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LIMIT THEOREM FOR PERTURBED RANDOM WALKS

We consider random walks perturbed at zero which behave like (possibly different) random walk with independent and identically distributed increments on each half lines and restarts at 0 whenever they cross that point. We show that the perturbed random walk, after being rescaled in a proper way, converges to a skew Brownian motion whose parameter is defined by renewal functions of the simple random walk and the transition probabilities from 0.

1. Introduction

Let $(S_n)_{n\geq 0}$ be a random walk (r.w. for short) on $\mathbb{Z}$ starting from 0 whose increments $\xi_k, k \geq 1$, are i.i.d with zero mean and variance $\sigma^2 := \mathbb{E}[\xi_k^2] \in (0, +\infty)$. A continuous time process $(S_t)_{t\geq 0}$ can be constructed from the sequence $(S_n)_{n\geq 0}$ by using the linear interpolation between the values at integer points. According to the well-known Donsker's theorem, the sequence of stochastic processes $(S_t^{(n)})_{n\geq 1}$, defined by

$$S_t^{(n)} := \frac{1}{\sigma\sqrt{n}} S_{nt}, \quad n \geq 0,$$

weakly converges as $n \to +\infty$ in the space continuous functions on $[0, 1]$ to the Brownian motion $(B_t)_{t\geq 0}$.

In this paper, we consider the more general cases of spatially inhomogeneous r.w. $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ which model some discrete time diffusion in a one dimensional space with two different media $\mathbb{Z}^-$ and $\mathbb{Z}^+$ and a barrier $\{0\}$. We prove an invariance principle for these r.w. towards the skew Brownian motion $(B_t^\alpha)_{t>0}$ on $\mathbb{R}$ with parameter $\alpha \in [0, 1]$. Recall that the process $(B_t^\alpha)_{t\geq 0}$ behaves like a Brownian motion except that the sign of each excursion is chosen using an independent Bernoulli random variable of parameter α; its transition probability density function is given by: for any $x, y \in \mathbb{R}$ and $t > 0$,

$$p_t^\alpha(x, y) = p_t(x, y) + (2\alpha - 1)\ \mathrm{sign}(y)\ p_t(0, |x| + |y|)$$

(see [21], page 87), where $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$ is the transition density of the Brownian motion. We refer to [21] for more details on the skew Brownian motion.

Let us now state the main result of the present article.

Theorem 1.1. *Assume that*

(1) $(\xi_n)_{n\geq 1}$ *and* $(\xi'_n)_{n\geq 1}$ *are sequences of* $\mathbb{Z}$*-valued, centered and i.i.d. random variables with finite second moments and respective variances* σ^2 *and* σ'^2*;*
(2) *the supports of the distribution of the* ξ_n *and* ξ'_n *are not included in the coset of a proper subgroup of* $\mathbb{Z}$ *(aperiodicity condition);*
(3) $(\eta_n)_{n\geq 1}$ *is a sequence of* $\mathbb{Z}$*-valued and i.i.d. random variables such that* $\mathbb{E}[|\eta_n|] < +\infty$*, and* $\mathbb{P}[\eta_n = 0] < 1$*;*
(4) *the sequences* $(\xi_n)_{n\geq 1}$*,* $(\xi'_n)_{n\geq 1}$ *and* $(\eta_n)_{n\geq 1}$ *are independent.*

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Let $(X_n)_{n\geq 0}$ be the $\mathbb{Z}$-valued process defined by: $X_0 = 0$ and, for $n \geq 1$,

$$X_n = \begin{cases} X_{n-1} + \xi_n & \text{if} \quad X_{n-1} > 0 \quad \text{and} \quad X_{n-1} + \xi_n > 0, \\ 0 & \text{if} \quad X_{n-1} > 0 \quad \text{and} \quad X_{n-1} + \xi_n \leq 0, \\ \eta_n & \text{if} \quad X_{n-1} = 0, \\ X_{n-1} + \xi'_n & \text{if} \quad X_{n-1} < 0 \quad \text{and} \quad X_{n-1} + \xi'_n < 0, \\ 0 & \text{if} \quad X_{n-1} < 0 \quad \text{and} \quad X_{n-1} + \xi'_n \geq 0. \end{cases} \tag{1}$$

Let $(X_t)_{t\geq 0}$ be the continuous time process constructed from the sequence $(X_n)_{n\geq 0}$ by linear interpolation between the values at integer points, and, for any $n \geq 1, t \geq 0$, set

$$X_t^{(n)} := \begin{cases} \dfrac{1}{\sigma\sqrt{n}} X_{nt} & \text{when} \quad X_{nt} \geq 0, \\ \dfrac{1}{\sigma'\sqrt{n}} X_{nt} & \text{when} \quad X_{nt} \leq 0. \end{cases}$$

Then, as $n \to +\infty$, the sequence of stochastic processes $(X_t^{(n)})_{n\geq 1}$ weakly converges in the space of continuous functions on $[0, +\infty)$ to the skew Brownian motion $(B^\alpha(t))_{t\geq 0}$ on $\mathbb{R}$ with parameter α depending on the distribution of the ξ_n, ξ'_n and η_n as follows:

$$\alpha = \frac{c_1 \mathbb{E}[h(\eta_1)\mathbf{1}_{\{\eta_1>0\}}]}{c_1 \mathbb{E}[h(\eta_1)\mathbf{1}_{\{\eta_1>0\}}] + c'_1 \mathbb{E}[h'(-\eta_1)\mathbf{1}_{\{\eta_1<0\}}]} \tag{2}$$

where

(1) *h is the "descending renewal function"* [1] *of the r.w. $S = (S_n)_{n\geq 0}$ with increments ξ_k;*
(2) $c_1 = \dfrac{\mathbb{E}[-S_{\ell_1}]}{\sigma\sqrt{2\pi}}$ *where ℓ_1 is the first strictly descending ladder epoch of the r.w. S;*
(3) *h' is the "ascending renewal function" of the r.w. $S' = (S'_n)_{n\geq 0}$ with increments ξ'_k;*
(4) $c'_1 = \dfrac{\mathbb{E}[S'_{\ell'_1}]}{\sigma'\sqrt{2\pi}}$ *where ℓ'_1 is the first strictly ascending ladder epoch of the r.w. S'.*

The Markov chain $(X_n)_{n\geq 0}$ has been the object of several studies in various contexts connected to random processes. It first appeared in [10] where the skew Brownian motion (SBM) on $\mathbb{R}$ is obtained as the weak limit of a normalized simple r.w. on $\mathbb{Z}$ which has special behavior only at the origin; in this seminal work, the authors consider the case when the ξ_n and ξ'_n are Bernoulli symmetric random variables and the variables η_k are also $\{-1, 1\}$ valued, but with respective probabilities $1 - \alpha$ and α. We may also cite [22] dealing with the multidimensional case and where, unlike the one-dimensional case, the limit is a Brownian motion.The key step of the proof in [10] relies on the reflection principle (but the convergence of the finite dimensional distributions is not done in details), which is valid only for Bernoulli symmetric r.w. Let us cite also the long and rich review article [15] by A. Lejay on the construction of the skew Brownian motion, where a slightly different argument is also presented: the approximation of the SBM(α) by Bernoulli symmetric r.w. is obtained starting from a trajectory of the SBM(α) and constructing recursively a suitable sequence of stopping times. Both proofs only work for Bernoulli symmetric r.w. Note also that Harrison and Shepp mentioned without a proof in [10] that such a result holds for arbitrary integrable random variables η_n; Theorem 1.1 covers (and extends) this case.

More recently, in [19], the SBM appears as the weak limit of a r.w. in $\mathbb{Z}$ whose transition probabilities coincide with those of a symmetric r.w. with unit steps throughout except for a fixed neighborhood $\{-m, \ldots, m\}$ of zero, called a "membrane". Assuming that,

[1] We refer to sections 2.1 and 3 for the definition of the"renewal function" of an oscillating r.w. .

from the membrane, the chain jumps to an arbitrary point of the set $\{-m-1, \ldots, m+1\}$, the authors describes the possible weak limits of the suitably scale process according to the fact that the sites $-m-1$ and $m+1$ are essential or not. All possible limits for the corresponding r.w. are described. In [12], it is proved that this convergence holds in fact for r.w. such that the absolute value $|\xi_k|$ of the steps outside the membrane are bounded by $2m+1$; this last condition ensures that the r.w. cannot jump over the membrane without passing through it. A. Iksanov and A. Pilipenko offer in [12] a new approach which is based on the martingale characterization of the SBM. In the present paper, the membrane is reduced to $\{0\}$ and the assumption of "bounded jumps out of the membrane" is replaced by the absorption condition at 0; nevertheless, Theorem 1.1 may be extended to a finite membrane, with similar conditions as in [19].

Let us mention that the expression of the parameter α is different of the one proposed in [19]; it takes into account in an explicit form the inhomogeneity of the r.w. on $\mathbb{Z}$. Others one dimensional non-homogeneous r.w. are studied in [17].

