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FIRST ORDER CONVERGENCE OF WEAK WONG–ZAKAI APPROXIMATIONS OF LÉVY-DRIVEN MARCUS SDES

For solutions $X = (X_t)_{t\in[0,T]}$ of a Lévy-driven Marcus (canonical) stochastic differential equation we study the Wong–Zakai type time discrete approximations $\bar X = (\bar X_{kh})_{0\leqslant k\leqslant T/h}$, $h > 0$, and establish the first order convergence $|\mathbf{E}_x f(X_T)-\mathbf{E}_x f(X^h_T)| \leqslant Ch$ for $f \in C^4_b$.

1. Introduction

Stochastic differential equations (SDE) driven by Lévy processes belong nowadays to a standard toolbox of researches working in Physics, Finance, Engineering etc. Under standard assumptions, a solution X of an SDE is a Markov (Feller) process containing a continuous diffusive component as well as (infinitely many) jumps which model instant change of the observable in the phase space.

From the point of view of applications, one often needs to determine averaged quantities of the type $\mathbf{E}_x f(X_T)$ for a fixed deterministic time $T > 0$ and a regular test function f. Calculation of such functionals is equivalent to solving a certain partial integro-differential equation that can be done by the method finite differences or finite elements, see, e.g. Cont and Tankov [4, Chapter 12]. In this paper we consider alternative approximations of $\mathbf{E}_x f(X_T)$ by means of simulation of effective approximations of the random process X.

The approximation problem for the functionals $\mathbf{E}_x f(X_T)$ for diffusions is nowadays a classical topic, see Kloeden and Platen [10]. The numerical methods have originated in the paper by Maruyama [24] who showed that for the Itô SDE $\mathrm{d}X = a(X)\,\mathrm{d}t + b(X)\,\mathrm{d}W$ driven by the Brownian motion, the Euler scheme $\bar X_{(k+1)h} = \bar X_{kh} + a(\bar X_{kh})h + b(\bar X_{kh})(W_{(k+1)h} - W_{kh})$ with the step size $h > 0$ converges to X_T in L^2-sense for each $T \geqslant 0$. Milstein [29] and Talay [38] showed that the Euler scheme yields weak convergence of the order $\mathcal{O}(h)$. Higher order approximation methods can be found in the papers by Mackevičius [21], Talay [38], Milstein [32], Talay and Tubaro [39], and Bally and Talay [2] as well as in the monographs by Milstein [30], Kloeden and Platen [10], and Milstein and Tretyakov [31].

Although the theory for diffusion models is well established, the presence of jumps typically requires an additional justification.

In various application areas, jumps appear quite naturally. For instance, in finance jumps can realistically model fluctuation of stock prices. In population biology jump processes appear as limits of Markov chains. Such models are well described by multivariate Lévy-driven Itô SDEs of the type $\mathrm{d}X = F(X)\,\mathrm{d}L$. The weak convergence of the Euler scheme for SDEs with a jump component of finite intensity was studied by Mikulevičius and Platen [26], and by Kubilius and Platen [14]. Protter and Talay [34]

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established the first order convergence of the Euler scheme in in case of C_b^4 coefficient F, a C_b^4 test function f, and tails of the Lévy measure having finite 8th moments (they also have results for increasing f; then more moments are needed). Further analysis was performed by Jacod *et al.* in [9]. Liu and Li [20] studied the SDE driven by a Brownian motion and a Poisson random measure under the assumptions hat all moments of X are finite. Recently weak approximations for SDEs with Hölder-continuous coefficients were studied by Mikulevičius and Zhang in [27, 25, 28]. A general class of high order weak approximation schemes for Lévy-driven Itô SDEs was studied by Kohatsu-Higa and Tankov [13], Tankov [40], Kohatsu-Higa and Ngo [11], and Kohatsu-Higa *et al.* [12].

There is however another, mechanical point of view on SDEs with jumps, which originates in the supposition that both the Brownian motion and the jump component are convenient mathematical idealizations of smooth real-world processes (e.g. mechanical motions). This paradigm goes back to Langevin who obtained a random motion of a heavy particle in a liquid as an integral of a correlated Gaussian velocity process.

