Theory of Stochastic Processes
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THE LIMIT BEHAVIOUR OF RANDOM WALKS WITH ARRESTS

Let $\tilde{S}$ be a random walk which behaves like a standard centred and square-integrable random walk except when hitting 0. Upon the i-th hit of 0 the random walk is arrested there for a random amount of time $\eta_i \geq 0$; and then continues its way as usual. The random variables $\eta_1,\ \eta_2,\ \ldots$ are assumed i.i.d. We study the limit behaviour of this process scaled as in the Donsker theorem. In case of $\mathbb{E}\eta_i < \infty$, weak convergence towards a Wiener process is proved. We also consider the sequence of processes whose arrest times are geometrically distributed and grow with n. We prove that the weak limit for the last model is either a Wiener process, a Wiener process stopped at 0 or a Wiener process with a sticky point.

1. Introduction

Let $\{S(n)\}_{n\in\mathbb{N}_0}$ be a random walk on $\mathbb{Z}$ with $S(0) = 0$ and centred jumps of finite variance σ^2. We define $S(t)$ for $t \geq 0$ by linear interpolation. Set

$$X_n(t) = \frac{S(nt)}{\sigma\sqrt{n}},\ n \in \mathbb{N}.$$

The well-known Donsker theorem (e.g. [2]) states the weak convergence of stochastic processes in $\mathrm{C}[0, \infty)$

$$X_n(t) \xrightarrow{w} W(t),\ n \to \infty,$$

where W is a Wiener process.

Upon changing transition probabilities at one point or a set of points (e.g. [7, 8, 12]) one obtains limit processes related to Brownian motion, for example, a skew Brownian motion, a Brownian motion with a sticky point, a Brownian motion with jump-exit from 0 (cf. [12]).

In this work we are concerned with the scaling limit of random walks with arrests. By arrest we mean adding a random delay at 0. Semi-Markov random walks with continuous-time and non-exponential arrests give rise to equations with fractional derivatives [10, 11]. For example, a process with jumps in $\mathbb{R}$ and lagged at each point for a random amount of time with a "heavy tail" distribution constitutes a sub-diffusion model. As remarked in [1] the processes with a sticky point could be used for modelling a financial market with governmental control. Sticky Brownian motion also arises while discussing storage processes that have different intensities in and out of zero, [6].

We consider a modified discrete random walk which is arrested for a random amount of time upon each visit to zero. It is shown that its scaling limit is a Brownian motion whenever the arrest time has a finite expectation. We also look at a triangular array of random walks with geometrically distributed times of arrest whose expectations depend on n. This model gives rise to a Brownian motion with a sticky point. For further discussion of this process see [1, 4, 5, 6, 9].

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2. Problem statement and results

Let $\{S(n)\}_{n\in\mathbb{N}_0}$ be a random walk generated by integer-valued i.i.d. random variables $\{\xi_n\}_{n\geq 1}$

$$S(n) = \sum_{i=1}^{n} \xi_i, \ n \in \mathbb{N} \text{ and } S(0) = 0.$$

Assume that $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 = \sigma^2 \in (0, \infty)$.

Extend S by linearity:

$$S(t) = S(n) + (t - n)(S(n+1) - S(n)), \ t \in [n, n+1],$$

for all $t \geq 0$.

Let also $\{\eta_n\}_{n\geq 1}$ be a sequence of non-negative i.i.d. random variables that is independent of $\{\xi_i\}_{n\geq 1}$.

