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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS WITH INTERACTION

Two-dimensional stochastic differential equation with interaction is considered. The large time behavior of the distance between two solutions starting from different points is studied. A nonzero limit that characterize this distance together with the analogue of the triangle inequality for the map that characterize the limit distance are obtained.

1. Introduction

The aim of the paper is to establish the asymptotic behavior of distribution of mass of interacting particles in the stochastic flow. The interest in such objects is caused in particular by the phenomena of intermittency. In short, intermittency represents the contrast between the realization and the mean characteristic of speed or density in a turbulent flow. The article [14] is one of the first articles devoted to the phenomena. An important case of intermittency is represented by the behavior of passive tracer, carried by a random field [15]. A stochastic differential equation for a random flow F based on a space time martingale U has the form

$$(1)\qquad \begin{cases} F_{st}(x) = x + \int\limits_s^t U\left(F_{sr}, dr\right), \quad t \geq s, \\ \mu_t(D) = \mu_0\left(\{x \in \mathbb{R}^d : \ F_{0t}(x) \in D\}\right), \ D \subseteq \mathcal{B}\left(\mathbb{R}^d\right), \end{cases}$$

$x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$, U is a Gaussian random vector-field on $\mathbb{R}^d$,

$$EU^i(x,t) = \int\limits_0^t u^i(x,r)dr, \qquad u^i(x,t) \in \mathbb{R}, \quad i = 1, ..., d,$$

$$Cov\left(U^i(x,s), U^j(y,t)\right) = a^{ij}(x,y)\min\{s,t\}, i,j = 1,...,d; \ x,y \in \mathbb{R}^d,$$

μ_0 is some finite measure that characterize the distribution of mass at the time 0. Suppose, that

$$|u(x,t)| \leq K_1(1+|x|), \quad |u(x,t) - u(y,t)| \leq K_2|x-y|, \quad |a^{ij}(x,y)| \leq K_3(1+|x|)(1+|y|),$$

$$|a^{ij}(x,y) - a^{ij}(x',y) - a^{ij}(x,y') + a^{ij}(x',y')| \leq K_4|x-x'||y-y'|$$

for all $x, x', y, y' \in \mathbb{R}^d$ and $t \in \mathbb{R}^+$, where K_1, K_2, K_3, K_4 are finite constants. Then by Theorem 4.2.5 from [12] there exists a flow $F = \{F_{st};\ 0 \leq s \leq t < \infty\}$ of homeomorphisms satisfying (1), such that the mapping $(x,t) \to F_t(x)$ is continuous almost surely and $F_{t_1t_2}$,...,$F_{t_{n-1}t_n}$ are independent for all $t_1 \leq t_2, \leq ... \leq t_n$. Due to the last property, F is called a Brownian flow.

A class of problems for such an equation (see, for example, [1]-[4],[9], [10], [15]-[18]) were formed by the investigation of the transition of mass in the isotropic Brownian

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flows. A Brownian flow F based on a Brownian motion U with drift u and covariance a is isotropic if and only if

$$u \equiv 0 \quad a(x,y) = b(x-y); \quad x, y \in R^d,$$

where b is such isotropic covariance tensor that $O^T b(Oz)O = b(z)$ for every orthogonal matrix O and every $z \in \mathbb{R}^d$. The behavior of the distance between particles in such a flow, started from two different points, was obtained in [1].

The concentration of mass and its spreading out can be described in terms of centroid $C_t = (C_t^i)$ and dispersion matrix $D_t = (D_t^{ij})$ that are defined by

$$C_t^i = \frac{1}{\mu_0(\mathbb{R}^d)} \int\limits_{\mathbb{R}^d} x^i \mu_t(dx),$$

$$D_t^{ij} = \frac{1}{\mu_0(\mathbb{R}^d)} \int\limits_{\mathbb{R}^d} (x^i - C_t^i)(x^j - C_t^j)\mu_t(dx).$$

For isotropic Brownian flows the limit behavior on infinity of $Cov(C_t^i, C_t^j)$, D_t^{ij}, $Var(C_t)$ can be found in [17, 18].