It turns out that the idealized diffusion dynamics in such an approach is correctly described by the Stratonovich SDEs that can be seen as a limit of random non-autonomous ordinary differential equations (ODE) in which the Brownian motion is replaced by its (piece-wise) smooth approximations, the so-called Wong–Zakai approximations, see Wong and Zakai [41, 42].

In the presence of jumps, the Marcus (canonical) SDEs are extensions of Stratonovich SDEs for diffusions. As Stratonovich equations, they have a lot of useful properties. For instance, the change of variables formula for solutions of Marcus SDEs looks like the deterministic Newton–Leibniz chain rule. They are also limits of continuous random ODEs obtained by pathwise approximations of the driving Lévy process by smooth functions (the Wong–Zakai technique). These properties justify their utilization in Physics and Engineering, see, e.g. Marcus [22], Di Paola and Falsone [5], Sun *et al.* [37], Chechkin and Pavlyukevich [3], Pavlyukevch *et al.* [33].

Roughly speaking, jumps in the Marcus setting should be understood as idealizations of very fast motions along certain trajectories determined by the physical parameters of the system.

Despite of their usefulness in applications, numerical methods for Marcus SDEs are not well-developed. Some partial results in this direction on the physical level of rigour were obtained by Li *et al.* [18, 19].

The goal of this paper is to fill this gap. We will construct an Euler–Maruyama (Wong–Zakai) type numerical scheme $\bar{X}$ on a discrete time grid of the size $h > 0$, and establish the first order weak approximations $|\mathbf{E}f(X_T) - \mathbf{E}f(\bar{X}_T)| \leqslant Ch$ for C_b^4-test functions f. The main difficulty will consist in the treatment of the non-linear jump dynamics, which involves the analysis of a certain family of non-linear ODEs and makes the approximation problem very different to the Itô case.

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2. Setting and the main result

On a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbf{P})$ satisfying the usual hypotheses consider an m-dimensional Brownian motion W and an independent m-dimensional pure jump Lévy process Z with a characteristic triplet $(0, 0, \nu)$. The Lévy process Z has the Lévy–Itô decomposition

$$Z_t = \int_0^t \int_{\|z\|\leqslant 1} z\,\tilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{\|z\|>1} z\,N(\mathrm{d}s,\mathrm{d}z), \tag{2.1}$$

where N is the Poisson random measure on $\mathbb{R}_+\times\mathbb{R}^m$ with the intensity measure $\mathrm{d}t\cdot\nu(\mathrm{d}z)$, and $\tilde{N}$ is the compensated Poisson random measure. The theory of Lévy processes is a classical topic nowadays, we refer the reader to e.g. Sato [36] and Applebaum [1].

For $d\geqslant 1$, consider a vector-valued function

$$a(x) = \begin{pmatrix} a^1(x), \\ \vdots \\ a^d(x) \end{pmatrix}, \tag{2.2}$$

and matrix-valued functions

$$b(x) = \begin{pmatrix} b^1_1(x) & \cdots & b^1_m(x) \\ \vdots & \ddots & \vdots \\ b^d_1(x) & \cdots & b^d_m(x) \end{pmatrix}, \qquad c(x) = \begin{pmatrix} c^1_1(x) & \cdots & c^1_m(x) \\ \vdots & \ddots & \vdots \\ c^d_1(x) & \cdots & c^d_m(x) \end{pmatrix}, \tag{2.3}$$

and denote

$$c^i(x) = \big(c^i_1(x),\dots,c^i_m(x)\big) \tag{2.4}$$

the i-th row of the matrix $c(x)$, $i=1,\dots,d$.