We construct a modified random walk $\{\tilde{S}(n)\}_{n\in\mathbb{N}_0}$ as follows. While keeping the excursions of $\tilde{S}$ to be the same as those of S, we introduce a random delay η_i between the $i-1$-th and i-th excursions of S. See Fig. 1 and 2. Although ξ_i may be equal to 0, we still define the excursion of S to be the interval between consecutive visits to zero: $e_0 = 0, \ e_1 = \inf\{t > e_0 : S(t) = 0\}, \ e_2 = \inf\{t > e_1 : S(t) = 0\}, \ \ldots$

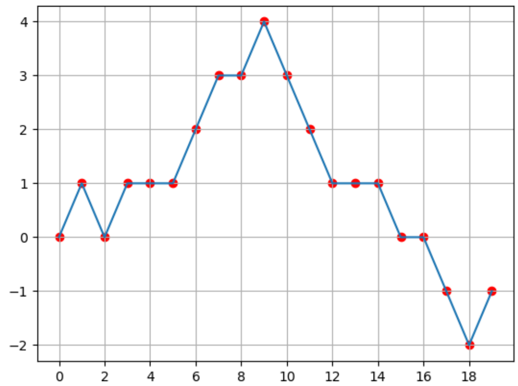

Figure 1. A plot of S where for simplicity $\xi_i \in \{-1, 0, 1\}$. Here it visits 0 at times 0, 2, 15 and 16.

The modification $\{\tilde{S}(n)\}_{n\in\mathbb{N}_0}$ can be defined formally. To this end, we put

$$\alpha(t) = t + \sum_{i=1}^{\tau_0(t)} \eta_i, \ t \geq 0.$$

where $\tau_0(t) = \#\{k \leq t : S(k) = 0\}$ for $t \geq 0$ is the number of visits to zero of the random walk S up to and including time t.

Denote by

$$\alpha^{(-1)}(t) = Inv[\alpha(\cdot)](t) = \inf\{x : \alpha(x) \geq t\}, \ t \geq 0,$$

a generalised inverse function of α. Observe that $\alpha^{(-1)}$ is continuous because α is non-decreasing.

The process $(\tilde{S}(t))_{t\geq 0}$ is then defined by

$$\tilde{S}(t) = S(\alpha^{(-1)}(t)), \ t \geq 0.$$

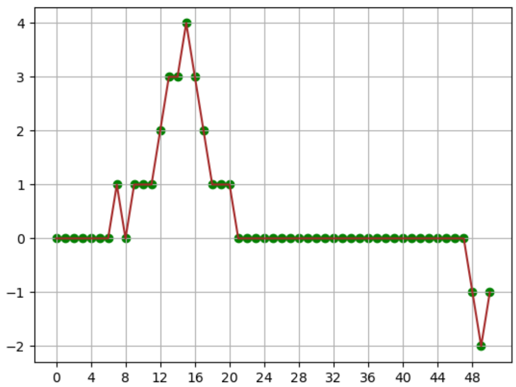

FIGURE 2. A plot of $\tilde{S}$ which corresponds to Fig. 1. Here $\eta_1 = 6,\ \eta_2 = 0,\ \eta_3 = 7,\ \eta_4 = 20$.

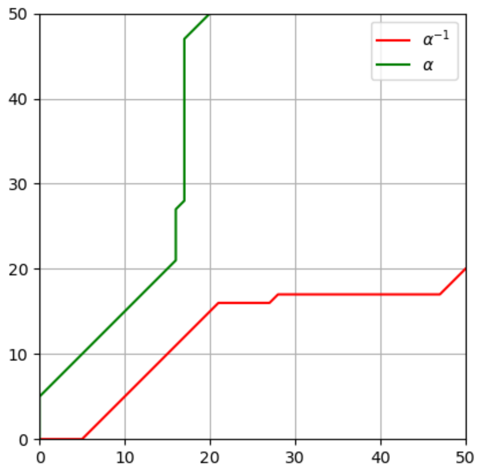

FIGURE 3. Plots of α and $\alpha^{(-1)}$ which correspond to Fig. 1 and 2.

Our goal is to study the limit behaviour of the sequence of processes $\left(\frac{\tilde{S}(nt)}{\sqrt{n}}\right)_{t\geq 0}$ as $n \to \infty$. Denote by $\mathrm{C}[0,\infty)$ the space of continuous functions on $[0,\infty)$ endowed with the topology of uniform convergence on finite intervals.