It is useful to study the limit behavior of μ_t on infinity to describe the evolution of mass distribution in a Brownian flow. In Theorems 4.3.9 and 4.3.10 from [12] the case, where the matrix $a(x,x)$ is strictly positive definite, is considered. If the one-point motion under F has an invariant distribution π with $\pi(\mathbb{R}^d) = 1$, the following limit relation was obtained for each Borel subset R of $\mathbb{R}^d$

$$\lim_{t\to\infty} \frac{1}{t} \int\limits_0^t \mu_s(R)ds = \pi(R) \quad \text{almost surely.}$$

In the case, where the one-point motion does not have a finite invariant measure, it was proved, that for each Borel subset R of $\mathbb{R}^d$

$$\lim_{t\to\infty} E\mu_t(R) = 0.$$

It is also useful to consider the process $M_t f = \int f(x)\mu_t(dx)$, $t \geq 0$ for various choices of f. The expression for joint quadratic variation of $M_t f$ and $M_t g$ was described in [15]. In the case, where μ_0 has compact support and $a(x,x)$ is uniformly elliptic with bounded second partial derivatives, it was proved in paper [16] that μ_t has the density $m_t \in C^{2,1}\left(\mathbb{R}^d \times (0,\infty)\right)$. The advection-diffusion equation for m_t also was obtained in the same paper.

The particles can move and at the same time interact with each other. In this case the coefficients of the corresponding stochastic differential equation (SDE) depend on some characteristic of positions of another particles. The following stochastic differential equation with interaction that describe such situation was introduced in [5]

$$\begin{cases} dx(u,t) = a(x(u,t), \mu_t, t)dt + \int\limits_{R^d} b(x(u,t), \mu_t, t, q)W(dt, dq) \\ x(u,0) = u, \ \mu_t = \mu_0 \circ x(\cdot, t)^{-1}. \end{cases} \tag{2}$$

Here W is a Brownian sheet, μ_0 is a probability measure, that plays a role of the distribution of mass of particles, $x(u,\cdot)$ is the trajectory of the particle, that left the point u at time zero, μ_t characterize the distribution of mass of particles at time t.

Definition 1.1. The random $\mathbb{R}^d$-valued $x(u,t)$, $u \in \mathbb{R}^d$, $t \in [0, +\infty)$ is called a (strong) solution to equation (2) with the coefficients a, b and initial measure μ_0 if the following conditions take place

- for all $t \geq 0$ the restriction of x to the interval $[0;t]$ is $\mathcal{B}_d \bigotimes \mathcal{B}_{[0,t]} \bigotimes \mathcal{F}_t$-measurable, where $\mathcal{F}_t = \sigma(W(s,\Delta))$, $\Delta \in \mathcal{B}_d$, $s \leq t$;
- for fixed $u \in \mathbb{R}^d$ for all $t \geq 0$ the integral form of (2) takes place with probability 1;
- $x(u,0) = u$ with probability 1 and all $u \in \mathbb{R}^d$.

In section 2 of [7] the conditions of existence and uniqueness of solutions to equation (2) were obtained, the properties of solutions were established.

The limit behavior of solutions to SDE with interaction in one-dimensional case was studied in [13]. The aim of the paper is to investigate the two-dimensional case.

Thus, the main object of the investigation in the paper is two-dimentional stochastic differential equation with interaction

$$\begin{cases} dx(u,t) = \int\limits_{\mathbb{R}^2} \varphi(x(u,t)-v)\mu_t(dv)dt + b(x(u,t),\mu_t)dw(t) \\ x(u,0) = u, \\ \mu_t = \mu_0 \circ x(\cdot,t)^{-1}. \end{cases} \tag{3}$$

The existence a. s. of a nonzero limit as t tends to infinity of the function, that characterize the distance between $x(u,t)$ and $x(v,t)$ for all $u,v \in \mathbb{R}^2$, is proved. Also the analogue of the triangle inequality for the limit map is obtained.

2. Main results

Let $\mathfrak{M}_2$ be a subspace of the space of all probability measures on $\mathbb{R}^2$ such that

$$\forall \mu \in \mathfrak{M}_2 \ \forall u \in \mathbb{R}^2 : \quad \int\limits_{\mathbb{R}^2} ||u-v||^2 \mu(dv) < +\infty.$$

For $\mu, \nu \in \mathfrak{M}_2$ the Wasserstein distance $\gamma_2(\mu,\nu)$ [6] is defined by the formula

$$\gamma_2(\mu,\nu) = \sqrt{\inf_{Q \in C(\mu,\nu)} \int\int\limits_{\mathbb{R}^2} ||u-v||^2 Q(du,dv)},$$

where $C(\mu,\nu)$ is the set of all probability measures on $\mathbb{R}^2$ which have μ and ν as their marginal projections.