In this paper we will work with a Marcus (canonical) SDE

$$X_t = X_0 + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\circ\mathrm{d}W_s + \int_0^t c(X_s)\diamond\mathrm{d}Z_s,\quad t\geqslant 0. \tag{2.5}$$

Canonical SDEs were introduced by Marcus [22] with the aim to construct jump-diffusions which at least formally obey the rules of ordinary calculus. It is well known that the chain rule for a solution of a Stratonovich SDE $\mathrm{d}X_t = b(X_t)\circ\mathrm{d}W_t$ coincides with the Newton–Leibniz formula $f(X_t) = f(X_0)+\int_0^t f'(X_s)b(X_s)\circ\mathrm{d}W_s$, $f\in C^2(\mathbb{R},\mathbb{R})$, where the latter integral has to be understood as the the stochastic Stratonovich integral, see, e.g. Protter [35, Chapter V.5]. To extend such a property to the jump case, one has to define the jumps of the process solution X of (2.5) properly. In the case of Marcus prescription, the jump $\Delta X_t = X_t - X_{t-}$ is obtained as a result of an infinitely fast motion along the integral curve of the vector field $c(\cdot)\Delta Z_t$. Indeed, for each $z\in\operatorname{supp}\nu\subseteq\mathbb{R}^m$, consider a non-linear ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}u}\phi^z(u;x) = c(\phi^z(u;x))z \\ \phi^z(0;x) = x,\quad u\in[0,1], \end{cases} \tag{2.6}$$

and define the so-called Marcus flow

$$\phi^z(x) := \phi^z(1;x). \tag{2.7}$$

Then by definition one sets $\Delta X_t = \phi^{\Delta Z_t}(X_{t-}) - X_{t-}$. To make the construction rigorous, it is convenient to rewrite (2.5) as an Itô SDE driven by a Brownian motion and a Poisson random measure. Written in the coordinates, it takes the form

$$
\begin{aligned}
X_t^i = x^i &+ \int_0^t a^i(X_s)\,\mathrm{d}s \\
&+ \sum_{j=1}^m \int_0^t b_j^i(X_s)\,\mathrm{d}W_s^j + \frac{1}{2}\sum_{j=1}^m\sum_{l=1}^d \int_0^t \frac{\partial}{\partial x^l} b_j^i(X_s) b_j^l(X_s)\,\mathrm{d}s \\
&+ \int_0^t \int_{\|z\|\leqslant 1} \Big(\phi^z(X_{s-}) - X_{s-}\Big)^i \tilde{N}(\mathrm{d}z,\mathrm{d}r) \\
&+ \int_0^t \int_{\|z\|\leqslant 1} \Big(\phi^z(X_s) - X_s - c(X_s)z\Big)^i \nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t \int_{\|z\|> 1} \Big(\phi^z(X_{s-}) - X_{s-}\Big)^i N(\mathrm{d}z,\mathrm{d}r), \quad i = 1,\dots,d.
\end{aligned}
\tag{2.8}
$$

For a complete account on Marcus SDEs see the works by Marcus [22, 23], Kurtz *et al.* [17], Kunita [16], and Applebaum [1]. Note that the Marcus integral $\int_0^t c(X_s) \diamond \mathrm{d}Z_s$ cannot be represented as a limit of Riemannian sums (opposite to the Itô or Stratonovich integrals), so that the SDE (2.5) should be understood via its Itô representation (2.8).

Lévy-driven Marcus SDEs possess a lot of useful properties. For instance, under sufficient smoothness assumptions on the coefficients a, b, and c, the their solutions forms flows of stochastic diffeomorphisms, see Fujiwara and Kunita [7]. Due to the coordinate free construction of the jump part and the Stratonovich diffusion part, Marcus SDEs can be defined on manifolds, see Fujiwara [6].

Finally, one can approximate solutions of Marcus SDEs by solutions of continuous random ordinary differential equations (ODEs) (the so-called Wong–Zakai approximations). For a time step $h > 0$, let us approximate W and Z by polygonal curves with knots at $\{kh, W_{kh}\}_{k\geqslant 0}$ and $\{kh, Z_{kh}\}_{k\geqslant 0}$ respectively. Namely, we define the piece-wise linear random processes

$$
\begin{aligned}
W_t^h &= W_{kh} + \frac{t-kh}{h}\Big(W_{(k+1)h} - W_{kh}\Big), \quad t\in[kh,(k+1)h), \quad k\geqslant 0, \\
Z_t^h &= Z_{kh} + \frac{t-kh}{h}\Big(Z_{(k+1)h} - Z_{kh}\Big), \quad t\in[kh,(k+1)h), \quad k\geqslant 0,
\end{aligned}
\tag{2.9}
$$

and consider a family of random ODEs

$$
\bar{X}_t^h = x + \int_0^t \Big(a(\bar{X}_s^h) + b(\bar{X}_s^h)\dot{W}_s^h + c(\bar{X}_s^h)\dot{Z}_s^h\Big)\,\mathrm{d}s, \quad t\geqslant 0.
\tag{2.10}
$$

