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WEAK UNIQUENESS OF MARTINGALE SOLUTIONS TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES

We prove the uniqueness of martingale solutions for stochastic partial differential equations generalizing the work in Mandrekar and Skorokhod (1998). The main idea used is to reduce this problem to the case in Mandrekar and Skorokhod using the techniques introduced in Filipović et al. (2010).

1. Introduction

The main purpose of the paper is to prove uniqueness of martingale solutions for stochastic partial differential equations (SPDE). We observe that for the finite dimensional case, the result for stochastic differential equations was first proved by Stroock and Varadhan (1969). Gikhman and Skorokhod (1979), in their book, give a proof based on a method from Harmonic Analysis using the Fourier transform of the infinitesimal generator. In Mandrekar and Skorokhod (1998), a generalization of the Stroock and Varadhan (1969) result to the infinite dimensional Hilbert space is given using the technique in Gikhman and Skorokhod (1979). However, this does not give the uniqueness of the martingale solution of a SPDE.

We use the idea in Filipović et al. (2010), which is to transform the SPDE to a SDE in a larger Hilbert space associated with the problem using Nagy's Theorem ((2010), Chapter I, Thm. 8.1). See also Tappe (2013). However the coefficients of the SDE are time-dependent and we need to generalize the Mandrekar and Skorokhod (1998) result to include this case. In Section 3, we show that this can be done using the ideas in Gikhman and Skorokhod (1979).

In Section 2, we state some preliminaries and definitions. In Section 4, we consider the martingale solution of the SPDE. Using Filipović et al. (2010) and the result in Section 3, we get as in Mandrekar and Skorokhod (1998), the uniqueness of the solution to the martingale problem for the transformed SDE in the larger Hilbert space. Finally, we combine this result with ideas from Filipović et al. (2010) to obtain the weak uniqueness of the Martingale solution to the SPDE.

2. Preliminaries and Definitions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$ be a filtered probability space with the filtration satisfying the usual conditions. Let K be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_K$ and norm $\| \cdot \|_K$. Let $L(K)$ denote the space of bounded linear operators from K to K and $L_1^+(K)$, the subspace of non-negative operators with finite trace. Let $Q \in L_1^+(K)$

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be a symmetric, self-adjoint operator and $\{f_j, 1, 2, \cdots\}$ be a complete orthonormal basis (ONB) in K diagonalizing Q, and let the corresponding eigenvalues be $\{\lambda_j, j = 1, 2, \cdots\}$. We assume that $\lambda_j > 0$ for all j. Let $\{w_j(t), t \geq 0\}$, $j = 1, 2, \cdots$, be a sequence of independent standard real-valued Wiener processes defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$. The process

$$W(t) = \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(t) f_j$$

is called a Q-Wiener process in K (Gawarecki and Mandrekar (2011, p. 20, 21)). The processes $\{w_j(t)\}$ are assumed to be continuous and hence the process $\{W(t)\}$ is continuous. Further, $E[W(t)] = 0$, and $E[\langle W(t), u\rangle_K \langle W(s), v\rangle_K] = \min\{t, s\}\langle Qu, v\rangle_K$ for every $u,\ v \in K,\ t, s \geq 0$.

Next we consider Itô's integral with respect to a Q-Wiener process following Gawarecki and Mandrekar (2011, Section 2.2). Let $K_Q = Q^{1/2}K$. Then K_Q equipped with the scalar product

$$\langle u, v\rangle_{K_Q} = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle u, f_j\rangle_K \langle v, f_j\rangle_K$$

is a separable Hilbert space with an ONB $\{\lambda_j^{1/2} f_j, j = 1, 2, \cdots\}$. Let H be another real separable Hilbert space and $L_2(K_Q, H)$ the space of Hilbert-Schmidt operators from K_Q to H, which is separable with respect to the Hilbert-Schmidt norm. Let $\Lambda_2(K_Q, H)$ be a class of $L_2(K_Q, H)$-valued processes $\{\Phi\}$, which are measurable as mappings from $([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F})$ to $(L_2(K_Q, H), \mathcal{B}(L_2(K_Q, H)))$, are adapted to the filtration $\{\mathcal{F}_t, t \leq T\}$ and satisfy the condition

$$E \int_0^T \|\Phi(t)\|^2_{L_2(K_Q, H)} dt < \infty,$$

where $\mathcal{B}$ denotes the Borel σ-field.

Definition 2.1. The stochastic integral of a process $\Phi \in \Lambda_2(K_Q, H)$ with respect to a K-valued Q-Wiener process $\{W(t)\}$ is the unique isometric linear extension of the mapping

$$\Phi(\cdot) \to \int_0^T \Phi(s) dW(s)$$

from the class of bounded elementary processes to $L^2(\Omega, H)$, to a mapping from $\Lambda_2(K_Q, H)$ to $L^2(\Omega, H)$, such that the image of $\Phi(t) = \phi I_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j I_{(t_j, t_{j+1}]}(t)$ is $\sum_{j=0}^{n-1} \phi_j (W(t_{j+1}) - W(t_j))$. The stochastic integral process $\int_0^t \Phi(s)(s) dW(s),\ 0 \leq t \leq T$ is defined by

$$\int_0^t \Phi(s)(s) dW(s) = \int_0^T \Phi(s) I_{[0,t]}(s) dW(s).$$

We give below the definition of a martingale solution for a SPDE and a SDE (Gawarecki and Mandrekar (2011, p. 75)).