The parameter α may be expressed differently, by using some estimates on fluctuations of the perturbed r.w. X (see section 2.2); indeed,

$$\alpha = \lim_{n\to\infty} \frac{\mathbb{P}\left[\tau_1^X > n, \eta_1 > 0\right]}{\mathbb{P}\left[\tau_1^X > n\right]}, \tag{3}$$

where τ^X is the first return time of the perturbed r.w. X at the origin. This expression highlights the connection with the construction of a skew Brownian motion from the point of view of Ito's synthesis theorem.

In the case when the two r.w. are Bernoulli symmetric [2], it holds $h(-x) = h'(x) = x$ for any $x \geq 0$ and $c_1 = c_1'$; thus, $\alpha = \frac{\mathbb{E}[(\eta_1)^+]}{\mathbb{E}[|\eta_1|]}$ as announced in [10].

When $\mathcal{L}(\xi_1) = \mathcal{L}(-\xi_1')$ and η_1 is symmetric (ie $\mathcal{L}(\eta_1) = \mathcal{L}(-\eta_1)$), the process $(|X_n|)_{n\geq 0}$ above coincides with the process $(Y_n)_{n\geq 0}$ defined by $Y_0 = 0$ and, for $n \geq 1$,

$$Y_n = \begin{cases} Y_{n-1} + \xi_n & \text{if} \quad Y_{n-1} > 0 \quad \text{and} \quad Y_{n-1} + \xi_n > 0, \\ 0 & \text{if} \quad Y_{n-1} > 0 \quad \text{and} \quad Y_{n-1} + \xi_n \leq 0, \\ \eta_n & \text{if} \quad Y_{n-1} = 0 \end{cases} \tag{4}$$

where $(\eta_k)_{k\geq 1}$ is a sequence of i.i.d. $\mathbb{N}$-valued random variables, independent of the sequence $(\xi_n)_{n\geq 1}$. This process $(Y_n)_{n\geq 0}$ is a variation of the so-called *Lindley process* $(L(n))_{n\geq 0}$, which appears in queuing theory and corresponds to the case when the random variables ξ_n and η_n are equal. Recall that the Lindley process is a fundamental example of processes governed by iterated functions systems ([7], [18]); indeed, setting $f_a(x) := \max(x+a, 0)$ for any $a, x \in \mathbb{R}$, the quantity $L_n(0)$ equals (recall $\xi_n = \eta_n$ for the Lindley process)

$$L_n(0) = f_{\xi_n} \circ \cdots \circ f_{\xi_2} \circ f_{\xi_1}(0).$$

The composition of the maps f_a is not commutative, this introduces several difficulties. Namely, between two consecutive visits of 0, the Lindley process behaves like a classical r.w. $(S_n)_{n\geq 1}$; thus, its study relies on fluctuations of r.w. on $\mathbb{Z}$. In the case of $(Y_n)_{n\geq 0}$, the fact that after each visit of 0 the increment is governed by another distribution, the one of the η_k, introduces some kind of inhomogeneity we have to control.

Corollary 1.1. *Assume that*

(1) $(\xi_n)_{n\geq 1}$ *is a sequence of* $\mathbb{Z}$*-valued i.i.d. random variables, and* $\mathbb{E}[\xi_n] = 0$ *and* $\mathbb{E}[\xi_n^2] = \sigma^2 < \infty$*;*

[2] The Bernoulli symmetric r.w. is not aperiodic and thus does not fall exactly within the scope of Theorem 1.1; nevertheless, a similar statement does exist in this case, taking into account the different cyclic classes.

(2) *the support of the distribution of ξ_n is not included in the coset of a proper subgroup of $\mathbb{Z}$;*
(3) *$(\eta_n)_{n\geq 1}$ is a sequence of $\mathbb{N}$-valued and i.i.d. random variables such that $\mathbb{E}[\eta_n] < +\infty$, and $\mathbb{P}[\eta_n = 0] < 1$;*
(4) *the sequences $(\xi_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ are independent.*

Let $(Y(t))_{t\geq 0}$ be the continuous time process constructed from the sequence $(Y_n)_{n\geq 0}$ by linear interpolation between the values at integer points. Then, as $n \to +\infty$, the sequence of stochastic processes $(Y_t^{(n)})_{n\geq 1}$, defined by

$$Y_t^{(n)} := \frac{1}{\sigma\sqrt{n}} Y(nt), \quad n \geq 1, 0 \leq t \leq 1,$$

weakly converges in the space of continuous functions on $[0,1]$ to the absolute value $(|B(t)|)_{t\geq 0}$ of the Brownian motion on $\mathbb{R}$.

The Lindley process, with "one-sided reflection", has been really studied in the literature, it is connected with queuing theory.

Let us conclude this introduction describing briefly our approach. In our model, the times of visits of 0 form a renewal process and excursions of $(X_n)_{n\geq 0}$ away from zero are positive with probability $\mathbb{P}[\eta_1 > 0]$ and negative with probability $\mathbb{P}[\eta_1 < 0]$. Furthermore, the distributions of the typical durations $\tau^{\pm}$ of these excursions satisfy the following tail property

$$\mathbb{P}[\tau^+ > n] \sim c_1 \frac{\mathbb{E}[h(\eta_1) \mid \eta_1 > 0]}{\sqrt{n}} \quad \text{and} \quad \mathbb{P}[\tau^- > n] \sim c_1' \frac{\mathbb{E}[h'(\eta_1) \mid \eta_1 < 0]}{\sqrt{n}}.$$

This implies that the probability $\mathbb{P}[X_n > 0]$ that the current excursion is positive converges to the value α given by formula (2). Now, the renewal theory applies and yields to precise informations on the limiting distribution of the time σ_n of last zero before time n; given σ_n, the quantity X_n is the value of an unexpired positive excursion at time $n - \sigma_n$ with probability $\mathbb{P}[\eta_1 > 0]$ and that of an unexpired negative excursion at time $n - \sigma_n$ with probability $\mathbb{P}[\eta_1 < 0]$.

We detail this approach using the classical strategy to prove invariance principles, described in the seminal book [4]: first, we check that the finite dimensional distribution of the processes $(X_n)_{n\geq 0}$ do converge to the suitable limit, then the tightness of these sequences of processes. This "pedestrian" approach is of interest as soon as we have a precise control of both the fluctuations of the r.w. on each half line $\mathbb{Z}^-$ and $\mathbb{Z}^+$ and the steps starting from 0. In particular, it is quite flexible and can be used as soon as the successive returns to a suitable reference set or to some configuration are sufficiently well controlled, i.e. when the probability to return exactly at time n to this subset or to this configuration behaves like a renewal sequence. Let us cite for instance

- processes whose trajectories cannot be decomposed exactly into independent pieces, as for instance the reflected r.w. on $\mathbb{Z}^+$ with elastic reflections at 0;
- Lindley processes in dimension ≥ 2 (see [18] and references therein);
- others inhomogeneous random processes obtained by local perturbation of r.w. [17], [22].

In these different cases, which are still open, many new difficulties do appear; hence, it is important to fix first the details of this approach for processes whose decomposition with renewal sequences is quite well understood.

The paper is organized as follows: in Section 2 we recall some classical results on the theory of fluctuations of r.w. and their consequences, Section 3 is devoted to the proof of Theorem 1.1.

2. Notations and auxiliary estimates

We consider sequences $\xi = (\xi_k)_{k\geq 1}$, $\xi' = (\xi'_k)_{k\geq 1}$ and $\eta = (\eta_k)_{k\geq 1}$ of i.i.d. $\mathbb{Z}$-valued random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the sequences ξ, ξ' and η are independent. We denote

- $\mathcal{F}_n$ is the σ-field generated by the random variables $\xi_1, \xi'_1, \eta_1, \ldots, \xi_n, \xi'_n, \eta_n$, for any $n \geq 1$. By convention $\mathcal{F}_0 = \{\emptyset, \Omega\}$;
- $S = (S_n)_{n\geq 0}$ the random walk starting from 0 and with jumps ξ_k, defined by

$$S_0 = 0 \quad \text{and} \quad S_n = \xi_1 + \ldots + \xi_n \quad \text{for} \quad n \geq 1,$$

and

- $(\ell_l)_{l\geq 0}$ its sequence of "descending ladder epochs" defined inductively by $\ell_0 = 0$ and,

$$\ell_{l+1} = \min\{n > \ell_l \mid S_n < S_{\ell_l}\}$$

for any $l \geq 1$ with the convention $\inf \emptyset = +\infty$;

- $S' = (S'_n)_{n\geq 0}$ the r.w. starting from 0 with jumps ξ'_k and $(\ell'_l)_{l\geq 0}$ its sequence of "ascending ladder epochs" defined inductively by $\ell'_0 = 0$ and, for any $l \geq 1$,

$$\ell'_{l+1} = \min\{n > \ell_l \mid S'_n > S'(\ell_l)\};$$

It is well known that, if $\mathbb{E}[|\xi_k|], \mathbb{E}[|\xi'_k|] < +\infty$ and $\mathbb{E}[\xi_k] = \mathbb{E}[\xi'_k] = 0$, then the r.w. S and S' are $\mathbb{P}$-a.s. oscillating between $+\infty$ and $-\infty$. Hence, the random variables ℓ_l and ℓ'_l are all $\mathbb{P}$-a.s. finite. Furthermore, the random variables $\ell_1, \ell_2 - \ell_1, \ell_3 - \ell_2, \ldots$ are i.i.d. (similarly, $\ell'_1, \ell'_2 - \ell'_1, \ell'_3 - \ell'_2, \ldots$ are i.i.d.).