It is well known (see, e.g. Marcus [22] and Kunita [15]) that the approximations $\bar{X}^h$ converge to X as $h\to 0$ in the sense of convergence of finite dimensional distributions.

In this paper we present a weak numerical scheme for (2.5) based on the Wong–Zakai approximations (2.10).

For $f\colon \mathbb{R}^d \mapsto \mathbb{R}$ we will use the uniform norm

$$
\|f\| = \sup_{x\in\mathbb{R}^d} |f(x)|.
\tag{2.11}
$$

For $x\in\mathbb{R}^d$ (and $\mathbb{R}^m$), we will work with the Euclidian norm $\|x\| = (x_1^2+\dots+x_d^2)^{1/2}$. For a function $f\colon\mathbb{R}^d\to\mathbb{R}$ denote by $\partial^\alpha f$ its partial derivative corresponding to a multiindex α. Let $Dc(x)$ be the gradient tensor of the mapping $x\mapsto c(x)$. For each $x\in\mathbb{R}^d$, we consider it as a linear operator $Dc(x)\colon \mathbb{R}^m \to \mathbb{R}^{d\times d}$ given by

$$Dc(x)z = \begin{pmatrix} \frac{\partial}{\partial x^1}\langle c^1(x), z\rangle & \cdots & \frac{\partial}{\partial x^d}\langle c^1(x), z\rangle \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x^1}\langle c^d(x), z\rangle & \cdots & \frac{\partial}{\partial x^d}\langle c^d(x), z\rangle \end{pmatrix} \tag{2.12}$$

and define

$$\|Dc(x)\| = \sup_{\|z\|\leqslant 1} \|Dc(x)z\|, \tag{2.13}$$

Also let

$$\|Dc\| = \sup_{x\in\mathbb{R}^d} \|Dc(x)z\|. \tag{2.14}$$

For practical needs it is sometimes convenient to use the maximum entry norm of the gradient tensor:

$$\|Dc\|_{\mathrm{e}} = \max_{\substack{1\leqslant i,k\leqslant d\\ 1\leqslant j\leqslant m}} \Big\|\frac{\partial}{\partial x_k} c^i_j(x)\Big\|. \tag{2.15}$$

Then we have

$$\|Dc(x)z\| \leqslant \|Dc\|\cdot\|z\| \leqslant d\sqrt{m}\cdot\|Dc\|_{\mathrm{e}}\cdot\|z\|. \tag{2.16}$$

In this paper we make the following assumptions on the coefficients a, b and c.

$\mathbf{H}_{a,b,c}$:

(2.17)

$$\begin{aligned}
&a\in C^4(\mathbb{R}^d,\mathbb{R}^d) \quad \text{and} && \|\partial^\alpha a^i\| < \infty, \quad 1\leqslant i\leqslant d, \quad 1\leqslant|\alpha|\leqslant 4;\\
&b\in C^4(\mathbb{R}^d,\mathbb{R}^{d\times m}) \quad \text{and} && \|\partial^\alpha b^i_j\| < \infty, \quad 1\leqslant i\leqslant d,\ 1\leqslant j\leqslant m, \quad 1\leqslant|\alpha|\leqslant 4,\\
& && \|b^i_j\cdot\partial^\alpha b^k_l\| < \infty, \quad 1\leqslant i,k\leqslant d,\ 1\leqslant j,l\leqslant m, \quad 2\leqslant|\alpha|\leqslant 4;\\
&c\in C^4(\mathbb{R}^d,\mathbb{R}^{d\times m}) \quad \text{and} && \|\partial^\alpha c^i_j\| < \infty, \quad 1\leqslant i\leqslant d,\ 1\leqslant j\leqslant m, \quad 1\leqslant|\alpha|\leqslant 4,\\
& && \|c^i_j\cdot\partial^\alpha c^k_l\| < \infty, \quad 1\leqslant i,k\leqslant d,\ 1\leqslant j,l\leqslant m, \quad 2\leqslant|\alpha|\leqslant 4.
\end{aligned}$$