Let H be a separable Hilbert space and $A : \mathcal{D}(A) \subset H \to H$ be the generator of a strongly continuous semigroup $(S_t, t \geq 0)$ on H. General forms of a SPDE and a SDE, respectively are

$$dX(t) = (AX(t) + \alpha(t, X(t)))dt + \sigma(t, X(t))dW(t),\ X(0) = x \in H, \tag{2.1}$$

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW(t),\ X(0) = x \in H, \tag{2.2}$$

where $\alpha : \Omega \times [0, T] \times C([0, T], H) \to H$, $\sigma : \Omega \times [0, T] \times C([0, T], H) \to L_2(K_Q, H)$ are specified continuous functions, and $\{W(t), t \geq 0\}$ is a Q- Wiener process in K.

Definition 2.2. A process $\{X(t)\}$ is said to be a martingale solution of the SPDE in (2.1) if there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$ and, on this probability space, a Q-Wiener process $\{W(t)\}$, relative to the filtration $\{\mathcal{F}_t\}$, such that

$$P\left(\int_0^\infty \|X(t)\|_H dt < \infty\right) = 1, \tag{2.3}$$

$$\int_0^t \{\|\alpha(u, X(u)))\|_H + \|\sigma(u, X(u))\|^2_{L_2(K_Q,H)}\}du < \infty \text{ for all, } t, P - a.s, \tag{2.4}$$

and

$$X(t) = S_{t-t_0}x + \int_{t_0}^t S_{t-t_0}\alpha(u, X(u)))du + \int_{t_0}^t S_{t-t_0}\sigma(u, X(u))dW(u), \text{ for all, } t, P - a.s.$$

Definition 2.3. A process $\{X(t)\}$ is said to be a martingale solution of the equation in (2.2) if (2.3) and (2.4) hold and if there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$ and, on this probability space, a Q-Wiener process $\{W(t)\}$ relative to the filtration $\{\mathcal{F}_t\}$, such that

$$X(t) = x + \int_{t_0}^t \alpha(u, X(u))du + \int_{t_0}^t \sigma(u, X(u))dW(u), \text{ for all, } t, P - a.s.$$

The martingale solution of a SDE is also called a weak solution.

Definition 2.4. Weak uniqueness of solutions of a SPDE (SDE) means that if for solutions $X_1(t)$ and $X_2(t)$ of the SPDE (SDE) on $[s, \infty)$ with initial values $X_1(s)$ and $X_2(s)$, the distributions of $X_1(s)$ and $X_2(s)$ coincide, then the distributions of $X_1(t)$ and $X_2(t)$ coincide for all $t \geq s$.

3. Weak uniqueness of martingale solutions of SDE

In this section we extend the results of Mandrekar and Skorokhod (1998) to SDEs with time dependent coefficients.

Let K and H be real separable Hilbert spaces and $\{W(t)\}$ be a $K-$valued Q-Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$ with the filtration satisfying the usual conditions. Let $< \cdot, \cdot >$ and $\| \cdot \|$ denote the inner product and the norm, respectively. Let $K_Q = Q^{1/2}(K)$ and $L_2(K_Q, H)$ denote the space of Hilbert-Schmidt operators from K_Q to H,

Consider the stochastic differential equation (SDE)

$$\begin{aligned} dX(t) &= a(t, X(t))dt + B(t, X(t))dW(t), \ t \geq s \\ X(s) &= x, \end{aligned} \tag{3.1}$$

where $a : \Omega \times [0, T] \times C([0, T], H) \to H$, $B : \Omega \times [0, T] \times C([0, T], H) \to L_2(K_Q, H)$ are specified continuous functions.

Let B_0 be a bounded linear operator from K to H, such that $(B_0QB_0^*)^{-1}$ exists, where A^* denotes the adjoint of the operator A. Set

$$B_1(t, x) = B(t, x)QB^*(t, x) - B_0QB_0^*$$

and $a_1(t, x) = B_0B_0^*a(t, x)$. We note that $B_1^*(t, x) = B_1(t, x)$, for all t and x.

Let μ_0 denote the measure on H induced by $B_0(W(1))$ and let F_λ denote the function which induces a measure on $[0, \infty)$ with Radon-Nikodym derivative $dF_\lambda/dm(t) = exp(-\lambda t)$, $\lambda > 0$, with respect to the Lebesgue measure m. We note that μ_0 is a Gaussian measure on H with mean zero and covariance $Q_1 = B_0QB_0^*$. Let $\mathcal{L}_2(F_\lambda \times \mu_0)$ denote the space of functions $g : [0, \infty) \times H \to R$ $(= (-\infty, \infty))$ for which $\int_0^\infty \int_H |g(t, x)|^2 d(F_\lambda \times$

$\mu_0)(t,x) < \infty$. Consider a basis in $\mathcal{L}_2(F_\lambda \times \mu_0)$ which consists of polynomial functions $\{\phi_k(t,x), k = 1, 2, \cdots\}$. Denote by $\hat{\mathcal{L}}_2$ the space of functions of the form

$$f(t,x) = e^{-\frac{1}{4}\|Q_1^{-1/2}x\|_H^2 - \frac{1}{2}\lambda t} \sum_k c_k \phi_k(t,x).$$

The norm of $f(t,x) \in \hat{\mathcal{L}}_2$ is defined by

$$\|f\|^2 = \int_0^\infty \int_H e^{\frac{1}{2}\|Q_1^{-1/2}x\|_H^2 + \lambda t} |f(t,x)|^2 d(F_\lambda \times \mu_0)(t,x).$$

We state below the conditions required for existence and uniqueness of a martingale solution.

