2}{2}} - e^{\sum_{j=1}^{[x\sqrt{m}]+1} \ln(1-\frac{j}{m})}|\}.$$

For $j \leq \lambda m$ we have

$$\ln(1-\frac{j}{m}) = -\sum_{l=1}^{\infty}(\frac{j}{m})^l \frac{1}{l} \geq -\frac{j}{m} - \frac{j^2}{m^2} - \frac{j^2}{m^2}\sum_{l=1}^{\infty}\frac{\lambda^l}{l} =$$

$$= -\frac{j}{m} - \frac{j^2}{m^2}(1-\ln(1-\lambda)).$$

Hence, for $0 \leq x < \lambda\sqrt{m}$

$$1 - e^{\frac{x^2}{2} + \sum_{j=1}^{[x\sqrt{m}]+1} \ln(1-\frac{j}{m})} \leq$$

$$\leq -\frac{x^2}{2} + \frac{1}{2}(x\sqrt{m}+1)(x\sqrt{m}+2)\frac{1}{m} +$$

$$+ \frac{(x\sqrt{m}+1)(x\sqrt{m}+2)(2x\sqrt{m}+3)}{6m^2} \cdot (1-\ln(1-\lambda)).$$

Consequently, there exist constants C_1, C_2 that do not depend on λ and m, such that

$$\sup_{0 \leq x < \lambda\sqrt{m}} e^{-\frac{x^2}{2}}(-\frac{x^2}{2} - \sum_{k=1}^{[x\sqrt{m}]+1} \ln(1-\frac{k}{m})) \leq \frac{C_1}{\sqrt{m}} - \frac{C_2}{\sqrt{m}}\ln(1-\lambda).$$

Finally, the Kolmogorov distance between the normalized length of the abridged version of LERW and the Rayleigh distribution is less or equal to

$$\max\{e^{-\frac{\lambda^2 m}{2}}, \frac{C_1}{\sqrt{m}} - \frac{C_2}{\sqrt{m}}\ln(1-\lambda)\}.$$

Let us estimate the Lévy distance between the normalized lengths of LERW and the abridged LERW, using the already constructed coupling. Consider $\frac{1}{n}\sum_{k=1}^{n}|l_k - l'_k|$, where l_k and l'_k are the lengths of the corresponding sequences on k-th step. Note that the set of numbers, for which $l_k \neq l'_k$ has the following structure. It is the union of disjoint intervals $I_j, j \geq 1$. Here $|I_j|, j \geq 1$ are independent geometrically distributed random variables with parameter $1-\lambda$. Note also that $l'_k = \lambda m$ when $k \in I_j$. Consequently,

$$E\sum_{k \in I_i}|l_k - l'_k| \leq \frac{1}{2}E|I_i|^2 = \frac{(1-\lambda)(2-\lambda)}{\lambda^2}.$$

From the ergodic theorem it follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{1}_{k \in \bigcup_{j=1}^{\infty} I_j} = \pi'_{\lambda m}.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|l_k - l'_k| \leq \frac{\pi'_{\lambda m}}{EI_1} \cdot \frac{(1-\lambda)(2-\lambda)}{\lambda^2} =$$

$$= \pi'_{\lambda m}\frac{\lambda}{1-\lambda}\frac{(1-\lambda)(2-\lambda)}{\lambda^2} = \pi'_{\lambda m}\frac{2-\lambda}{\lambda} \leq 2\pi'_{\lambda m}\frac{1}{\lambda}.$$

Finally, the estimation

$$\pi'_{\lambda m} = \prod_{k=1}^{\lambda m+1}(1-\frac{k}{m}) = e^{\sum_{k=1}^{\lambda m+1}\ln(1-\frac{k}{m})} \leq$$

$$\leq e^{-\sum_{k=1}^{\lambda m+1}\frac{k}{m}} = e^{-\frac{1}{2}\frac{(\lambda m+1)(\lambda m+2)}{m}} \leq e^{-\frac{1}{2}\lambda^2 m}$$

gives the desired result after optimization with respect to λ. $\square$

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