Consider SDE (3) with $\mu_0 \in \mathfrak{M}_2$,

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where w_k are one-dimensional independent Wiener processes,

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \varphi_k : \mathbb{R}^2 \to \mathbb{R}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

$b_k : \mathbb{R}^2 \times \mathfrak{M}_2 \to \mathbb{R}$ $(k = 1,2)$ being globally Lipschitz and for some $\alpha_1, \alpha_2, B_1, B_2 > 0$ and for all $u, v \in \mathbb{R}^2$, $t > 0$, $\mu_t \in \mathfrak{M}_2$

$$-\alpha_1 \|u-v\|^2 \leq (u-v, \varphi(u)-\varphi(v)) \leq -\alpha_2 \|u-v\|^2; \tag{4}$$

$$B_1\|u-v\|^2 \leq (u-v, b(u,\mu_t)-b(v,\mu_t)); \quad \|b(u,\mu_t)-b(v,\mu_t)\| \leq B_2\|u-v\|; \tag{5}$$

where

$$\alpha_k - B_k^2 \geq 0 \ \ (k = 1,2). \tag{6}$$

Then by Theorems 2.1.1 and 2.1.2 from [7] there exists a unique strong solution of (3) such that x is a flow of homeomorphisms. Moreover, we have from (3) the following representation

$$
\begin{aligned}
||x(u,t)-x(v,t)|| = ||u-v|| + \int\limits_0^t \Phi\left(x(v,s),x(u,s),\mu_s\right)ds + \\
+\sum_{k=1}^{2}\int\limits_0^t P_k\left(x(v,s),x(u,s),\mu_s\right)dw_k(s),
\end{aligned} \tag{7}
$$

where

$$
\Phi\left(r,q,\mu_s\right) = \frac{1}{||q-r||}\int\limits_{\mathbb{R}^2}\left(q-r,\varphi(q-v)-\varphi(r-v)\right)\mu_s(dv) +
$$

$$
+\frac{||b(q,\mu_s)-b(r,\mu_s)||^2}{2||q-r||} - \frac{\left(q-r,b(q,\mu_s)-b(r,\mu_s)\right)^2}{2||q-r||^3},
$$

$$
P_k(r,q,\mu_s) = \frac{(q_k-r_k)(b_k(q,\mu_s)-b_k(r,\mu_s))}{||q-r||},\quad q=\begin{pmatrix} q_1\\ q_2\end{pmatrix},\quad r=\begin{pmatrix} r_1\\ r_2\end{pmatrix},\ k=1,2.
$$

Due to Lemma 1 from [13] we obtain, that the martingale part in equalities of the type (7) may have the limit.

The following result also characterizes the distance between trajectories of different particles on infinity.

Theorem 2.1. *For all $u,v\in\mathbb{R}^2$ there exists*

$$
\lim_{t\to\infty}\left(||x(u,t)-x(v,t)|| - \int\limits_0^t \Phi\left(x(v,s),x(u,s),\mu_s\right)ds\right)\quad a.e. \tag{8}
$$

Proof. At the beginning let us consider the sum

$$
\sum_{k=1}^{2}\int\limits_0^t P_k\left(x(v,s),x(u,s),\mu_s\right)dw_k(s).
$$

If we put

$$
I_k(t) = \int\limits_0^t P_k\left(x(v,s),x(u,s),\mu_s\right)dw_k(s),\quad k\in\{1,2\},
$$

then we have from the mutual independence of w_1 and w_2 that I_1, I_2 are continuous martingales with quadric variation

$$
\langle I_1+I_2\rangle_t = \sum_{k=1}^{2}\int\limits_0^t P_k^2\left(x(v,s),x(u,s),\mu_s\right)ds.
$$