The same property holds for the random variables $S_{\ell_1}, S_{\ell_2} - S_{\ell_1}, S_{\ell_3} - S_{\ell_2}, \ldots$ on the one hand and $S'_{\ell'_1}, S'_{\ell'_2} - S'_{\ell'_1}, S'_{\ell'_3} - S'_{\ell'_2}, \ldots$ on the other hand. In other words, the processes $(\ell_l)_{l\geq 0}$ and $(S_{\ell_l})_{l\geq 0}$ (resp. $(\ell'_l)_{l\geq 0}$ and $(S'_{\ell_l})_{l\geq 0}$) are random walks on $\mathbb{N}$ and $\mathbb{Z}$ with distribution $\mathcal{L}(\ell_1)$ and $\mathcal{L}(S_{\ell_1})$ (resp. $\mathcal{L}(\ell'_1)$ and $\mathcal{L}(S'_{\ell'_1})$).

At last, when $\mathbb{E}[\xi_k^2] < +\infty$ (resp. $\mathbb{E}[(\xi')_k^2] < +\infty$), the random variables $\mathbb{S}_{\ell_1}$ (resp. $\mathbb{S}'_{\ell'_1}$) has finite first moment. In the next section, we recall some general results on fluctuations of r. w. on $\mathbb{Z}$; the useful tools are introduced in good time.

Now, let us introduce some notations concerning the process X. Since the random variables ξ_k and ξ'_k are centered, this process $(X_n)_{n\geq 0}$ visits 0 infinitely often; the successive visit times of 0 are defined by $\tau_0^X = 0$ and, for any $l \geq 0$,

$$\tau_{l+1}^X = \inf\{n > \tau_l^X \mid X_n = 0\}.$$

The $\tau_l^X, l \geq 0$, are stopping times with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$.

In order to establish an invariance principle for the process $(X_n)_{n\geq 0}$, we control the excursions of this process between two visits of 0; after each visit, the first transition jump has distribution $\mathcal{L}(\eta_1)$, it is thus natural to introduce the following random variables $U_{k,n}$ and $U'_{k,n}, 0 \leq k \leq n$, defined by:

$$\text{for} \quad n > k \geq 1, \qquad \begin{cases} U_{k,n} &= \eta_{k+1} + \xi_{k+2} + \ldots + \xi_n \text{ when } \eta_{k+1} > 0; \\ U'_{k,n} &= \eta_{k+1} + \xi'_{k+2} + \ldots + \xi'_n \text{ when } \eta_{k+1} < 0. \end{cases}$$

The jumps η_k introduce some spatial inhomogeneity and force us to take into account that the random walks S and S' may start from another point than 0; hence, for any $x \geq 0$, we set

$$\tau^S(x) := \inf\{n \geq 1 : x + S_n \leq 0 \quad \text{and} \quad \tau^{S'}(-x) := \inf\{n \geq 1 : -x + S'_n \geq 0\}.$$

At last, not to encumber the text, we use the following notations. For any sequences of positive reals $u = (u_n)_{n\geq 0}$ and $v = (v_n)_{n\geq 0}$, we write

- $u \overset{c}{\preceq} v$ (or simply $u \preceq v$) when $u_n \leq c v_n$ for some constant $c > 0$ and n large enough;

- $u_n \sim v_n$ if u is “similar” to v i.e. $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$.
- $u_n \approx v_n$ if u is ”approximately equal” to v i.e. $\lim_{n\to\infty}(u_n - v_n) = 0$.

2.1. **On the fluctuations of random walks.** The excursions of $(X_n)_{n\geq 0}$ between two visits of 0 coincide with some parts of the trajectories of $(S_n)_{n\geq 0}$ or $(S'_n)_{n\geq 0}$, suitably shifted. Their study is based on the theory of fluctuations of these random walks.

Let h be the Green function of the r.w. $(S_{\ell_j})_{j\geq 0}$, called sometimes the “descending renewal function” of S, defined by

$$h(x) = \begin{cases} 1 + \displaystyle\sum_{j=1}^{+\infty} \mathbb{P}[S_{\ell_j} \geq -x] & \text{if} \quad x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The function h is harmonic for the r.w. $(S_n)_{n\geq 0}$ killed when it reaches the negative half line $(-\infty; 0]$; namely, for any $x \geq 0$,

$$\mathbb{E}[h(x + \xi_1); x + \xi_1 > 0] = h(x).$$

This holds for any oscillating r.w., possibly without finite second moment.

Similarly, the “ascending renewal function” h' of S' is defined by

$$h'(x) = 1 + \sum_{j=1}^{+\infty} \mathbb{P}[S'_{\ell'_j} \leq x] \quad \text{if} \quad x \geq 0 \quad \text{and} \quad h'(x) = 0 \quad \text{otherwise.}$$

The function h' is increasing and $h'(x) = O(x)$ as $x \to \infty$.

Let us now state some classical results in the theory of fluctuations; we refer to [14], [16] and [9] for elements of the proofs. Recall that $c_1 = \dfrac{\mathbb{E}[-S_{\ell_1}]}{\sigma\sqrt{2\pi}}$ and $c'_1 = \dfrac{\mathbb{E}[-S'_{\ell'_1}]}{\sigma\sqrt{2\pi}}$.

Lemma 2.1. *For any $x \geq 0$, it holds, as $n \to +\infty$,*

$$\mathbb{P}[\tau^S(x) = n] \sim c_1 \, \frac{h(x)}{n^{3/2}}, \quad \textit{and} \quad \mathbb{P}[\tau^{S'}(-x) = n] \sim c'_1 \, \frac{h'(x)}{n^{3/2}}.$$

Furthermore, there exists a constant $C_1 > 0$ such that, for any $x \geq 0$ and $n \geq 1$,

$$\mathbb{P}[\tau^S(x) = n] \leq C_1 \frac{h(x)}{n^{3/2}} \quad \textit{and} \quad \mathbb{P}[\tau^{S'}(-x) = n] \leq C_1 \frac{h(x)}{n^{3/2}}.$$

As a direct consequence, it holds for any $x \geq 0$,

$$\mathbb{P}[\tau^S(x) > n] \sim 2c_1 \, \frac{h(x)}{\sqrt{n}}, \quad \text{and} \quad \mathbb{P}[\tau^{S'}(-x) > n] \sim 2c'_1 \, \frac{h'(x)}{\sqrt{n}}.$$

The following local limit theorem provides a more precise result. We only state it for the r.w. S and denote by $\tilde{h}$ the ascending renewal function of S, that is also the descending renewal function of the r.w. $\tilde{S} = -S$ defined as in (5).

Lemma 2.2. *For any $x, y \geq 0$, as $n \to +\infty$,*

$$\mathbb{P}[\tau^S(x) > n, x + S_n = y] \sim \frac{1}{\sigma\sqrt{2\pi}} \frac{h(x)\tilde{h}(y)}{n^{3/2}},$$

and there exists a constant $C_2 > 0$ such that, for any $x, y \geq 0$ and $n \geq 1$,

$$\mathbb{P}[\tau^S(x) > n, x + S_n = y] \leq C_2 \frac{h(x)\tilde{h}(y)}{n^{3/2}}.$$

2.2. **On the fluctuations of the perturbed random X.** The sequence $(\tau_l^X)_{l\geq 0}$ is a strictly increasing r.w. on $\mathbb{N}$, with i.i.d. increments distributed as τ_1^X. Thus, its potential $\sum_{l=1}^{+\infty}\mathbb{P}[\tau_l^X = n]$ is finite for any $n \geq 0$. In this subsection, we describe the asymptotic behavior of the distribution of τ_1^X and the corresponding Green function.