Under these conditions there is a unique global solution ϕ^z of (2.6) whose properties are studied in Appendix A.

We recall now the definition (2.9) of the processes W^h and Z^h and introduce the discrete time scheme $\bar X = (\bar X_{kh})_{k\geqslant 0}$ as follows.

For $\tau\geqslant 0$, $w,z\in\mathbb{R}^m$, consider the ordinary differential equation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}u}\psi(u) = a(\psi(u))\tau + b(\psi(u))w + c(\psi(u))z,\\ &\psi(0) = x, \quad u\in[0,1], \end{aligned} \tag{2.18}$$

which has a unique global solution under assumptions $\mathbf{H}_{a,b,c}$. Let

$$\psi(x) = \psi(x;\tau,w,z) := \psi(1;x,\tau,w,z). \tag{2.19}$$

The properties of the mapping ψ are studied in Appendix B.

For $x\in\mathbb{R}$ and the time step $h>0$, consider the non-linear Euler type scheme

$$\begin{aligned} &\bar X_0 = x,\\ &\bar X_{(k+1)h} = \psi(\bar X_{kh}; h, W_{(k+1)h} - W_{kh}, Z_{(k+1)h} - Z_{kh}), \quad k\geqslant 0. \end{aligned} \tag{2.20}$$

The goal of this paper is to establish the weak convergence rate of this numerical scheme. It is assumed that the increments of the Brownian motion and of the pure jump process Z can be simulated exactly. We also do not take into account numerical errors which may arise while solving the ODE (2.18).

Now we formulate further assumptions and the main result of this paper.

$\mathbf{H}_\nu$**:** Assume that the tails of the Lévy measure ν satisfy

$$\int_{\|z\|>1} \|z\|^3 \cdot \mathrm{e}^{8\|Dc\|\cdot\|z\|}\,\nu(\mathrm{d}z) < \infty. \tag{2.21}$$

In view of (2.16), Assumption $\mathbf{H}_\nu$ is guaranteed by the following condition which is easier to verify in practice.

$\mathbf{H}'_\nu$**:**

$$\int_{\|z\|>1} \|z\|^3 \cdot \mathrm{e}^{8d\sqrt{m}\cdot\|Dc\|_{\mathrm{e}}\cdot\|z\|}\,\nu(\mathrm{d}z) < \infty. \tag{2.22}$$

Theorem 2.1. *Assume that conditions $\mathbf{H}_{a,b,c}$ and $\mathbf{H}_\nu$ hold true. Then for any $T>0$, there is a constant C_T such that for any $x \in \mathbb{R}$ the following holds.*
1. There is a unique strong solution $X=(X_t)_{t\in[0,T]}$ such that

$$\mathbf{E}_x \sup_{t\in[0,T]} \|X_t\|^4 \leqslant C_T(1+\|x\|^4). \tag{2.23}$$

2. For any $h>0$, the numerical scheme $\{\bar X_{kh}\}_{0\leqslant kh\leqslant T}$ satisfies

$$\mathbf{E}_x \sup_{0\leqslant kh\leqslant T} \|\bar X_{kh}\|^4 \leqslant C_T(1+\|x\|^4). \tag{2.24}$$

Proof. See Section 4. □

The following result is interesting on its own. Assume

$\mathbf{H}_{\nabla\phi,\nu}$**:**

$$\begin{aligned}
&\int_{|z|>1} \|\nabla_x\phi^z\|^4\,\nu(\mathrm{d}z) < \infty,\\
&\int_{|z|>1} \|\nabla_x^2\phi^z\|^2\,\nu(\mathrm{d}z) < \infty,\\
&\int_{|z|>1} \|\nabla_x^3\phi^z\|^{4/3}\,\nu(\mathrm{d}z) < \infty,\\
&\int_{|z|>1} \|\nabla_x^4\phi^z\|\,\nu(\mathrm{d}z) < \infty.
\end{aligned} \tag{2.25}$$