Assumptions A: The coefficients satisfy the following:

(A.1) The functions $a(t,x)$ and $B(t,x)$ are jointly adapted and continuous functions.

(A.2) For every $t \le T$ and $x \in C([0,T], H)$, there exits a K_T such that

$$\|a(t,x)\|_H + \|B(t,x)\|_{L_2(K_Q,H)} \le K_T(1 + \sup_{0\le t\le T} \|x(t)\|_H).$$

(A.3) There exists a Hilbert space H_0 such that the embedding $J : H_0 \to H$ is a compact operator with representation

$$J(x) = \sum_{i=1}^\infty \lambda_n < x, e_n >_{H_0} h_n,\ \lambda_n > 0, n = 1, 2, \cdots,$$

where $\{e_n, n = 1, 2, \cdots\}$ and $\{h_n, n = 1, 2, \cdots\}$ are orthonormal basis of H_0 and H respectively. We identify H_0 with $J(H_0)$. In particular $e_n = \lambda_n h_n$ and $\lambda_n < x, e_n >_{H_0} = < x, h_n >_H$.

(A.4) The functions $a(t,x)$ and $B(t,x)$ restricted to H_0 satisfy

$$a : [0,T] \times C([0,T], H_0) \to H_0,\ B : [0,T] \times C([0,T], H_0) \to L_2(K_Q, H_0).$$

and for every $t \le T$ and $x \in C([0,T], H_0)$, there exits a K'_T such that

$$\|a(t,x)\|_{H_0} + \|B(t,x)\|_{L_2(K_Q,H_0)} \le K'_T(1 + \sup_{0\le t\le T} \|x(t)\|_{H_0}).$$

(A.5) The functions $a_1(s,x)$ and $B_1(s,x)$ satisfy

$$\sup_{s,x}[\|a_1(s,x)\|_H^2 + Trace(B_1(s,x)^2) = q_1^2 < \infty.$$

(**A′.5**) The functions $a_1(s,x)$ and $B_1(s,x)$ satisfy

$$\sup_{s,x}[\|a_1(s,x)\|_H^2 + (Trace B_1(s,x))^2 = q_2^2 < \infty.$$

Remark 3.1. The assumption A.5 can be replaced by the following: For all $s_0 \in [0, \infty)$

$$\sup_{s\in\{s:|s-s_0|<\rho\},x} [\|a_1(s,x)\|_H^2 + Trace(B_1(s,x)^2) = q_3^2 < \infty,$$

for some $\rho > 0$. (For a proof of the above statement, we refer to the proof of Theorem 6 in Gikhman and Skorokhod ((1979), Chapter 3, Section 3, page 298.)

Remark 3.2. If assumptions $A.1$, $A.2$, $A.3$ and $A.4$ hold, then a Martingale (weak) solution of equation (3.1) exists. For a proof we refer to Gawarecki and Mandrekar (2011) (Theorem 3.12, pp.131).

Below, following the techniques in Mandrekar and Skorokhod (1998), we show that a martingale solution of (3.1) is weakly unique.

Define

$$R_\lambda(f(s,x)) = E\int_s^\infty e^{-\lambda(t-s)} f(t, x + B_0(W(t) - W(s)))dt.$$

Then

$$R_\lambda(f(s,x)) = \int f(v+s, x+\sqrt{v}u)d(F_\lambda \times \mu_0)(v,u),$$

where the integration is over $[0,\infty) \times H$.

Lemma 3.1. *Let $q(s,x)$ be a polynomial function on $[0,\infty) \times H$ and $V_\lambda(s,x) = R_\lambda e^{-\frac{1}{4}\|Q_1^{-1/2}x\|_H^2 - \frac{\lambda s}{2}} q(s,x)$. Then $V_\lambda(s,x) \in \hat{\mathcal{L}}_2$.*

Proof.

$$V_\lambda(s,x) = e^{-\frac{1}{4}\|Q_1^{-1/2}x\|_H^2 - \frac{\lambda s}{2}} \int \Big\{ e^{-\frac{1}{2}\langle Q_1^{-1/2}x, \sqrt{v}Q_1^{-1/2}u\rangle - \frac{1}{4}\|\sqrt{v}Q_1^{-1/2}u)\|_H^2 - \frac{\lambda v}{2}} \times q(v+s, x+\sqrt{v}u)\Big\} d(F_\lambda \times \mu_0)(v,u). \tag{3.2}$$

Therefore

$$\int (e^{\frac{1}{4}\|Q_1^{-1/2}x\|_H^2 + \frac{\lambda s}{2}} V_\lambda(s,x))^2 d(F_\lambda \times \mu_0)(s,x) = \int\int\int \Big\{ e^{-\frac{1}{2}\langle Q_1^{-1/2}x, Q_1^{-1/2}(\sqrt{v}u + \sqrt{v'}u'\rangle} \times e^{-\frac{1}{4}\|\sqrt{v}Q_1^{-1/2}u)\|_H^2 - \frac{1}{4}\|\sqrt{v'}Q_1^{-1/2}u'\|^2 - \frac{\lambda(v+v')}{2}} q(v+s, x+\sqrt{v}u) \times q(v'+s, x+\sqrt{v'}u')\Big\} d(F_\lambda \times \mu_0)(v,u) d(F_\lambda \times \mu_0)(v',u') d(F_\lambda \times \mu_0)(s,x).$$