Theorem 1. *Let $\{\tilde{S}(n)\}_{n\in\mathbb{N}_0}$ be a modified random walk and $\mathbb{E}\eta_1 < \infty$. For the sequence of processes $\{\tilde{X}_n(\cdot) = \frac{\tilde{S}(n\cdot)}{\sigma\sqrt{n}},\ n \geq 1\}$ weak convergence in $\mathrm{C}[0,\infty)$ holds:*

$$\tilde{X}_n(\cdot) \xrightarrow{w} W(\cdot),\ n \to \infty,$$

where W is a Wiener process.

Remark 1. For $p \in (0,1]$ and an integer-valued random variable ξ with $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 < \infty$, put $p_{ij} = \mathbb{P}\{\xi = i - j\}$ for $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$, $p_{0j} = (1-p)\mathbb{P}\{\xi = j\}$ for $j \in \mathbb{N}$ and $p_{00} = p + \mathbb{P}\{\xi = 0\}$. The distribution of a Markov chain which starts at 0 and has transition probabilities $(p_{ij},\ i \in \mathbb{N},\ j \in \mathbb{N})$ is the same as the distribution of $\tilde{S}$ in which η_1 has a geometric distribution with mean $\frac{1-p}{p}$. Thus Theorem 1 can be applied to this Markov chain.

Let us consider more closely the random walk from Remark 1. Denote it by $S^{(p)}$. We will show that the sequence of processes $\left\{X_n^{(p_n)}\right\}_{n\geq 1}$, where

$$X_n^{(p_n)}(t) = \frac{S^{(p_n)}(nt)}{\sigma\sqrt{n}},\ t \geq 0,$$

and

$$p_n = \frac{\rho}{n^\gamma},\ n \geq 1$$

has different weak limits with respect to γ as $n \to \infty$. Theorem 2 below describes all possible modes.

Denote by $(W_{\beta\text{-sticky}}(t))_{t\geq 0}$ a Brownian motion with a sticky point defined by

$$W_{\beta\text{-sticky}}(t) = W(A_\beta^{(-1)}(t)),\ t \geq 0$$

where

$$A_\beta(t) = t + \beta L(t), \quad A_\beta^{(-1)} \text{ is a generalised inverse of } A_\beta$$

and

$$L(t) = \mathbb{P}\text{-}\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t 1_{\{W(s)\in[-\varepsilon,\varepsilon]\}}ds$$

is a local time of a Brownian motion W at zero. As opposed to a usual Brownian motion, this one spends a positive amount of time at zero, yet there is no interval of positive length that it remains there.

Theorem 2. *The weak convergence in* $\mathrm{C}[0,\infty)$ *holds:*

$$\text{if } 0 \leq \gamma < 0.5,\ \text{then } X_n^{(p_n)}(t) \xrightarrow{w} W(t),\ n \to \infty,$$
$$\text{if } \gamma > 0.5,\ \text{then } X_n^{(p_n)}(t) \xrightarrow{w} 0,\ n \to \infty,$$
$$\text{if } \gamma = 0.5,\ \text{then } X_n^{(p_n)}(t) \xrightarrow{w} W_{\rho^{-1}\text{-sticky}}(t),\ n \to \infty.$$

3. Proofs

The following two lemmas can be found in [13] (Proposition 3.2).