Lemma 2.3. *It holds, as $n \to +\infty$,*

$$\mathbb{P}[\tau_1^X = n] \sim \frac{c_1\mathbb{E}[h(\eta_1)\mathbf{1}_{\{\eta_1>0\}}] + c_1'\mathbb{E}[h'(-\eta_1)\mathbf{1}_{\{\eta_1<0\}}]}{n^{3/2}}. \tag{6}$$

Proof. For any $n \geq 2$,

$$\begin{aligned}\mathbb{P}[\tau_1^X = n] &= \mathbb{P}[U_{0,1} = \eta_1 > 0, U_{0,2} > 0, \dots, U_{0,n-1} > 0, U_{0,n} \leq 0]\\ &\quad + \mathbb{P}[U'_{0,1} = \eta_1 < 0, U'_{0,2} < 0, \dots, U'_{0,n-1} < 0, U'_{0,n} \geq 0]\\ &= \sum_{x\geq 1}\mathbb{P}[\eta_1 = x]\mathbb{P}[\tau^S(x) = n-1] + \sum_{x\geq 1}\mathbb{P}[\eta_1 = -x]\mathbb{P}[\tau^{S'}(-x) = n-1]\end{aligned}$$

and the statement is a direct consequence of Lemma 2.1. $\square$

Lemmas 2.1 and 2.3 yield to the alternative representation of α, announced in the introduction (see (3)) :

$$\alpha = \lim_{n\to\infty}\frac{\mathbb{P}\left[\tau_1^X > n, \eta_1 > 0\right]}{\mathbb{P}\left[\tau_1^X > n\right]}.$$

Lemma 2.3 also leads to the behavior at infinity of Green function of the r.w. $(\tau_l)_{l\geq 0}$. We set $\Sigma_n^X := \sum_{l=0}^{+\infty}\mathbb{P}[\tau_l^X = n]$ for any $n \geq 0$.

Lemma 2.4. *As $n \to +\infty$,*

$$\Sigma_n^X \sim \frac{1}{c_1\mathbb{E}[h(\eta_1)\mathbf{1}_{\{\eta_1>0\}}] + c_1'\mathbb{E}[h'(-\eta_1)\mathbf{1}_{\{\eta_1<0\}}]}\frac{1}{2\pi\sqrt{n}}. \tag{7}$$

Proof. We apply Theorem B in [8] [3] and have to check that

$$\sup_{n\geq 0}\frac{n\mathbb{P}[\tau_1^X = n]}{\mathbb{P}[\tau_1^X > n]} < +\infty.$$

This is a direct consequence of Lemma 2.3. $\square$

2.3. **Conditional limit theorem.** Let $(S_t)_{t\geq 0}$ be the continuous time process constructed from the sequence $(S_n)_{n\geq 0}$ by using the linear interpolation between the values at integer points.Then, by Lemma 2.3 in [2],

$$\mathcal{L}\Big(\left(\frac{S_{[nt]}}{\sigma\sqrt{n}}\right)_{0\leq t\leq 1}|\tau^S(x) > n\Big) \Rightarrow \mathcal{L}(L^+) \quad \text{as } n \to +\infty,$$

where L^+ is the Brownian meander (the symbol " $\Rightarrow$ " means "weak convergence").

As a direct consequence, for any bounded Lipschitz continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ and any $t \in [0,1]$ and $x \geq 0$,

$$\lim_{n\to+\infty}\mathbb{E}\left[\varphi\left(\frac{x + S_{[nt]}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x) > [nt]\right] = \frac{1}{t}\int_0^{+\infty}\varphi(u)\ u\ e^{-u^2/2t}du. \tag{8}$$

Let us also state the Caravenna-Chaumont's result [6] concerning limit theorem of random bridges conditioned to stay positive. This is here that the fact that η and ξ are

[3]This corresponds to Theorem B in [8] where the RHS term of (1.10) should be replaced by $1/\eta(\alpha)\eta(1-\alpha)$

integer valued is used. We could also consider the case when η and ξ have absolutely continuous distributions as well; in both cases, it is possible to fix the arrival point at time n of the r.w. S and to consider its rescaled limit as $n \to +\infty$. By Corollary 2.5 in [6], the r.w. bridge conditioned to stay positive, starting at $x \geq 0$ and ending at $y \geq 0$, under linear interpolation and diffusive rescaling, converges in distribution on $C([0,1], \mathbb{R})$ toward the normalized Brownian excursion $\mathcal{E}^+$. In other words,

$$\mathcal{L}\Big(\left(\frac{S_{[nt]}}{\sigma\sqrt{n}}\right)_{0\leq t\leq 1} | \tau^S(x) > n, S_n = y\Big) \Rightarrow \mathcal{L}(\mathcal{E}^+) \quad \text{as } n \to +\infty,$$

As a direct consequence, for any bounded Lipschitz continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ and any $t \in [0,1]$ and $x, y \geq 0$,

$$\begin{aligned}\lim_{n\to+\infty} \mathbb{E}\Big[\varphi\left(\frac{x+S_{[nt]}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x) > n, x+S_n = y\Big]& \\ = \frac{2}{\sqrt{2\pi}\sqrt{t^3(1-t)^3}} \int_0^{+\infty} \varphi(u) u^2 e^{-u^2/2t(1-t)} du.& \end{aligned} \tag{9}$$

This yields also to the following convergence, which is not explicitly stated in [6]. For the sake of completeness, we detail the proof.

Lemma 2.5. *For any bounded Lipschitz continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ and any $t \in [0,1]$ and $x \geq 0$,*

$$\lim_{n\to+\infty} \mathbb{E}\Big[\varphi\left(\frac{x+S_{[nt]}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x) = n\Big] = \frac{2}{\sqrt{2\pi}\sqrt{t^3(1-t)^3}} \int_0^{+\infty} \varphi(u) u^2 e^{-u^2/2t(1-t)} du.$$

Proof. For any $n \geq 1$, it holds

$$\begin{aligned}
&\mathbb{E}\Big[\varphi\left(\frac{x+S_{[nt]}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x) = n\Big] \\
&= \frac{1}{\mathbb{P}[\tau^S(x) = n]} \sum_{y=1}^{+\infty} \mathbb{P}[y + \xi_n \leq 0] \times \mathbb{E}\left[\varphi\left(\frac{x+S_{[nt]}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x) > n-1, x + S(n-1) = y\right] \\
&\qquad = \frac{1}{\mathbb{P}[\tau^S(x) = n]} \sum_{y=1}^{+\infty} \mathbb{P}[y + \xi_1 \leq 0] \mathbb{P}[\tau^S(x) > n-1, x + S(n-1) = y] \\
&\qquad\qquad \times \mathbb{E}\left[\varphi\left(\frac{x+S_{[nt]}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x) > n-1, x + S(n-1) = y\right]
\end{aligned}$$

Now, the sequence $\left(\mathbb{E}\left[\varphi\left(\frac{x+S_{[nt]}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x) > n-1, x + S(n-1) = y\right]\right)_{n\geq 0}$ is bounded by $|\varphi|_\infty$; it also converges as $n \to +\infty$ to the RHS term in (9).
Furthermore, by Lemmas 2.2, as $n \to +\infty$,

$$\frac{1}{\mathbb{P}[\tau^S(x) = n]} \mathbb{P}[y + \xi_1 \leq 0] \mathbb{P}[\tau^S(x) > n-1, x + S(n-1) = y] \longrightarrow \frac{\tilde{h}(y)}{c_1\sigma\sqrt{2\pi}} \mathbb{P}[y + \xi_1 \leq 0]$$

and there exists $C(x) > 0$ such that

$$\frac{1}{\mathbb{P}[\tau^S(x) = n]} \mathbb{P}[y + \xi_1 \leq 0] \mathbb{P}[\tau^S(x) > n-1, x + S(n-1) = y] \leq C(x)\tilde{h}(y) \mathbb{P}[y + \xi_1 \leq 0]$$

with $\sum_{y\geq 1} \tilde{h}(y)\mathbb{P}[y+\xi_n \leq 0] < +\infty$. We achieve the proof, combining the dominated convergence theorem with the identity, valid for any $x \geq 0$ and $n \geq 1$:

$$\mathbb{P}[\tau^S(x) = n] = \sum_{y\geq 1} \mathbb{P}[y+\xi_1 \leq 0]\mathbb{P}[\tau^S(x) > n-1, x+S(n-1) = y].$$

□

3. Proof of Theorem 1.1

The proof uses the classical approach for weak convergence in the space $C[0,1]$ of continuous functions on $[0,1]$ (see [4], chapter 2). Firstly, we prove that the finite dimensional distributions of the process $(X_t^{(n)}; 0 \leq t \leq 1)$ converge weakly to those of the skew Brownian motion $(B^\alpha(t); 0 \leq t \leq 1)$ on $\mathbb{R}$, then the tightness of the distributions of this sequence of processes.

Throughout this section, the functions φ, φ_1 and φ_2 are non negative and Lipschitz continuous and have a compact support; we denote $[\varphi]$, $[\varphi_1]$, and $[\varphi_2]$ their respective Lispchitz coefficient.

We assume $\sigma = \sigma'$, in order to avoid to have to consider several cases at any step, according to the sign of the excursion we study.

3.1. Convergence of the one-dimensional distributions.

In this section, we prove the following lemma.

Lemma 3.1. *For any $t \in [0,1]$, it holds*

$$\lim_{n\to+\infty} \mathbb{E}\left[\varphi\left(X_t^{(n)}\right)\right] = \int_{-\infty}^{+\infty} \varphi(u) p_t^\alpha(0,u)du. = \int_{-\infty}^{+\infty} \tilde{\varphi}(u)\frac{2e^{-u^2/2t}}{\sqrt{2\pi t}}du,$$

where $\tilde{\varphi}(u) := \alpha\varphi(u)\mathbf{1}_{[u>0]} + (1-\alpha)\varphi(u)\mathbf{1}_{[u<0]}$.