If $h(s)$ is a polynomial on $[0,\infty)$ then

$$\int_0^\infty h(s^{1/2}) dF_\lambda(s) \le C,$$

for some constant C. Further if $g(x)$ is a polynomial of degree p on H, since μ_0 is a zero mean Gaussian measure on H, we have

$$|\int e^{\langle Q_1^{-1/2}x, z\rangle} g(x) d\mu_0(x)| \le C_1(1 + \|z\|_H)^p e^{\frac{1}{2}\langle z,z\rangle},$$

where C_1 is some constant. Using the above two inequalities and that $q(s,x)$ is a polynomial of degree p, we obtain

$$\int (e^{\frac{1}{4}\|Q_1^{-1/2}x\|_H^2 + \frac{\lambda s}{2}} V_\lambda(s,x))^2 d(F_\lambda \times \mu_0)(s,x) \le C_2 \int\int \Big\{ e^{-\frac{1}{8}\|Q_1^{-1/2}(\sqrt{v}u - \sqrt{v'}u')\|_H^2} (1 + \|\sqrt{v}u)\|_H + \|\sqrt{v'}u'\|_H)^{2m}\Big\} d(F_\lambda \times \mu_0)(v,u) d(F_\lambda \times \mu_0)(v',u'), \tag{3.3}$$

where C_2 is some constant. Now, using the properties of the measures dF_λ and μ_0, it can be shown that the right hand side of the inequality in (3.3) is finite. □

Consider a function $u(t,x)$ defined on $[0,\infty) \times H$ with continuous bounded partial derivatives u'_t (a R-valued function), u'_x (a H-valued function), u'_{xx} (a $L(H)$-valued function). Let

$$L_1 u(t,x) = \langle a_1(t,x), u'_x(t,x)\rangle + \frac{1}{2} Trace B_1(t,x) u''_{xx}(t,x).$$

Lemma 3.2. *For $\lambda > 0$, $L_1 V_\lambda(s,x) \in \hat{\mathcal{L}}_2$.*

Proof. (We give the proof assuming $a(t,x)=0$ for the sake of convenience of the notation. Proof for $a(t,x)\neq 0$ follows by similar arguments.)

$L_1V_\lambda(s,x)=\frac{1}{2}TraceB_1(s,x)(V_\lambda)''_{xx}(s,x)$. From (3.2)

$$\begin{aligned}
2L_1V_\lambda(s,x) &= e^{-\frac{1}{4}\|Q_1^{-1/2}x\|_H^2-\frac{\lambda s}{2}}\int e^{-\frac{1}{2}\langle Q_1^{-1/2}x,Q_1^{-1/2}\sqrt{v}u\rangle-\frac{1}{4}\|Q_1^{-1/2}\sqrt{v}u)\|_H^2}\\
&\times\ [q(v+s,x+\sqrt{v}u)\left(-\frac{1}{2}Trace(B_1(s,x)Q_1^{-1})\right.\\
&+\ \left.\frac{1}{4}\langle Q_1^{-1}(x+\sqrt{v}u)(\cdot)B_1(s,x),Q_1^{-1}(x+\sqrt{v}u)(\cdot)\rangle\right)\\
&-\ \langle Q_1^{-1}(x+\sqrt{v}u)(\cdot),q_x(v+s,x+\sqrt{v}u)B_1(s,x)\rangle\\
&+\ TraceB_1(s,x)q_{xx}(v+s,x+\sqrt{v}u)]d(F_\lambda\times\mu)(v,u),
\end{aligned}$$

where $Q_1^{-1}(x+\sqrt{v}u)(\cdot)$ denotes a linear map from H to R defined by $Q_1^{-1}(x+\sqrt{v}u)(h)=\langle Q_1^{-1}(x+\sqrt{v}u),h\rangle$.

Using the boundedness of $Trace(B_1(s,x))^2$, of Q_1^{-1} and of the derivatives q_x, and q_{xx} in the ball $\{x|\|x\|\leq 1\}$ and arguing as in the proof of Lemma3.1, we obtain

$$\int(e^{\frac{1}{4}\|Q_1^{-1/2}x\|_H^2+\frac{\lambda s}{2}}L_1V_\lambda(s,x))^2d(F_\lambda\times\mu_0)(s,x)<\infty.$$

□

The following Lemma is from Gikhman and Skorokhod ((1979), Lemma 1, Chapter 3, Section 3, page 281) for finite dimensional Euclidean spaces.