Lemma 1. *Let* $\{\xi_n(t)\}_{n\geq 1},\ t \in [0,T]$ *be a sequence of random processes such that*

(a) *for each* n *the process* $\xi_n(t)$ *is a.s. monotone;*

(b) *for every* t

$$\xi_n(t) \xrightarrow{\mathbb{P}} \xi(t),\ n \to \infty;$$

(c) *the limiting process* $\xi(t)$ *is continuous a.s.*

Then the uniform convergence in probability holds

$$\sup_{t\in[0,T]} |\xi_n(t) - \xi(t)| \xrightarrow{\mathbb{P}} 0,\ n \to \infty.$$

Lemma 2. *Let* $\{\xi_n(t)\}_{n\geq 1},\ t \in [0,T]$ *be a sequence of random processes satisfying (a), (b), (c) of Lemma 1 and*

(d) *for each* n

$$\xi_n(0) = 0,\ \xi_n(\infty) = \infty.$$

Then for any $T > 0$ *the uniform convergence in probability holds*

$$\sup_{t\in[0,T]} |\xi_n^{(-1)}(t) - \xi^{(-1)}(t)| \xrightarrow{\mathbb{P}} 0,\ n \to \infty.$$

3.1. **Proof of Theorem 1.** Set

$$h_n(t) = \frac{\alpha^{(-1)}(nt)}{n},\ t \geq 0.$$

From the definition of $\tilde{S}$ one has

$$\tilde{X}_n(t) = \frac{\tilde{S}(nt)}{\sqrt{n}} = \frac{S(\alpha^{(-1)}(nt))}{\sigma\sqrt{n}} = \frac{S(n\frac{\alpha^{(-1)}(nt)}{n})}{\sigma\sqrt{n}} = X_n(h_n(t)).$$

Hence we will prove that

$$X_n(h_n(\cdot)) \xrightarrow{w} W(\cdot),\ n \to \infty. \tag{1}$$

We claim that

$$\sup_{t\in[0,T]} |h_n(t) - t| = \sup_{t\in[0,T]} \left|\frac{\alpha^{(-1)}(nt)}{n} - t\right| \xrightarrow{\mathbb{P}} 0,\ n \to \infty. \tag{2}$$

To check this we intend to show that for any $t \geq 0$:

$$\frac{\alpha(nt)}{n} \xrightarrow{\mathbb{P}} t,\ n \to \infty. \tag{3}$$

This is obvious for $t = 0$. For $t > 0$

$$\frac{\alpha(nt)}{n} = t + \frac{1}{n}\sum_{i=1}^{\tau_0(nt)} \eta_i = t + \frac{\tau_0(nt)}{n}\frac{1}{\tau_0(nt)}\sum_{i=1}^{\tau_0(nt)} \eta_i. \tag{4}$$

For a fixed $t > 0$ one has $\mathbb{P}\{\lim_{n\to\infty} \tau_0(nt) = \infty\} = 1$, see e.g. [15] (Proposition I.2.3 and I.2.8). Thus, by the strong law of large numbers

$$\frac{1}{\tau_0(nt)}\sum_{i=1}^{\tau_0(nt)} \eta_i \to \mathbb{E}\eta_1 < \infty,\ n \to \infty \text{ a.s.}$$

Here $\frac{\tau_0(nt)}{\sqrt{n}}$ converges in distribution to an absolute value of a normally distributed random variable as $n \to \infty$, see for example [3]. So

$$\frac{\tau_0(nt)}{n} \xrightarrow{\mathbb{P}} 0,\ n \to \infty,$$

and thereupon

$$\frac{\alpha(nt)}{n} \xrightarrow{\mathbb{P}} t,\ n \to \infty. \tag{5}$$

Since the functions $\{\frac{\alpha(n\cdot)}{n}\}_{n\geq 1}$ are nondecreasing a.s. and their sequence converges to the continuous limit, we invoke Lemmas 1 and 2 to conclude that (2) holds.

The following is well-known, e.g. [2] (Theorem 4.4).