Proof. We fix $t \in (0,1)$. Notice first that

$$\left|\mathbb{E}\left[\varphi\left(\frac{X_{[nt]}}{\sigma\sqrt{n}}\right)\right] - \mathbb{E}\left[\varphi\left(X_t^{(n)}\right)\right]\right| \leq [\varphi]\mathbb{E}\left[\left|\frac{X_{[nt]}}{\sigma\sqrt{n}} - X_t^{(n)}\right|\right]$$

with

$$\mathbb{E}\left[\left|\frac{X_{[nt]}}{\sigma\sqrt{n}} - X_t^{(n)}\right|\right] \leq \frac{\mathbb{E}\left[|\xi_{[nt]+1}| + |\eta_{[nt]+1}| + |\xi'_{[nt]+1}| + |\eta'_{[nt]+1}|\right]}{\sigma\sqrt{n}} \to 0 \text{ as } n \to +\infty.$$

Hence, it suffices to prove that, as $n \to +\infty$,

$$A_n^+ := \mathbb{E}\left[\varphi\left(\frac{X_{[nt]}}{\sigma\sqrt{n}}\right); X_{[nt]} > 0\right] \longrightarrow \alpha \int_0^{+\infty} \varphi(u)\frac{2e^{-u^2/2t}}{\sqrt{2\pi t}}du$$

and

$$A_n^- := \mathbb{E}\left[\varphi\left(\frac{X_{[nt]}}{\sigma'\sqrt{n}}\right); X_{[nt]} < 0\right] \longrightarrow (1-\alpha) \int_{-\infty}^0 \varphi(u)\frac{2e^{-u^2/2t}}{\sqrt{2\pi t}}du.$$

We decompose A_n^+ as

$$\begin{aligned}
A_n^+ &= \sum_{k=0}^{[nt]}\sum_{l=0}^{+\infty}\mathbb{E}\left[\varphi\left(\frac{X_{[nt]}}{\sigma\sqrt{n}}\right); \tau_l^X = k, X_{k+1}>0,\dots,X_{[nt]}>0\right]\\
&\approx \sum_{k=0}^{[nt]-1}\sum_{l=0}^{+\infty}\mathbb{E}\left[\varphi\left(\frac{X_{[nt]}}{\sigma\sqrt{n}}\right); \tau_l^X = k, X_{k+1}>0,\dots,X_{[nt]}>0\right]\\
&= \sum_{k=0}^{[nt]-1}\sum_{l=0}^{+\infty}\mathbb{E}\Big[\varphi\left(\frac{\eta_{k+1}+\xi_{k+2}+\ldots+\xi_{[nt]}}{\sigma\sqrt{n}}\right); \tau_l^X = k, \eta_{k+1}>0,\\
&\qquad\qquad \eta_{k+1}+\xi_{k+2}>0,\dots,\eta_{k+1}+\xi_{k+2}+\ldots+\xi_{[nt]}>0\Big].
\end{aligned}$$

The event $[\tau_l^X = k]$ is independent on the variables $\eta_{k+1}, \xi_{k+2}, \dots, \xi_{[nt]}$; furthermore

$$\mathcal{L}(\eta_{k+1}, \xi_{k+2}, \dots, \xi_{[nt]}) = \mathcal{L}(\eta_1, \xi_1, \dots, \xi_{[nt]-k-1}).$$

Hence, recalling that $\Sigma_k^X = \sum_{l=0}^{+\infty}\mathbb{P}[\tau_l^Y = k]$,

$$\begin{aligned}
A_n^+ &\approx \sum_{k=0}^{[nt]-1}\Sigma_k^X\mathbb{E}\Big[\varphi\left(\frac{\eta_1+S([nt]-k-1)}{\sigma\sqrt{n}}\right); \eta_1>0, \eta_1+S(1)>0,\dots,\\
&\qquad\qquad \eta_1+S_{[nt]-k-1}>0\Big]\\
&= \sum_{k=0}^{[nt]-1}\Sigma_k^X\mathbb{E}\left[\mathbb{E}\left(\varphi\left(\frac{x+S_{[nt]-k-1})}{\sigma\sqrt{n}}\right); \tau^S(x)>[nt]-k-1\right)\Big|_{\eta_1=x>0}\right]\\
&= \sum_{k=0}^{[nt]-1}\Sigma_k^X\mathbb{E}\Big[\mathbb{E}\left(\varphi\left(\frac{x+S_{[nt]-k-1})}{\sigma\sqrt{n}}\right)\Big|\tau^S(x)>[nt]-k-1\right)\\
&\qquad\qquad \times\mathbb{P}[\tau^S(x)>[nt]-k-1]\Big|_{\eta_1=x>0}\Big],
\end{aligned}$$

with the notation $\mathbb{E}[\phi(x)|_{\eta_1=x>0}] := \mathbb{E}[\phi(\eta_1)\mathbf{1}_{[\eta_1>0]}]$ for any bounded function $\phi:\mathbb{R}\to\mathbb{R}$. By Lemmas 2.1 and 2.4, each term in this sum is $\mathcal{O}(\frac{1}{\sqrt{n}})$, hence,

$$A_n^+ = \int_0^t f_n^+(s)ds + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$$

where the function $f_n : (0,t)\to\mathbb{R}$ is defined by:

- for $2\le k\le \dots, [nt]-4$ and $s\in[\frac{k}{n}, \frac{k+1}{n})$,

$$\begin{aligned}
f_n^+(s) &= n\Sigma_k^X\mathbb{E}\Big[\mathbb{E}\left(\varphi\left(\frac{x+S_{[nt]-k-1})}{\sigma\sqrt{n}}\right)\Big|\tau^S(x)>[nt]-k-1\right)\\
&\qquad\qquad \times\mathbb{P}[\tau^S(x)>[nt]-k-1]\Big|_{\eta_1=x>0}\Big]\\
&= n\Sigma_k^X\mathbb{E}\Big[\mathbb{E}\left(\varphi\left(\frac{x+S_{[nt]-[ns]-1}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x)>[nt]-[ns]-1\right)\\
&\qquad\qquad \times\mathbb{P}[\tau^S(x)>[nt]-[ns]-1]\Big|_{\eta_1=x>0}\Big],
\end{aligned}$$

- $f_n^+(s) = 0$ on $[0, \frac{2}{n})$ and $[\frac{[nt]-1}{n}, t)$.

By using Lemmas 2.1, 2.4, equality (8) and Lebesgue dominated convergent theorem, we get: for any $s \in (0,t)$,

$$\lim_{n\to+\infty} f_n^+(s) = \frac{\alpha}{\pi}\,\frac{1}{t\sqrt{s(t-s)}}\frac{1}{t-s}\int_0^{+\infty}\varphi(y)ye^{-y^2/2(t-s)}dy.$$

On the other hand, for $2 \le [ns] \le [nt]-4$,

$$\begin{aligned}|f_n^+(s)| &\le C_1|\varphi|_\infty \mathbb{E}[h(\eta_1)]\frac{n}{\sqrt{[ns]([nt]-[ns]-1)}}\sqrt{[ns]}\Sigma^X_{[ns]}\\ &\preceq \frac{n}{\sqrt{[ns]([nt]-[ns]-1)}}\\ &\le \frac{2}{\sqrt{s(t-s)}}\end{aligned}$$

(it is in the last inequality that we use the fact that $2 \le [ns] \le [nt]-4$). Therefore, the Lebesgue dominated convergence theorem yields

$$\begin{aligned}\lim_{n\to+\infty} A_n^+ &= \lim_{n\to+\infty}\int_0^t f_n^+(s)ds\\ &= \frac{\alpha}{\pi}\int_0^t \frac{1}{\sqrt{s(t-s)}}\left(\frac{1}{t-s}\int_0^{+\infty}\varphi(y)ye^{-y^2/2(t-s)}dy\right)ds.\end{aligned}$$

By using the change of variable $a = 1-\frac{s}{t}$ with $0<a<1$, we obtain

$$\lim_{n\to+\infty} A_n^+ = \frac{\alpha}{\pi t}\int_0^{+\infty}\varphi(y)y\left(\int_0^1 \frac{1}{\sqrt{a^3(1-a)}}e^{-y^2/2at}da\right)dy.$$

Note that, for any $\lambda,\mu>0$

$$\int_0^{+\infty}\frac{1}{\sqrt{t}}\exp\Big(-\lambda t-\frac{\mu}{t}\Big)dt = \sqrt{\frac{\pi}{\lambda}}e^{-2\sqrt{\lambda\mu}} \tag{10}$$

(see for instance [13] p.17). Using this identity and the change of variable $x = \frac{1-a}{a}$, we get $\displaystyle\int_0^1 \frac{1}{\sqrt{a^3(1-a)}}e^{-\lambda/a}da = \sqrt{\frac{\pi}{\lambda}}e^{-\lambda}$ for any $\lambda>0$. Hence,

$$\lim_{n\to+\infty} A_n^+ = \alpha\int_0^{+\infty}\varphi(y)\frac{2e^{-y^2/2t}}{\sqrt{2\pi t}}dy.$$