Lemma 3.3. *Let $\{\tilde{w}(t)\}$ be a Wiener process on R^d and $g(s,u)$ a square integrable function on $[0,\infty)\times R^d$ w.r.t the Lebesgue measure. Let*

$$\tilde{R}_\lambda f(s,u)=E\int_s^\infty e^{-\lambda(t-s)}f(t,u+\tilde{w}(t)-\tilde{w}(s))dt$$

and

$$Lg(s,u)=\langle(s,u),g'_u\rangle+TraceC(s,u)g'''_{uu}$$

be a differential operator with coefficients $b(s,u)$ and $C(s,u)$, defined on $[0,\infty)\times R^d$, taking values in R^d and $L(R^d)$ (space of linear operators on R^d), respectively, being measurable, and satisfying the inequalities

$$\|b(s,u)\|_{R^d}\leq\delta \text{ and } \mathrm{Trace(C(s,u)C^*(s,u)}\leq\theta^2.$$

Then for each $\epsilon>0$ and all sufficiently large $\lambda>0$,

$$\int\int(L\tilde{R}_\lambda g(s,u))^2dsdu\leq(\theta^2+\epsilon)\int\int g^2(s,u)dsdu.$$

We note that the choice of λ in the above Lemma does not depend on the dimension d. Using the above three Lemmas, we can prove the following Theorem.

Theorem 3.1. *Suppose Assumptions A.1, A.2, and A.5, given above, hold. Then for sufficiently large $\lambda>0$, the operator L_1R_λ is defined on the space $\hat{\mathcal{L}}_2$ and*

$$\|L_1R_\lambda\|\leq q_0.$$

Proof. For the sake of convenience of notation we assume $a(s,x)=0$. Let Π_n denote a projection on a (finite) n-dimensional subspace of H.

Consider a function $f(s,x)$ such that $f(s,\Pi_n x)$ is square integrable with respect to the Lebesgue measure $m\times m_n$ on $[0,\infty)\times \Pi_n H$. Then Lemma 3.3 implies, for all sufficiently large $\lambda>0$,

$$\int\int (L_1R_\lambda f(s,\Pi_n x))^2 dm(s)dm_n(\Pi_n x)\le q_0^2\int\int f^2(s,\Pi_n x)dm(s)dm_n(\Pi_n x),$$

which from the definition of F_λ and μ_0 implies

$$\begin{aligned}(3.4)\qquad &\int (e^{\frac{1}{4}\|Q_1^{-1/2}\Pi_n x\|_H^2+\frac{\lambda s}{2}}L_1R_\lambda f(s,\Pi_n x))^2 d(F_\lambda\times\mu_0)(s,x)\\ &\le q_0^2\int (e^{\frac{1}{4}\|Q_1^{-1/2}\Pi_n x\|_H^2+\frac{\lambda s}{2}} f(s,\Pi_n x))^2 d(F_\lambda\times\mu_0)(s,x).\end{aligned}$$

Let the sequence Π_n converge to I, the identity operator and let

$$f(s,x)=e^{-\frac{1}{4}\|Q_1^{-1/2}x\|^2-\frac{\lambda s}{2}}q(s,x),$$

where $q(s,x)$ is a polynomial on $[0,\infty)\times H$. Using an argument similar to proofs of Lemma 3.1 and Lemma 3.2 to show boundedness, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\int (e^{\frac{1}{4}\|Q_1^{-1/2}\Pi_n x\|_H^2+\frac{\lambda s}{2}}L_1R_\lambda f(s,\Pi_n x))^2 d(F_\lambda\times\mu_0)(s,x)\\ (3.5)\qquad &=\int (e^{\frac{1}{4}\|Q_1^{-1/2}x\|_H^2+\frac{\lambda s}{2}}L_1R_\lambda f(s,x))^2 d(F_\lambda\times\mu_0)(s,x),\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\int (e^{\frac{1}{4}\|Q_1^{-1/2}\Pi_n x\|_H^2+\frac{\lambda s}{2}} f(s,\Pi_n x))^2 d(F_\lambda\times\mu_0)(s,x)\\ (3.6)\qquad &=\int (e^{\frac{1}{4}\|Q_1^{-1/2}x\|_H^2+\frac{\lambda s}{2}} f(s,x))^2 d(F_\lambda\times\mu_0)(s,x).\end{aligned}$$

Thus the inequality (3.4) holds for all functions of the form

$$f(s,x)=e^{-\frac{1}{4}\|x\|_H^2-\lambda s/2}q(s,x),$$

with $q(s,x)$ a polynomial. This class of functions is dense in $\hat{\mathcal{L}}_2$. Thus we conclude that

$$\|L_1R_\lambda\|\le q_0.$$

□

Corollary 3.1. *If $\|L_1R_\lambda\|<1$, then for $f\in\hat{\mathcal{L}}_2$, the solution of $g_\lambda(s,x)-L_1R_\lambda g_\lambda(s,x)=f(s,x)$ is given by*

$$g_\lambda(s,x)=(I-L_1R_\lambda)^{-1}f(s,x).$$

Suppose $X(t)$ is a solution of (3.1) on $[s,\infty)$. Let

$$(3.7)\qquad R_\lambda^X f(s,x)=E_{s,x}\int_s^\infty e^{-\lambda(t-s)}f(t,X(t))dt.$$

Let $f(t,x)$ be a function with continuous bounded derivatives f'_t, f'_x and f''_{xx}, and let

$$(3.8)\qquad L_X f(t,x)=f'_t(t,x)+(a(t,x),f'_x(t,x)))+\frac{1}{2}Trace B(t,x)QB^*(t,x)f''_{xx}(t,x).$$

From Itô's formula, we obtain

$$\begin{aligned}(3.9)\, E_{s,x}f(t,X(t)) &= f(s,x)+E_{s,x}\int_s^t [f'_u(u,X(u))+(a_1(u,X(u)),f'_x(u,X(u)))\\ &+ \frac{1}{2}Trace B(t,x)QB^*(t,x)f''_{xx}(u,X(u))]du.\end{aligned}$$

From (3.7), (3.8) and (3.9) we get,

$$R_\lambda^X f(s,x) = \frac{1}{\lambda} f(s,x) + \frac{1}{\lambda} E_{s,x} \int_s^\infty e^{-\lambda(t-s)} L_X f(t, X(t)) dt.$$

That is,

$$f(s,x) = R_\lambda^X [\lambda f - L_X f](s,x). \tag{3.10}$$

The following lemma and its proof is similar to Lemma 2 (with its proof) of Gikhman and Skorokhod ((1979), Chapter 3, Section 3, page 283.