Lemma 3. *Let E be a Polish space, $\{X_n, n \geq 1\}, X, \{h_n, n \geq 1\}$ be random elements with values in E, and $h \in E$ be non-random. Assume that $X_n \xrightarrow{w} X$ and $h_n \xrightarrow{w} h$. Then the pairs of random variables converge weakly*

$$(X_n, h_n) \xrightarrow{w} (X, h),\ n \to \infty.$$

As $X_n(\cdot) \xrightarrow{w} W(\cdot)$ and $h_n(\cdot) \xrightarrow{w} h(\cdot)$, where $h(t) = t$ for $t \geq 0$, Lemma 3 yields $(X_n, h_n) \xrightarrow{w} (W, h)$. Due to the Skorokhod representation theorem [2] there exists a probability space which supports random elements $\bar{X}_n$ and $\bar{h}_n$ such that in C$[0,\infty)$:

$$(\bar{X}_n, \bar{h}_n) \stackrel{\mathrm{d}}{=} (X_n, h_n),$$

and for any $T > 0$ the uniform convergence on $[0,T]$ holds

$$\bar{X}_n(t) \rightrightarrows \bar{W}(t) \ \text{ and } \ \bar{h}_n(t) \rightrightarrows t,\ n \to \infty \text{ a.s.}$$

Thus $\bar{X}_n(\bar{h}_n(\cdot)) \to \bar{W}(\cdot),\ n \to \infty$ a.s., hence

$$X_n(h_n(\cdot)) \xrightarrow{w} \bar{W}(\cdot).$$

3.2. **Proof of Theorem 2.** We recall that now, for each $n \geq 1$, $\{\eta_i^{(n)}\}_{i\geq 1}$ is a sequence of independent geometrically distributed random variables with mean $\frac{1-p_n}{p_n}$. As before, for $t \geq 0$, $\tau_0(t)$ is the number of visits to zero of the random walk S up to and including time t. Let

$$\alpha_n(t) = t + \sum_{i=1}^{\tau_0(t)} \eta_i^{(n)},\ t \geq 0,$$

and $\alpha_n^{(-1)}$ be its generalised inverse. Set

$$h_n(t) = \frac{\alpha_n^{(-1)}(nt)}{n},$$

hence

$$X_n^{(p_n)}(t) = \frac{S^{(p_n)}(nt)}{\sqrt{n}} = \frac{S(\alpha_n^{(-1)}(nt))}{\sigma\sqrt{n}} = \frac{S(n\frac{\alpha_n^{(-1)}(nt)}{n})}{\sigma\sqrt{n}} = X_n(h_n(t)).$$

Let us start with discussing the behaviour of

$$\frac{\alpha_n(nt)}{n} = t + \frac{1}{n}\sum_{i=1}^{\tau_0(nt)} \eta_i^{(n)}. \tag{6}$$

Observe that

$$\frac{\alpha_n(nt)}{n} = t + \frac{n^\gamma}{\sqrt{n}}\frac{\tau_0(nt)}{\sqrt{n}}\frac{1}{\tau_0(nt)}\sum_{i=1}^{\tau_0(nt)} \frac{\eta_i^{(n)}}{n^\gamma}. \tag{7}$$

Theorem 3 ([3])**.** *Let W be a Brownian motion in $\mathbb{R}$, L be its local time. Then in* $\mathrm{C}[0,\infty)$

$$\Big(\frac{\tau_0(nt)}{\sqrt{n}},\ \frac{S(nt)}{\sigma\sqrt{n}}\Big) \xrightarrow{w} (L(t),\ W(t)),\ n\to\infty.$$

For each $n \geq 1$ by the Skorokhod theorem we can construct a probability space which supports random elements $\bar{\tau}_0^{(n)}$ and $\bar{S}^{(n)}$ such that in $\mathrm{C}[0,\infty)$:

$$\Big(\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}},\ \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}}\Big)_{t\geq 0} \overset{\mathrm{d}}{=} \Big(\frac{\tau_0(nt)}{\sqrt{n}},\ \frac{S(nt)}{\sqrt{n}}\Big)_{t\geq 0}, \tag{8}$$

and for any $T > 0$ the uniform convergence on $[0,T]$ holds

$$\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{L}(t) \text{ and } \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{W}(t) \text{ as } n\to\infty \text{ a.s.} \tag{9}$$

Assume that, for each $n \geq 1$, $\{\eta_i^{(n)}\}_{i\geq 1}$ is defined on the same probability space as $\bar{S}^{(n)}$, $\bar{\tau}_0^{(n)}$, $\bar{L}$ and $\bar{W}$, and is independent of these.