Similarly, $\displaystyle\lim_{n\to+\infty} A_n^- = (1-\alpha)\int_{-\infty}^0\varphi(y)\frac{2e^{-y^2/2t}}{\sqrt{2\pi t}}dy$. This achieves the proof. □

3.2. **Finite-dimensional distribution.** The convergence of the finite-dimensional distributions of $(X_t^{(n)})_{n\ge1}$ is more delicate. We detail the argument for two-dimensional ones, the general case may be treated in a similar way. We fix two functions $\varphi_1,\varphi_2 : \mathbb{R}\to\mathbb{R}$ with compact support and set, for any $0<s<t$ and $n\ge1$,

$$\kappa = \kappa(n,s) = \min\{k>[ns] : X_k = 0\}.$$

We write

$$\mathbb{E}\left[\varphi_1(X_s^{(n)})\varphi_2(X_t^{(n)})\right] \approx \mathbb{E}\left[\varphi_1(X_{[ns]})\varphi_2(X_{[nt]})\right] = A_n + B_n, \tag{11}$$

where

$$A_n = \sum_{k=[ns]+1}^{[nt]}\mathbb{E}\left[\varphi_1(X_{[ns]})\varphi_2(X_{[nt]})\mathbf{1}_{[\kappa=k]}\right]$$

and
$$B_n = \mathbb{E}\left[\varphi_1(X_{[ns]})\varphi_2(X_{[nt]})\mathbf{1}_{[\kappa>[nt]]}\right].$$
The term A_n deals with the trajectories of $X_k, 0 \leq k \leq n$, which visit 0 between $[ns]+1$ and $[nt]$ while B_n concerns the others trajectories.

3.2.1. *Estimate of A_n.* As in the previous section, based on the sign of $X_{[ns]}$, we decompose A_n as $A_n^+ + A_n^-$, with
$$\begin{aligned} A_n^+ = &\sum_{k_1=0}^{[ns]} \Sigma_{k_1}^X \sum_{k_2=[ns]+1}^{[nt]} \mathbb{E}\left[\varphi_2\left(\frac{X_{[nt]-k_2}}{\sigma\sqrt{n}}\right)\right] \\ &\times \mathbb{E}\left[\mathbb{E}\left[\varphi_1\left(\frac{x+S_{[ns]-k_1-1}}{\sigma\sqrt{n}}\right); \tau^S(x) = k_2-k_1-1\right]\Big|_{\eta_1=x>0}\right], \end{aligned}$$

and
$$\begin{aligned} A_n^- = &\sum_{k_1=0}^{[ns]} \Sigma_{k_1}^X \sum_{k_2=[ns]+1}^{[nt]} \mathbb{E}\left[\varphi_2\left(\frac{X_{[nt]-k_2}}{\sigma'\sqrt{n}}\right)\right] \\ &\times \mathbb{E}\left[\mathbb{E}\left[\varphi_1\left(\frac{x+S'_{[ns]-k_1-1}}{\sigma'\sqrt{n}}\right); \tau^{S'}(x) = k_2-k_1-1\right]\Big|_{\eta_1=x<0}\right]. \end{aligned}$$
Let us focus on the term A_n^+. It holds
$$\begin{aligned} A_n^+ \approx &\sum_{l=0}^{+\infty}\sum_{k_1=0}^{[ns]-1} \sum_{k_2=[ns]+1}^{[nt]} \mathbb{E}\left[\varphi_2\left(\frac{X_{[nt]-k_2}}{\sigma\sqrt{n}}\right)\right] \\ &\times \mathbb{E}\left[\varphi_1\left(\frac{S_{[ns]})}{\sigma\sqrt{n}}\right); \tau_l^X = k_1, \kappa = k_2; \eta_{k_1+1} > 0, U_{k_1,k_1+2} > 0, \ldots, U_{k_1,[ns]} > 0\right] \\ = &\sum_{l=0}^{+\infty}\sum_{k_1=0}^{[ns]-1} \sum_{k_2=[ns]+1}^{[nt]} \mathbb{P}[\tau_l^X = k_1]\mathbb{E}\left[\varphi_2\left(\frac{X_{[nt]-k_2}}{\sigma\sqrt{n}}\right)\right] \\ &\times \mathbb{E}\left[\varphi_1\left(\frac{X_{[ns]-k_1}}{\sigma\sqrt{n}}\right); \eta_1 > 0, U_{0,2} > 0, \ldots, U_{0,k_2-k_1-1} > 0, U_{0,k_2-k_1} \leq 0\right] \\ = &\sum_{k_1=0}^{[ns]-1} \Sigma_{k_1}^X \sum_{k_2=[ns]+1}^{[nt]} \mathbb{E}\left[\varphi_2\left(\frac{X_{[nt]-k_2}}{\sigma\sqrt{n}}\right)\right] \\ &\times \sum_{x\geq 1} \mathbb{E}\left[\varphi_1\left(\frac{x+S_{[ns]-k_1-1}}{\sigma\sqrt{n}}\right); \tau^S(x) = k_2-k_1-1\right]\mathbb{P}[\eta_1 = x]. \end{aligned}$$
For any $2 \leq k_1 < [ns]-6$ and $[ns] \leq k_2 \leq [nt]$ and any $s_1 \in [\frac{k_1}{n}, \frac{k_1+1}{n})$ and $s_2 \in [\frac{k_2}{n}, \frac{k_2+1}{n})$, we write
$$\begin{aligned} f_n^+(s_1,s_2) = &n^2\Sigma_{k_1}^X \mathbb{E}\left[\varphi_2\left(\frac{X_{[nt]-[ns_2]}}{\sigma\sqrt{n}}\right)\right] \\ &\times \sum_{x\geq 1} \mathbb{E}\left[\varphi_1\left(\frac{x+S_{[ns]-[ns_1]-1}}{\sigma\sqrt{n}}\right); \tau^S(x) = [ns_2]-[ns_1]-1\right]\mathbb{P}[\eta_1 = x] \end{aligned}$$
and $f_n^+(s_1,s_2) = 0$ for the others values of k_1, such that $0 \leq k_1 \leq [ns]$. Hence,
$$A_n^+ = \int_0^s ds_1 \int_s^t ds_2\ f_n^+(s_1,s_2) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$$
and we have to study the point wise limit of the sequence $(f_n^+)_{n\geq 1}$ as $n \to +\infty$.

On the one hand, by Lemma 3.1,

$$\lim_{n\to+\infty}\mathbb{E}\left[\varphi_2\left(\frac{X_{[nt]-[ns_2]}}{\sigma\sqrt{n}}\right)\right]=\int_{-\infty}^{+\infty}\varphi_2(z)p^{\alpha}_{t-s_2}(0,z)dz=\int_{-\infty}^{+\infty}\tilde{\varphi}_2(z)\frac{2e^{-z^2/2(t-s_2)}}{\sqrt{2\pi(t-s_2)}}dz.$$

where we set $\tilde{\varphi}_2(z)=\alpha\varphi_2(z)+(1-\alpha)\varphi_2(-z)$. On the other hand, one may write

$$\begin{aligned}
n^{3/2}\mathbb{E}\Big[\varphi_1&\left(\frac{x+S_{[ns]-[ns_1]-1}}{\sigma\sqrt{n}}\right);\tau^S(x)=[ns_2]-[ns_1]-1\Big]\\
&=\mathbb{E}\Big[\varphi_1\left(\frac{x+S_{[ns]-[ns_1]-1}}{\sigma\sqrt{n}}\right)\Big|\tau^S(x)=[ns_2]-[ns_1]-1\Big]\\
&\qquad\qquad\qquad\times n^{3/2}\mathbb{P}[\tau^S(x)=[ns_2]-[ns_1]-1\Big].
\end{aligned}$$

By Lemmas 2.4 and 2.5, as $n\to+\infty$, this quantity converges to

$$c_1h(x)\frac{2}{\sqrt{2\pi}}\frac{1}{\sqrt{(s-s_1)^3(s_2-s)^3}}\int_0^{+\infty}\varphi_1(y)y^2e^{-y^2/2\frac{(s-s_1)(s_2-s)}{(s_2-s_1)^2}}dy$$

and is dominated by $h(x)|\varphi|_\infty$, up to a multiplicative constant.