Lemma 3.4. *Suppose there exists a* $c > 0$ *such that for all* $x \in H$ *and* $s \in [0, \infty)$ *the following inequalities hold:*

$$\|(a(s,x)\| \le \frac{1}{c}, \ Trace(B_1(s,x)B_1^*(s,x)) \le 1 - c.$$

Then for λ *sufficiently large,*

$$R_\lambda^X f(s,x) = R_\lambda (I - L_1 R_\lambda)^{-1} f(s,x).$$

There exits constants λ_0 *and* M *depending on c only such that for* $\lambda > \lambda_0$,

$$\|R_\lambda^X\| \le M.$$

Proof. In (3.10), substitute $f = R_\lambda g$ for $g \in \hat{\mathcal{L}}_2$. Then we obtain

$$R_\lambda g(s,x) = R_\lambda^X [\lambda R_\lambda g - L_X R_\lambda g](s,x). \tag{3.11}$$

Using the form of $R_\lambda g$ and Itô's Lemma,

$$\frac{\partial}{\partial s} R_\lambda g + \frac{1}{2} Trace B_0 Q B_0^* R_\lambda g''_{xx} = R_\lambda \left[\frac{\partial g}{\partial s} + \frac{1}{2} Trace B_0 Q B_0^* g''_{xx} \right] = -g + \lambda R_\lambda g.$$

Thus

$$L_X R_\lambda g - L_1 R_\lambda g = -g + \lambda R_\lambda g.$$

Using the above equality and (3.11) we get

$$R_\lambda g = R_\lambda^X (g - L_1 R_\lambda g). \tag{3.12}$$

Under the conditions of the theorem and from Corollary(3.1), we get for $f \in \mathcal{L}_2$ and for sufficiently large λ, the equation

$$g - L_1 R_\lambda g = f$$

has a solution given by

$$g = (I - L_1 R_\lambda)^{-1} f.$$

Substituting this g in (3.12), we get

$$R_\lambda^X f(s,x) = R_\lambda (I - L_1 R_\lambda)^{-1} f(s,x).$$

□

The above results are used to prove that a solution of equation (3.1) is weakly unique in Gikhman and Skorokhod ((1979), Theorem 6, Chapter 3, Section 3, page 298) for finite dimensional diffusion processes. The proof extends to Hilbert space valued processes and is given below.

Theorem 3.2. *Under the assumptions* $A.1$, $A.2$,*and* $A.5$ *a martingale solution of equation (3.1) on* $[0, \infty)$ *with the initial condition* $X(0)$ *is weakly unique for any* $X(0)$.

Proof. Suppose $X_{s,x}(t)$ is a solution of equation (3.1) on $[s,\infty)$ with the initial condition $X_{s,x}(s) = x$ and

$$P_{s,x}(t,D) = P[X_{s,x}(t) \in D].$$

Let $\mathcal{F}_t^X$ denote the σ-algebra generated by $\{X(u), 0 \le u \le t\}$. The Laplace transform

$$\begin{aligned} \int_s^\infty e^{-\lambda(t-s)} P_{s,x}(t,D)dt &= E_{s,x} \int_s^\infty e^{-\lambda(t-s)} I_D(X(t))dt \\ &= e^{\delta s} E_{s,x} \int_s^\infty e^{-(\lambda-\delta)(t-s)} e^{-\delta t} I_D(X(t))dt \\ &= e^{\delta s} R^X_{\lambda-\delta}(e^{-\delta s} I_D(x)), \end{aligned} \tag{3.13}$$

where $I_D(x)$ denotes the indicator function of the set D.

Next we show that, with probability 1,

$$E[I_D(X(t))|\mathcal{F}_s^X] = P_{s,X(s)}(t,D) \tag{3.14}$$

by showing that both have the same Laplace transform. That is,

$$\int_0^\infty e^{-\lambda t} E[I_D(X(t))|\mathcal{F}_s^X]dt = \int_0^\infty e^{-\lambda t} P_{s,X(s)}(t,D)dt,$$

equivalently

$$E[\int_0^\infty e^{-\lambda t} I_D(X(t))dt|\mathcal{F}_s^X] = \int_0^\infty e^{-\lambda t} P_{s,X(s)}(t,D)dt. \tag{3.15}$$

Now from (3.13) and Lemma 3.4 and since $W(t) - W(s)$ is independent of $\mathcal{F}_s^X$, for $A \in \mathcal{F}_s^X$,