Theorem 4. *For every $T > 0$*

$$\sup_{t\in[0,T]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}\bar{L}(t)}\frac{\eta_i^{(n)}}{n^\gamma} - \frac{\bar{L}(t)}{\rho}\right| \xrightarrow{\mathbb{P}} 0,\ n\to\infty, \tag{10}$$

where $\sum_{i=1}^{x}$ means $\sum_{i=1}^{[x]}$.

We need some auxiliary results.

Proposition 1. *For any fixed $t \geq 0$ we have*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}t} \frac{\eta_i^{(n)}}{n^\gamma} \xrightarrow{\mathbb{P}} \frac{t}{\rho},\ n \to \infty.$$

Proof. Since

$$\mathbb{E} \frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}t} \frac{\eta_i^{(n)}}{n^\gamma} = \frac{[\sqrt{n}t]}{\sqrt{n}\rho} \to \frac{t}{\rho},\ n \to \infty,$$

it suffices to verify that the variances converge to 0. The summands are independent, thus

$$\mathbb{V}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}t} \frac{\eta_i^{(n)}}{n^\gamma} \right) = \frac{1}{n} \sum_{i=1}^{\sqrt{n}t} \frac{\mathbb{V}\eta_i^{(n)}}{n^{2\gamma}} = \frac{1}{n} \sum_{i=1}^{\sqrt{n}t} \frac{1 - \frac{\rho}{n^\gamma}}{\frac{\rho^2}{n^{2\gamma}} n^{2\gamma}} = \frac{[\sqrt{n}t]}{n} \frac{1 - \frac{\rho}{n^\gamma}}{\rho^2} \to 0,\ n \to \infty.$$

□

Proposition 2. *For every $T > 0$ and for any $\varepsilon > 0$ we have*

$$\sup_{t \in [0,T]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}t} \frac{\eta_i^{(n)}}{n^\gamma} - \frac{t}{\rho} \right| \xrightarrow{\mathbb{P}} 0,\ n \to \infty.$$

Proof. The sum is monotonous in t and due to Proposition 1 it has a continuous limit. Thus this proposition follows from Lemma 1. □

Proof of Theorem 4. Let $\delta > 0$ be a fixed number. Find T' such that the set $\Omega_\delta = \{\bar{L}(T) < T'\}$ satisfies $\mathbb{P}(\Omega_\delta) > 1 - \delta$. Note that for any $t \in [0,T]$ it holds that $\bar{L}(t) \leq \bar{L}(T)$. Hence on the set Ω_δ

$$\sup_{t \in [0,T]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}\bar{L}(t)} \frac{\eta_i^{(n)}}{n^\gamma} - \frac{\bar{L}(t)}{\rho} \right| \leq \sup_{y \in [0,T']} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}y} \frac{\eta_i^{(n)}}{n^\gamma} - \frac{y}{\rho} \right|.$$

Denote by

$$A_{n,\varepsilon} = \left\{ \sup_{t \in [0,T]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}\bar{L}(t)} \frac{\eta_i^{(n)}}{n^\gamma} - \frac{\bar{L}(t)}{\rho} \right| > \varepsilon \right\}$$

and write down

$$\mathbb{P}(A_{n,\varepsilon}) = \mathbb{P}(A_{n,\varepsilon} \cap \Omega_\delta) + \mathbb{P}(A_{n,\varepsilon} \cap \bar{\Omega}_\delta).$$

From Proposition 2

$$\overline{\lim_{n \to \infty}}\, \mathbb{P}(A_{n,\varepsilon}) \leq 0 + \delta.$$

As δ and ε are arbitrary, the last inequality proves the theorem. □

Now suppose that Ω is a set where (9) holds with probability 1. Let ε be fixed, then for N large enough find the set $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_d elta) > 1 - \delta$ on which the event

$$\sup_{t \in [0,T]} \left| \bar{L}(t) - \frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}} \right| < \varepsilon$$

holds for each $n > N$ and $\mathbb{P}(\Omega_\delta) > 1 - \delta$.