Finally, since $\mathbb{E}[h(\eta_1);\eta_1>0]<+\infty$, by the dominated convergence theorem, $(f_n^+)_{n\geq 1}$ converges to the function f^+ given by: for all $0<s_1<s_2<t$,

$$\begin{aligned}
f^+(s_1,s_2)=\frac{\alpha}{\pi^2\sqrt{s_1}}&\int_0^{+\infty}\varphi_1(y)\exp\left(-\frac{y^2}{2\frac{(s_2-s)(s-s_1)}{s_2-s_1}}\right)\frac{y^2}{\sqrt{(s-s_1)^3(s_2-s)^3}}dy\\
&\times\int_{-\infty}^{+\infty}\tilde{\varphi}_2(z)\frac{2e^{-z^2/2(t-s_2)}}{\sqrt{2\pi(t-s_2)}}dz.
\end{aligned}$$

The f_n^+ are also dominated as follows: for $2\leq k_1<[ns]-6$, $[ns]\leq k_2\leq[nt]$ and $s_1\in[\frac{k_1}{n},\frac{k_1+1}{n})$ and $s_2\in[\frac{k_2}{n},\frac{k_2+1}{n})$

$$|f_n^+(s_1,s_2)|\preceq\frac{n^2}{\sqrt{[ns_1]}([ns_2]-[ns_1]-2)^{3/2}}\preceq\frac{1}{\sqrt{s_1}(s_2-s_1)^{3/2}}$$

with $\displaystyle\int_0^s ds_1\int_s^t ds_2\frac{1}{\sqrt{s_1}(s_2-s_1)^{3/2}}<+\infty$. Hence

$$\begin{aligned}
A^+:=\lim_{n\to+\infty}A_n^+&=\int_0^s ds_1\int_s^t ds_2 f^+(s_1,s_2)\\
&=\frac{\alpha}{\pi^2}\int_0^s\frac{ds_1}{\sqrt{s_1}}\int_s^t ds_2\int_0^{+\infty}\varphi_1(y)\exp\left(-\frac{y^2}{2\frac{(s_2-s)(s-s_1)}{s_2-s_1}}\right)\\
&\qquad\times\frac{y^2}{\sqrt{(s-s_1)^3(s_2-s)^3}}dy\int_{-\infty}^{+\infty}\tilde{\varphi}_2(z)\frac{2e^{-z^2/2(t-s_2)}}{\sqrt{2\pi(t-s_2)}}dz
\end{aligned}$$

Setting $a=s_1/s$ and $b=(s_2-s)/(t-s)$yields

$$\begin{aligned}
A^+=\frac{\alpha}{\pi^2}\frac{1}{s(t-s)}&\int_0^{+\infty}dy\int_0^{+\infty}dz\int_0^1\frac{da}{\sqrt{a(1-a)^3}}\int_0^1\frac{db}{\sqrt{b^3(1-b)}}\\
\times\varphi_1(y)y^2&\exp\Big(-\frac{y^2}{2(t-s)b}\Big)\exp\Big(-\frac{y^2}{2s(1-a)}\Big)\tilde{\varphi}_2(z)\exp\Big(-\frac{z^2}{2(t-s)(1-b)}\Big).
\end{aligned}$$

With the change of variable $x=\frac{a}{1-a}$ and the identity (10),

$$\int_0^1\frac{1}{\sqrt{a(1-a)^3}}y\exp\Big(-\frac{y^2}{2s(1-a)}\Big)da=\sqrt{2\pi s}e^{-y^2/2s}. \tag{12}$$

Setting first $\beta = b/1-b$, then $c = 1/\beta$,

$$\begin{aligned} I &:= \int_0^1 \frac{y}{\sqrt{b^3(1-b)}} \exp\Big(-\frac{y^2}{2(t-s)b}\Big) \exp\Big(-\frac{z^2}{2(t-s)(1-b)}\Big) db \\ &= \exp\Big(-\frac{y^2+z^2}{2(t-s)}\Big) \int_0^{+\infty} \frac{y}{\beta^{3/2}} \exp\Big(-\frac{y^2}{2(t-s)\beta} - \frac{z^2\beta}{2(t-s)}\Big) d\beta \\ &= \exp\Big(-\frac{y^2+z^2}{2(t-s)}\Big) \int_0^{+\infty} \frac{y}{c^{1/2}} \exp\Big(-\frac{y^2 c}{2(t-s)c} - \frac{z^2}{2(t-s)c}\Big) dc. \end{aligned}$$

Using the identity (10) again yields

$$I = \exp\Big(-\frac{y^2+z^2}{2(t-s)}\Big)\sqrt{2(t-s)\pi}\exp\Big(-\frac{yz}{t-s}\Big) = \sqrt{2\pi(t-s)}\exp\Big(-\frac{(y+z)^2}{2(t-s)}\Big).$$

Hence,

$$A^+ = \frac{2\alpha}{\pi\sqrt{s(t-s)}} \int_0^{+\infty}\int_{-\infty}^{+\infty} \varphi_1(y)\tilde{\varphi}_2(z) e^{-y^2/2s} e^{-\frac{(y+z)^2}{2(t-s)}} dydz.$$

Similarly,

$$A^- := \lim_{n\to\infty} A_n^- = \frac{2(1-\alpha)}{\pi\sqrt{s(t-s)}} \int_{-\infty}^{0}\int_{-\infty}^{+\infty} \varphi_1(y)\tilde{\varphi}_2(z) e^{-y^2/2s} e^{-\frac{(y+z)^2}{2(t-s)}} dydz.$$

Eventually

$$A = \lim_{n\to+\infty} A_n = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \tilde{\varphi}_1(y)\tilde{\varphi}_2(z) e^{-y^2/2s} e^{-\frac{(y+z)^2}{2(t-s)}}. \tag{13}$$

3.2.2. *Estimate of B_n.* As above, we decompose B_n as $B_n^+ + B_n^-$, where

$$\begin{aligned} B_n^+ = \sum_{k\le[ns]}\sum_{l\ge 0} \mathbb{E}\Big[\varphi_1\left(\frac{X_{[ns]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt]}}{\sigma\sqrt{n}}\right);& \\ \tau_l^X = k, \eta_{k+1} > 0, U_{k,k+2} > 0, \dots, U_{k,[nt]} > 0\Big]& \end{aligned}$$

and

$$\begin{aligned} B_n^- = \sum_{k\le[ns]}\sum_{l\ge 0} \mathbb{E}\Big[\varphi_1\left(\frac{X_{[ns]}}{\sigma'\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt]}}{\sigma'\sqrt{n}}\right);& \\ \tau_l^X = k, \eta_{k+1} < 0, U'_{k,k+2} < 0, \dots, U'_{k,[nt]} < 0\Big]& \end{aligned}$$

(recall that $U_{k,n} = \eta_{k+1} + \xi_{k+2} + \dots + \xi_n$ and $U'_{k,n} = \eta_{k+1} + \xi'_{k+2} + \dots + \xi'_n$ for any $0 \le k \le n$.) We focus on the term B_n^+ and write

$$\begin{aligned} B_n^+ &= \mathbb{E}\Big[\varphi_1\left(\frac{U_{k,[ns]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{U_{k,[nt]}}{\sigma\sqrt{n}}\right); \tau_l^X = k, \eta_{k+1} > 0, U_{k,k+2} < 0, \dots, U_{k,[nt]} < 0\Big] \\ &= \sum_{k\le[ns]} \Sigma_k^X \mathbb{E}\Bigg[\mathbb{E}\Big[\varphi_1\left(\frac{x+S_{[ns]-k-1}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{x+S_{[nt]-k-1}}{\sigma\sqrt{n}}\right); \\ &\qquad\qquad \tau^S(x) > [nt]-k-1\Big]\Bigg|_{\eta_1 = x > 0}\Bigg]. \end{aligned}$$

For $u \in (0, s]$, we set

$$g_n(u) = n\Sigma_{[nu]}^X \mathbb{E}\Big[\mathbb{E}\Big[\varphi_1\left(\frac{x + S_{[ns]-[nu]-1)}}{\sigma\sqrt{n}}\right) \times \varphi_2\left(\frac{x + S_{[nt]-[nu]-1)}}{\sigma\sqrt{n}}\right); \tau^S(x) > [nt] - [nu] - 1\Big]\Big|_{\eta_1 = x > 0}\Big].$$

By Lemmas 2.1 and 2.4, it is clear that $0 \leq g_n(u) \preceq \frac{1}{\sqrt{u(t-u)}}$. To compute the pointwise limit on $(0, s]$ of the sequence $(g_n)_{n\geq 1}$, we first write

$$\begin{aligned} g_n(u) =& n\Sigma_{[nu]}^X \mathbb{E}\left[\mathbb{E}\Big[\varphi_1\left(\frac{x + S_{[ns]-[nu]-1)}}{\sigma\sqrt{n}}\right) \right. \\ & \left. \times \varphi_2\left(\frac{x + S_{[nt]-[nu]-1}}{\sigma\sqrt{n}}\right); \tau^S(x) > [nt] - [nu] - 1\Big]\Bigg|_{\eta_1 = x > 0}\right] \\ =& n\Sigma_{[nu]}^X \mathbb{E}\left[\mathbb{E}\Big[\varphi_1\left(\frac{x + S_{[ns]-[nu]-1}}{\sigma\sqrt{n}}\right) \varphi_2\left(\frac{x + S_{[nt]-[nu]-1}}{\sigma\sqrt{n}}\right) \Big| \right. \\ & \left. \tau^S(x) > [nt] - [nu] - 1\Big] \mathbb{P}[\tau^S(x) > [nt] - [nu] - 1]\Bigg|_{\eta_1 = x > 0}\right]. \end{aligned}$$