Hence by Theorem 18.4 from [11]

$$
I_1(t)+I_2(t) = w_{u,v}\left(\int\limits_0^t\left(P_1^2\left(x(v,s),x(u,s),\mu_s\right)+P_2^2\left(x(v,s),x(u,s),\mu_s\right)\right)ds\right), \tag{9}
$$

where $w_{u,v}$ is a Wiener process. By conditions (4) and (5) we have for all $t\geq 0$ the following inequality

$$
\int\limits_0^t \Phi\left(x(v,s),x(u,s),\mu_s\right)ds \leq \left(-\alpha_2+\frac{1}{2}\left(B_2^2-B_1^2\right)\right)\int\limits_0^t ||x(u,s)-x(v,s)||ds; \tag{10}
$$

As constants α_2, B_2 satisfy the condition (6), we have, that

$$
-\alpha_2+\frac{1}{2}\left(B_2^2-B_1^2\right) < 0, \tag{11}
$$

and then by (10) the following inequality takes place for all $t \geq 0$

$$||x(u,t) - x(v,t)|| - \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds \geq 0. \tag{12}$$

The functions $x(u,\cdot)$ and $x(v,\cdot)$ are continuous and for $u \neq v$ we have, that $||x(u,t) - x(v,t)|| > 0$ for small t. So, for $t > 0$, $u, v \in \mathbb{R}^2$, $u \neq v$, $\int_0^t ||x(u,s) - x(v,s)||ds > 0$ and taking into account (10) and (12) we have, that for all $t \geq 0$

$$||x(u,t) - x(v,t)|| - \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds > 0. \tag{13}$$

Then, because of (9), we have that the equality

$$||x(u,t) - x(v,t)|| - \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds = ||u - v|| + I_1(t) + I_2(t) =$$

$$= ||u - v|| + w_{u,v}\left(\int_0^t \left(P_1^2\left(x(v,s), x(u,s), \mu_s\right) + P_2^2\left(x(0,s), x(u,s), \mu_s\right)\right) ds\right),$$

and the following inequality takes place

$$\int_0^{+\infty} \left(P_1^2\left(x(v,s), x(u,s), \mu_s\right) + P_2^2\left(x(v,s), x(u,s), \mu_s\right)\right) ds \leq \tau_{u,v} < +\infty, \tag{14}$$

where $\tau_{u,v}$ is the time of the first hitting of $||u - v||$ by $w_{u,v}$. So, the following equality is valid with probability 1

$$\begin{aligned} &\lim_{t\to\infty}\left(||x(u,t) - x(v,t)|| - \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds\right) = ||u - v|| + \\ &+ w_{u,v}\left(\int_0^{+\infty} \left(P_1^2\left(x(v,s), x(u,s), \mu_s\right) + P_2^2\left(x(v,s), x(u,s), \mu_s\right)\right) ds\right). \end{aligned} \tag{15}$$

The Theorem is proved. □

Lemma 2.1. *For all $u, v \in \mathbb{R}^2$, $u \neq v$*

$$\lim_{t\to\infty}\left(||x(u,t) - x(v,t)|| - \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds\right) > 0 \quad a.e. \tag{16}$$

Proof. According to the proof of Theorem 2.1, we have, that for all $u, v \in \mathbb{R}^2$, $u \neq v$

$$\lim_{t\to\infty}\left(||x(u,t) - x(v,t)|| - \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds\right) \geq 0 \quad a.e. \tag{17}$$

Now let us suppose that the opposite of (16) is true, that is for some $u, v \in \mathbb{R}^2$, $u \neq v$ there exists some set $A \in \mathfrak{F}$, $P(A) > 0$ such that

$$\forall \omega \in A: \lim_{t\to\infty}\left(||x(u,t) - x(v,t)|| - \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds\right) = 0. \tag{18}$$

By (10), (11) and (6) both terms in the above limit are nonnegative, therefore it follows from (18), that

$$(19) \qquad \forall\omega \in A: \lim_{t\to\infty} ||x(u,t) - x(v,t)|| = 0 \text{ and } \lim_{t\to\infty} \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds = 0.$$

But then, from the second equality of (19), (10) and (11), we have, that

$$\forall\omega \in A: \lim_{t\to\infty} \int_0^t ||x(u,s) - x(v,s)|| ds = 0.$$

Here we have a contradiction, because $u \neq v$, $x(u,0) = u$ and $x(v,0) = v$. The Lemma is proved. □

Let us now consider the function

$$(20) \qquad F(u,v) = \lim_{t\to\infty} \left(||x(u,t) - x(v,t)|| - \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds \right).$$

It follows from Lemma 2.1, that for all $u, v \in \mathbb{R}^2$, $u \neq v$

$$F(u,v) > 0.$$

The next result gives us the analogue of triangle inequality.