Consider the difference

$$\sup_{t \in [0,T]} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{\sqrt{n}\bar{L}(t)} \frac{\eta_i^{(n)}}{n^\gamma} - \sum_{i=1}^{\bar{\tau}_0^{(n)}(nt)} \frac{\eta_i^{(n)}}{n^\gamma} \right|. \tag{11}$$

We show that it converges to 0 in probability and so the limits of the sums should coincide. Since $\{\eta_i^{(n)}\}_{i\geq 1}$ are independent of $(\bar{L}, \bar{\tau}_0^{(n)})$, the last expression is equal in distribution to

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}\sup_{t\in[0,T]}|\bar{L}(t)-\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}}|} \frac{\eta_i^{(n)}}{n^\gamma}.$$

Now on the set Ω_δ for $n > N$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}\sup_{t\in[0,T]}|\bar{L}(t)-\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}}|} \frac{\eta_i^{(n)}}{n^\gamma} \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}\varepsilon} \frac{\eta_i^{(n)}}{n^\gamma},$$

and Proposition 2 implies the convergence of the latter to $\frac{\varepsilon}{\rho}$. Since the probability of the complement of Ω_δ is less or equal δ and ε was arbitrary, one sees that (11) converges in probability to 0. Now due to Theorem 4

$$\sup_{t\in[0,T]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\bar{\tau}_0^{(n)}(nt)} \frac{\eta_i^{(n)}}{n^\gamma} - \frac{\bar{L}(t)}{\rho} \right| \xrightarrow{\mathbb{P}} 0,\ n\to\infty. \tag{12}$$

3.2.1. *Proof of the theorem in case $\gamma < 0.5$.* Recall (7):

$$\frac{\alpha_n(nt)}{n} = t + \frac{n^\gamma}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_0(nt)} \frac{\eta_i^{(n)}}{n^\gamma}. \tag{13}$$

In case $\gamma < 0.5$, the right hand side of (13) converges to t in probability. Now Lemmas 1 and 2 assure that for every $T > 0$:

$$\sup_{t\in[0,T]} |h_n(t) - t| = \sup_{t\in[0,T]} \left| \frac{\alpha_n^{(-1)}(nt)}{n} - t \right| \xrightarrow{\mathbb{P}} 0,\ n\to\infty. \tag{14}$$

The last limit is non random, thus for each $n \geq 1$ we use Lemma 3 and the Skorokhod theorem to construct a probability space which supports random variables $(\bar{\tau}_0^{(n)},\ \bar{S}^{(n)},\ \bar{h}_n)$ such that in C$[0,\infty)$:

$$\left(\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}},\ \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}},\ \bar{h}_n(t)\right)_{t\geq 0} \stackrel{\mathrm{d}}{=} \left(\frac{\tau_0(nt)}{\sqrt{n}},\ \frac{S(nt)}{\sqrt{n}},\ h_n(t)\right)_{t\geq 0},$$

and for any $T > 0$ the uniform convergence on $[0, T]$ holds

$$\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{L}(t),\ \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{W}(t) \ \text{ and } \ \bar{h}_n(t) \rightrightarrows t \ \text{ as } \ n\to\infty \text{ a.s.}$$