By Theorem 3.2 in [5] and Theorems 2.23 and 3.4 in [11], [4]

$$\begin{aligned} &\lim_{n\to+\infty} \mathbb{E}\Big[\varphi_1\left(\frac{x + S_{[ns]-[nu]-1}}{\sigma\sqrt{n}}\right) \\ &\quad \times \varphi_2\left(\frac{x + S_{[nt]-[nu]-1}}{\sigma\sqrt{n}}\right) \Big| \tau^S(x) > [nt] - [nu] - 1\Big] \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \frac{\sqrt{t-u}}{(s-u)^{3/2}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(y)\varphi_2(z) y e^{-\frac{y^2}{2(s-u)}} \\ &\qquad \times \left(e^{-\frac{(z-y)^2}{2(t-s)}} - e^{-\frac{(z+y)^2}{2(t-s)}}\right) dydz. \end{aligned}$$

Hence, by Lemma 2.1, Lemma 2.4 and the Lebesgue dominated convergence theorem, the sequence $(g_n)_{n\geq 1}$ pointwise converges towards the function g defined by

$$\begin{aligned} g(u) =& \frac{\alpha}{\pi\sqrt{2\pi(t-s)}} \frac{1}{\sqrt{u(s-u)^3}} \\ & \times \int_0^{+\infty} \int_0^{+\infty} \varphi_1(y)\varphi_2(z) y e^{-\frac{y^2}{2(s-u)}} \left(e^{-\frac{(z-y)^2}{2(t-s)}} - e^{-\frac{(z+y)^2}{2(t-s)}}\right) dydz. \end{aligned}$$

[4] In [11] the author needed the third order moment of the increment is finite; in fact, it only requires finite second moment [5].

Finally,

$$
\begin{aligned}
B^+ := \lim_{n\to+\infty} B_n^+ &= \frac{\alpha}{\pi\sqrt{2\pi(t-s)}}\int_0^s du\int_0^{+\infty} dy\int_0^{+\infty} dz \\
&\quad\times \varphi_1(y)\varphi_2(z)\frac{y}{\sqrt{u(s-u)^3}}e^{-\frac{y^2}{2(s-u)}}\left(e^{-\frac{(z-y)^2}{2(t-s)}}-e^{-\frac{(z+y)^2}{2(t-s)}}\right) \\
&= \frac{\alpha}{\pi s\sqrt{2\pi(t-s)}}\int_0^{+\infty} dy\int_0^{+\infty} dz\varphi_1(y)\varphi_2(z)\left(e^{-\frac{(z-y)^2}{2(t-s)}}-e^{-\frac{(z+y)^2}{2(t-s)}}\right) \\
&\quad\times\left(\int_0^1 \frac{y}{\sqrt{a(1-a)^3}}e^{-\frac{y^2}{2s(1-a)}}da\right) \\
&= \frac{1}{\pi\sqrt{s(t-s)}}\int_0^{+\infty} dy\int_0^{+\infty} dz\ \varphi_1(y)\varphi_2(z)e^{-y^2/2s}\left(e^{-\frac{(z-y)^2}{2(t-s)}}-e^{-\frac{(z+y)^2}{2(t-s)}}\right)
\end{aligned}
$$

where the last equality follows from (12). Similarly

$$
\begin{aligned}
B^- &:= \lim_{n\to+\infty} B_n^- \\
&= \frac{1}{\pi\sqrt{s(t-s)}}\int_{-\infty}^0 dy\int_{-\infty}^0 dz\ \varphi_1(y)\varphi_2(z)e^{-y^2/2s}\left(e^{-\frac{(z-y)^2}{2(t-s)}}-e^{-\frac{(z+y)^2}{2(t-s)}}\right).
\end{aligned}
$$

Eventually

$$
\begin{aligned}
B &= \lim_{n\to+\infty} B_n \\
&= \frac{1}{\pi\sqrt{s(t-s)}}\int_{(\mathbb{R}^-)^2\cup(\mathbb{R}^+)^2} dydz\ \varphi_1(y)\varphi_2(z)e^{-y^2/2s}\left(e^{-\frac{(z-y)^2}{2(t-s)}}-e^{-\frac{(z+y)^2}{2(t-s)}}\right).
\end{aligned} \tag{14}
$$

3.2.3. *End of the proof.* Combining (11), (13) and (14) yield to the following: for any $0 \le s < t$,

$$
\lim_{n\to\infty}\mathbb{E}\left[\varphi_1\left(Y_s^{(n)}\right)\varphi_2\left(Y_t^{(n)}\right)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde\varphi_1(y)\tilde\varphi_2(z)p_s^\alpha(0,y)p_{t-s}^\alpha(y,z)dydz.
$$

3.3. **Tightness.** In this section, we prove that the sequence of processes $(X_t^{(n)})_{n\ge0}$ is tight. By Theorem 7.3 in [4], it is sufficient to check that

(i) for any $\eta > 0$, there exist $a > 0$ and $n_0 \ge 1$ such that
$$
\mathbb{P}\left[|X_0^{(n)}| \ge a\right] \le \eta, \quad n \ge n_0;
$$

(ii) for any $\epsilon, \eta > 0$, there exist $\delta \in (0,1)$ and $n_0 \ge 1$ such that
$$
\mathbb{P}\left[w_{X^{(n)}}(\delta) \ge \epsilon\right] \le \eta, \quad n \ge n_0
$$
where $w_{X^{(n)}}(\delta) = \sup\{|X_s^{(n)} - X_t^{(n)}| : t, s \in [0,1], |s-t| \le \delta\}$.

The first condition is clear since $X_0^{(n)} = X_0 = 0$ for any $n \ge 0$. For the second condition, we write

$$
\begin{aligned}
w_{X^{(n)}}(\delta) \le \frac{3}{(\sigma\wedge\sigma')\sqrt{n}}&\left(\sup_{\substack{1\le i<j\le n\\|i-j|\le n\delta}}|S_i - S_j| + \sup_{\substack{1\le i<j\le n\\|i-j|\le n\delta}}|S'_i - S'_j|\right) \\
&+ \frac{1}{(\sigma\wedge\sigma')\sqrt{n}}\sup_{1\le l\le n}|\eta_{\tau_l^X+1}|\mathbf{1}_{\{\tau_l^X\le n\}}.
\end{aligned}
$$

Denote $N_n = \sum_{l=1}^{\infty} \mathbf{1}_{\{\tau_l^X \leq n\}}$ the number of times that $X_k, 1 \leq k \leq n$, visits zeros. For any $\epsilon > 0$, it holds

$$\begin{aligned}
&\mathbb{P}\left[\frac{1}{\sqrt{n}} \sup_{1\leq l\leq n} |\eta_{\tau_l^X+1}| \mathbf{1}_{\{\tau_l^X \leq n\}} \geq \epsilon\right] \\
&= \mathbb{P}\left[\sup_{1\leq l\leq N_n} |\eta_{\tau_l^X+1}| \geq \sqrt{n}\epsilon\right] \\
&\leq \mathbb{P}\left[\sup_{1\leq l\leq a\sqrt{n}} |\eta_{\tau_l^X+1}| \geq \sqrt{n}\epsilon\right] + \mathbb{P}\left[N_n > a\sqrt{n}\right], \qquad (15)
\end{aligned}$$

for any positive constant a. By using the inequality$(1-x)^\alpha \geq 1-\alpha x$, valid for any $\alpha \geq 1$ and $x \in (0,1)$, we get

$$\begin{aligned}
\mathbb{P}\left[\sup_{1\leq l\leq a\sqrt{n}} |\eta_{\tau_l^X+1}| \geq \sqrt{n}\epsilon\right] &= 1 - \left(1 - \mathbb{P}[|\eta_1| \geq \sqrt{n}\epsilon]\right)^{[a\sqrt{n}]} \\
&\leq a\sqrt{n}\mathbb{P}[|\eta_1| \geq \sqrt{n}\epsilon],
\end{aligned}$$

which tends to zeros as $n \to \infty$, since η_1 is integrable.

Now, let us control the second term on the RHS in (15). Lemma 2.4 yields $\mathbb{E}[N_n] \preceq \sqrt{n}$, so that,

$$\sup_{n\geq 1} \mathbb{P}[N_n > a\sqrt{n}] \leq \sup_{n\geq 1} \frac{\mathbb{E}[N_n]}{a\sqrt{n}} \preceq \frac{1}{a} \to 0 \quad \text{as} \quad a \to +\infty.$$

Moreover, by a classical argument (see for instance [4], Chapter 7),

$$\lim_{\delta\to 0} \lim_{n\to\infty} \mathbb{P}\left[\frac{1}{\sigma\sqrt{n}} \sup_{\substack{1\leq i<j\leq n \\ |i-j|\leq n\delta}} |S_i - S_j| \geq \epsilon\right] + \mathbb{P}\left[\frac{1}{\sigma'\sqrt{n}} \sup_{\substack{1\leq i<j\leq n \\ |i-j|\leq n\delta}} |S'_i - S'_j| \geq \epsilon\right] = 0.$$

Finally, the sequence of processes $(X^{(n)}(.))$ is tight. This achieves the proof of Theorem 1.1.

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