(3.16)

$$\begin{aligned} E[I_A \int_s^\infty e^{-\lambda(t-s)} P_{s,X(s)}(t,D)dt] &= E[I_A e^{\delta s} R_{\lambda-\delta}(I - L_1 R_{\lambda-\delta})^{-1}(e^{-\delta s} I_D(X(s)))] \\ = E[I_A e^{\delta s} E_{s,X(s)}\{\int_s^\infty & e^{-(\lambda-\delta)(t-s)}(I-L_1R_{\lambda-\delta})^{-1}(e^{-\delta t} I_D(X(s)+B_0(W(t)-W(s))))dt\}] \\ = E[I_A e^{\delta s} E_{s,x}\{\int_s^\infty & e^{-(\lambda-\delta)(t-s)}(I-L_1R_{\lambda-\delta})^{-1}(e^{-\delta t} I_D(x+B_0(W(t)-W(s))))dt|\mathcal{F}_s^X\}] \\ = E[E_{s,x}\{I_A e^{\delta s} \int_s^\infty & e^{-(\lambda-\delta)(t-s)}(I-L_1R_{\lambda-\delta})^{-1}(e^{-\delta t} I_D(x+B_0(W(t)-W(s))))dt|\mathcal{F}_s^X\}] \\ = E[I_A e^{\delta s} \int_s^\infty & e^{-(\lambda-\delta)(t-s)}(I - L_1R_{\lambda-\delta})^{-1}(e^{-\delta t} I_D(x + B_0(W(t) - W(s))))dt] \\ &= e^{\delta s} R_{\lambda-\delta}(I - L_1R_{\lambda-\delta})^{-1}(e^{-\delta s} I_A I_D(x)) \end{aligned}$$

Further, from Lemma 3.4

$$\begin{aligned} E[I_A \int_0^\infty e^{-\lambda t} I_D(X(t))dt] &= E_{s,x}[\int_s^\infty e^{-\lambda(t-s)} I_A I_D(X(t))dt]] \\ &= e^{\delta s} R_{\lambda-\delta}(I - L_1R_{\lambda-\delta})^{-1}(e^{-\delta s} I_A I_D(x)). \end{aligned} \tag{3.17}$$

From (3.16) and (3.17) we obtain (3.15) and thus (3.14). From (3.14), we conclude that each solution of equation (3.1) is a Markov process with transition probability $P_{s,x}(t,D)$. From (3.16), it can be seen that $P_{s,x}(t,D)$ depends only on the coefficients of equation (3.1). Thus the measure associated with $X(t)$ is uniquely determined by the distribution of $X(0)$. Thus the theorem is proved.. □

4. Stochastic partial differential equation and martingale solution

Let H be a separable Hilbert space and A the generator of a strongly continuous semigroup $(S_t, t \geq 0)$ on H. We note that $\mathcal{D}(A)$, domain of A is dense in H. Consider the stochastic partial differential equation (SPDE)

$$dr(t) = (Ar(t) + \alpha(t, r(t)))dt + \sigma(t, r(t))dW(t), \ r(0) = h, \tag{4.1}$$

where $\alpha : \Omega \times [0, \infty) \times C([0, \infty), H) \to H$, $\sigma : \Omega \times [0, \infty) \times C([0, \infty), H) \to L_2(K_Q, H)$, are specified functions satisfying assumptions $A.1$ to $A.4$ and $A'.5$ (with $a(s, x)$ replaced by $\alpha(s, x)$ and $B(s, x)$ by $\sigma(s, x)$.)

In the approach suggested by Filipović et al. (2010), the SPDE is transformed to a SDE by using a time-dependent transformation $r \to S_{-t}r$, where the semigroup is extended by $S_{-t} := S_t$ for $t \geq 0$. The SDE is solved and the solution process is transformed by $r \to S_t r$ in order to obtain a martingale solution of the original SPDE. In order to transform the above SPDE to a SDE, Filipović et al. (2010) make the following assumption.

Assumption B There exist another separable Hilbert space $\mathcal{H}$, a C_0-semigroup $(U_t)_{t\in R}$ on $\mathcal{H}$ and continuous linear operators $l : H \to \mathcal{H}$ and $\pi : \mathcal{H} \to H$ such that for every $t \in [0, \infty)$,

$$\pi U_t l = S_t.$$

In particular, $\pi l = I$ (the identity operator).

Filipović et al. (2010) show that if the semigroup $\{S_t, t \geq 0\}$ is pseudo-contractive, then the Assumption B holds. Moreover $l : H \to \mathcal{H}$ is an isometric embedding and $\pi = l^* : \mathcal{H} \to H$ is the orthogonal projection from $\mathcal{H}$ to H.

Consider the transformed SDE for a $\mathcal{H}$ valued process $\{Y(t)\}$:

$$\begin{aligned} dY(t) &= \tilde{\alpha}(t, Y(t))dt + \tilde{\sigma}(t, Y(t))dW(t) \\ Y(0) &= lh, \end{aligned} \tag{4.2}$$

where $\tilde{\alpha}(t, Y(t)) = U_{-t}l\alpha(t, \pi U_t Y(t))$, $\tilde{\sigma}(t, Y(t)) = U_{-t}l\sigma(t, \pi U_t Y(t))$ and $lh \in \mathcal{H}$. Let $\tilde{\sigma}_1(t, x) = \tilde{\sigma}(t, x)Q\tilde{\sigma}^*(t, x) - B_0 Q B_0^*$, where B_0 is a bounded linear operator from K to H. Now to prove the weak uniqueness of the solution to the martingale problem for the transformed SDE, we use results of Section 3 with arguments similar to the ones in Mandrekar and Skorokhod (1998).