Recall that in Theorem 1 we had the similar situation. So analogously one obtains that the limit is a Brownian motion

$$X_n^{(p_n)}(\cdot) \xrightarrow{w} W(\cdot),\ n\to\infty.$$

3.2.2. *Proof of the theorem in case $\gamma > 0.5$.* In this case the expression (13) converges to ∞ in probability for every $t > 0$. Since for each $n \geq 1$ the functions $\frac{\alpha_n(n\cdot)}{n}$ are nondecreasing, we have

$$\forall\delta > 0\ \forall M\ \exists N\ \forall t \in [\delta,\infty)\ \forall n > N \quad \mathbb{P}\left(\frac{\alpha_n(nt)}{n} > M\right) > 1-\delta,$$

which loosely may be interpreted as $\frac{\alpha_n(nt)}{n} \overset{\mathbb{P}}{\rightrightarrows} \infty$ on a set $[\delta, \infty)$. This implies the uniform convergence in probability on the compact subsets of $[0, \infty)$ for $h_n(t)$:

$$h_n(t) = \frac{\alpha_n^{(-1)}(nt)}{n} \overset{\mathbb{P}}{\rightrightarrows} 0, \ n \to \infty.$$

Once again this limit is non random. By Lemma 3 and the Skorokhod theorem, we construct a probability space which supports random variables $(\bar{\tau}_0^{(n)}, \ \bar{S}^{(n)}, \ \bar{h}_n)$ such that in $\mathrm{C}[0, \infty)$:

$$\Big(\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}}, \ \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}}, \ \bar{h}_n(t)\Big)_{t\ge 0} \overset{\mathrm{d}}{=} \Big(\frac{\tau_0(nt)}{\sqrt{n}}, \ \frac{S(nt)}{\sqrt{n}}, \ h_n(t)\Big)_{t\ge 0},$$

and the uniform convergence on the compact subsets of $[0, \infty)$ holds

$$\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{L}(t), \ \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{W}(t) \ \text{ and } \ \bar{h}_n(t) \rightrightarrows 0 \ \text{ as } \ n \to \infty \text{ a.s.}$$

Thus

$$X_n(h_n(t)) \overset{w}{\to} 0, \ n \to \infty.$$

3.2.3. *Proof of the theorem in case $\gamma = 0.5$.* In this case $\frac{n^\gamma}{\sqrt{n}} = 1$ and so from (12) one sees that (13) has a non-trivial limit

$$h_n(t) = \frac{\alpha_n(nt)}{n} \overset{w}{\to} t + L(t)/\rho, \ n \to \infty.$$

Furthermore, we may consider the copies of random variables that we constructed after stating Theorem 3. For them we proved (12) and so for any $T > 0$

$$\sup_{t\in[0,T]} \left| \frac{\bar{\alpha}_n(nt)}{n} - t - \frac{\bar{L}(t)}{\rho} \right| \overset{\mathbb{P}}{\to} 0, \ n \to \infty. \tag{15}$$

For each $n \ge 1$ the functions $\frac{\bar{\alpha}_n(n\cdot)}{n}$ are a.s. monotone and their limit is continuous (because the local time is continuous, e.g. [5]). Thus Lemma 2 implies (16). Recall that we denoted a generalised inverse of a function as Inv.

$$\sup_{t\in[0,T]} \left| \frac{\bar{\alpha}_n^{(-1)}(nx)}{n} - Inv[t + \bar{L}(t)/\rho](x) \right| \overset{\mathbb{P}}{\to} 0, \ n \to \infty. \tag{16}$$

And hence the convergence in $\mathrm{C}[0, \infty)$ is proved

$$\bar{X}_n(\bar{h}_n(\cdot)) \to \bar{W}(Inv[t + \bar{L}(t)/\rho](\cdot)), \ n \to \infty \text{ a.s.}$$

Thus in $\mathrm{C}[0, \infty)$

$$X_n^{(p_n)}(\cdot) \overset{w}{\to} \bar{W}(Inv[t + \bar{L}(t)/\rho](\cdot)), \ n \to \infty.$$

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