Lemma 2.2. *For all* $u_1, u_2, u_3 \in \mathbb{R}^2$

$$(21) \qquad F(u_1,u_2) + F(u_2,u_3) \geq \frac{2\alpha_2 - B_2^2 + B_1^2}{2\alpha_1 - B_1^2 + B_2^2} F(u_1,u_3) \quad a.e.$$

Proof. Using the Cauchy-Schwarz inequality and (5) we obtain the inequalities

$$B_2\|u - v\|^2 \geq |(u - v, b(u,\mu_t) - b(v,\mu_t))|; \quad \|b(u,\mu_t) - b(v,\mu_t)\| \geq B_1\|u - v\|.$$

It follows from this inequalities and (5) that for all $t \geq 0$

$$(22) \qquad \int_0^t \Phi\left(x(v,s), x(u,s), \mu_s\right) ds \geq \left(-\alpha_1 + \frac{1}{2}\left(B_1^2 - B_2^2\right)\right) \int_0^t ||x(u,s) - x(v,s)|| ds.$$

Then by (10) we have, that

$$0 < \alpha_2 - \frac{1}{2}\left(B_2^2 - B_1^2\right) \leq \alpha_1 - \frac{1}{2}\left(B_1^2 - B_2^2\right).$$

Taking into account (20), (10) and (22) we obtain, that for all $u_1, u_2, u_3 \in \mathbb{R}^2$ and all $t > 0$

$$F(u_1,u_2) + F(u_2,u_3) \geq ||x(u_1,t) - x(u_2,t)|| +$$

$$+ \left(\alpha_2 - \frac{1}{2}\left(B_2^2 - B_1^2\right)\right) \int_0^t ||x(u_1,s) - x(u_2,s)|| ds +$$

$$+ ||x(u_2,t) - x(u_3,t)|| + \left(\alpha_2 - \frac{1}{2}\left(B_2^2 - B_1^2\right)\right) \int_0^t ||x(u_2,s) - x(u_3,s)|| ds.$$

Then we use the triangle inequality for the norm and (22)

$$F(u_1,u_2) + F(u_2,u_3) \geq ||x(u_1,t) - x(u_3,t)|| +$$

$$+\left(\alpha_2 - \frac{1}{2}\left(B_2^2 - B_1^2\right)\right)\int_0^t ||x(u_1,s) - x(u_3,s)||ds \geq$$

$$\geq ||x(u_1,t) - x(u_3,t)|| - \frac{2\alpha_2 - B_2^2 + B_1^2}{2\alpha_1 - B_1^2 + B_2^2}\int_0^t \Phi\left(x(u_1,s), x(u_3,s), \mu_s\right) ds.$$

The Lemma is proved. □

The next example shows the situation for the linear case.

Example 2.1. Let

$$\varphi(u) = \begin{pmatrix} q & q_{12} \\ -q_{12} & q \end{pmatrix}\int_{R^2}(u-v)\mu_t(dv), \quad b(u,\mu_t) = \begin{pmatrix} p & -p \\ p & p \end{pmatrix}\int_{R^2}(u-v)\mu_t(dv),$$

$$p > 0, \quad q + 2p^2 \leq 0.$$

Then, $(u-v, \varphi(u) - \varphi(v)) = q||u-v||^2$, $(u-v, b(u,\mu_t) - b(v,\mu_t)) = p||u-v||$, $\|b(u,\mu_t) - b(v,\mu_t)\| = p\sqrt{2}\|u-v\|$ and therefore $\alpha_1 = \alpha_2 = -q$, $B_1 = p$, $B_2 = \sqrt{2}p$. So, by Lemma 2.2, in this case the inequality (16) becomes

$$F(u_1,u_2) + F(u_2,u_3) \geq \frac{-2q - p^2}{-2q + p^2}F(u_1,u_3).$$

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