Theorem 4.1. *Suppose Assumption $A.3$ and Assumption B hold and the coefficients of the SPDE in (4.1) satisfy Assumptions $A.1$, $A.2$, $A.4$ and $A'.5$. Then the equation (4.1) has a weakly unique martingale solution.*

Proof: For the SDE (4.2), the Assumption $A.3$ holds with the compact operator J replaced by the compact operator lJ. Its coefficients satisfy $A.1$, $A.2$ and $A.4$ since $U_t h$ is continuous, $\|U_t h\| \leq e^{\beta|t|}$ for some real number β, and π is a projection (see Filipović et al. (2010)). Assumption A.5 follows from Assumption $A'.5$, the trace inequalities $Trace(\tilde{\sigma}_1^2(t, x)) \leq (Trace(\tilde{\sigma}_1(t, x)))^2 \leq (\|U_{-t}l\| trace(\sigma(t, \pi U_t x)))^2$, $\|U_t h\| \leq e^{\beta|t|}$, and Remark 3.1.

Hence a martingale solution to SDE (4.2) exists (Gawarecki and Mandrekar (2011), Theorem 3.12, pp.131). Let $Y(t)$ denote this solution with $Y(0) = lh$ with $\{W(t)\}$ the corresponding Q-Wiener process. Then arguing as in the proof of Theorem 8.8 of Filipović et al. (2010) (using only (8.1)), we get that $r(t) = \pi U_t Y(t)$ is a solution for the SPDE (4.1). This can be seen from the following argument. Since $Y(t)$ is a solution of the SDE in (4.2),

$$Y(t) = lh + \int_0^t U_{-s}l\alpha(s, \pi U_s Y(s))ds + \int_0^t U_{-s}l\sigma(s, \pi U_s Y(s))dW(s). \tag{4.3}$$

Let $r(t) = \pi U_t Y(t)$. Then from (4.3),

$$\begin{aligned} r(t) &= \pi U_t \left(lx + \int_0^t U_{-s} l\alpha(s, \pi U_t^{t_0} Y(s))ds + \int_0^t U_{-s} l\sigma(s, \pi U_t^{t_0} Y(s))dW(s) \right) \\ &= S_t h + \int_0^t S_{t-s}\alpha(s, r(s))ds + \int_0^t S_{t-s}\sigma(s, r(s))dW(s), \end{aligned}$$

which shows that $r(t)$ is a martingale solution of (4.1) and thus a weak solution also (Filipović et al. (2010)).

This shows that $\{r(t)\}$ is a martingale solution with respect to the filtration $\{\mathcal{F}_t^Y\}$, where $\mathcal{F}_t^Y$ denotes the $\sigma - field$ generated by $\{Y(s), 0 \le s \le t\}$. However using Stroock and Varadhan (1969), we get that for $\theta \in H$

$$exp\left(\langle \theta, r(t) - S_t h \rangle - \int_0^t \langle \theta, S_{t-s}\alpha(s, r(s)) \rangle ds - \frac{1}{2} \int_0^t \langle \theta, (S_{t-s}\sigma(s, r(s))^* Q S_{t-s}\sigma(s, r(s))\theta \rangle ds \right) \tag{4.4}$$

is a martingale with respect to $\{\mathcal{F}_t^Y\}$. Since $\{\mathcal{F}_t^r, t \ge 0\} \subseteq \{\mathcal{F}_t^Y, t \ge 0\}$ we get that the expression (4.4) is a martingale with respect to $\{\mathcal{F}_t^r, t \ge 0\}$. Thus $\{r(t)\}$ is a martingale solution with respect to the filtration $\{\mathcal{F}_t^r\}$.

Now suppose that $r(t)$ is a martingale solution of the SPDE (4.1). Then

$$r(t) = S_t h + \int_0^t S_{t-s}\alpha(s, r(s))ds + \int_0^t S_{t-s}\sigma(s, r(s))dW(s). \tag{4.5}$$

From Assumption B,

$$r(t) = \pi U_t \left(lh + \int_0^t U_{-s} l\alpha(s, r(s))ds + \int_0^t U_{-s} l\sigma(s, r(s))dW(s) \right),$$

For $t \ge 0$, let

$$X(t) = lh + \int_0^t U_{-s} l\alpha(s, r(s))ds + \int_0^t U_{-s} l\sigma(s, r(s))dW(s)$$

and $X(0) = lh$. Thus $r(t) = \pi U_t(X(t))$ up to indistinguishability. Therefore we have,

$$\begin{aligned} X(t) &= lh + \int_0^t U_{-s} l\alpha(s, r(s))ds + \int_0^t U_{-s} l\sigma(s, r(s))dW(s) \\ &= lh + \int_0^t U_{-s} l\alpha(s, \pi U_s(X(s)))ds + \int_0^t U_{-s} l\sigma(s, \pi U_s(X(s)))dW(s) \\ &= lh + \int_0^t \tilde{\alpha}(s, (X(s)))ds + \int_0^t \tilde{\sigma}(s, (X(s)))dW(s), \end{aligned}$$

That is, $X(t)$ is a martingale solution of (4.2) with $X(0) = lh$.

From the Theorem 3.2 above, we conclude that that $X(t)$ is the weakly unique solution of the SDE, which implies that $r(t)$ is a weakly unique solution of the SPDE.

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