Theorem 2.2. *Under conditions $\mathbf{H}_{a,b,c}$ and $\mathbf{H}_{\nabla\phi,\nu}$, for any $f\in C_b^4$, any $T>0$, there is $C>0$ such that for each $x\in\mathbb{R}^d$, $t\in[0,T]$ and any multiindex α*

$$\|\partial^\alpha \mathbf{E}_x f(X_t)\| \leqslant C, \quad 1\leqslant|\alpha|\leqslant 4. \tag{2.26}$$

Proof. See Section 7. □

Remark 2.3. Under assumptions $\mathbf{H}_{a,b,c}$, it follows from Lemma A.2 that $\mathbf{H}_\nu$ implies $\mathbf{H}_{\nabla\phi,\nu}$.

The main result of this paper is the first order weak convergence rate of the Euler type (Wong–Zakai) scheme (2.20).

Theorem 2.4. *Let conditions $\mathbf{H}_{a,b,c}$ and $\mathbf{H}_\nu$ hold true. Then for any $f\in C_b^4(\mathbb{R},\mathbb{R})$ and any $T>0$ there is a constant $C=C(T,f)$ such that for any $n\in\mathbb{N}$ and $h>0$ such that $nh\leqslant T$*

$$|\mathbf{E}_x f(X_{nh}) - \mathbf{E}_x f(\bar X_{nh})| \leqslant C\cdot nh^2\cdot(1+\|x\|^4), \quad x\in\mathbb{R}^d. \tag{2.27}$$

The proof of this theorem will be given in the following Sections.

Eventually we comment on conditions $\mathbf{H}_{a,b,c}$ and $\mathbf{H}_\nu$, and the applicability of the numerical scheme.

Remark 2.5. Conditions $\mathbf{H}_{a,b,c}$ are less restrictive than the assumptions in Protter and Talay [34] and Jacod *et al.* [9] where the coefficients of the SDE are C_b^4 or smoother.

Remark 2.6. Assumption $\mathbf{H}_\nu$ (or $\mathbf{H}'_\nu$) requires existence of exponential moments of the Lévy measure ν and looks more restrictive than the assumptions in [34] and [9] where existence of high absolute moments (up to the 32-th or higher ones) is demanded. Exponential moments appear due to the non-linear nature of the Marcus ODE (2.6). Recall that the jump size of an Itô SDE $\mathrm{d}X_t = c(X_{t-})\,\mathrm{d}Z_t$ is $\Delta X_t = c(X_{t-})\Delta Z_t$ and hence is a linear function of ΔZ_t. On the contrary, the jump size of the Marcus SDE $\mathrm{d}X_t = c(X_t)\diamond\mathrm{d}Z_t$ equals to $\Delta X_t = \phi^{\Delta Z_t}(X_{t-})-X_{t-}$ and is determined by the non-linear ODE (2.6). The best generic estimate for the size of this jump is given by the Gronwall inequality. Hence exponential moments in the Marcus case serve as a natural analog of the conventional moments in the Itô scheme. For instance, assumptions $\mathbf{H}_\nu$ and $\mathbf{H}'_\nu$ are always satisfied for a Lévy process Z with bounded jumps.

In particular cases one can find less restrictive assumptions on the moments of the Lévy measure. For instance one can show that in dimensions $d = m = 1$ for the equation $\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\circ\mathrm{d}W_t + MX_t\diamond\mathrm{d}Z_t$, with $a, b \in C_b^4$ and $M > 0$, convergence (2.27) holds for any spectrally negative Lévy process Z with $\nu((0,+\infty)) = 0$, and in particular for a spectrally negative stable Lévy process. However we were not able to find similar tractable sufficient conditions for convergence in general, especially in the multivariate case.

Remark 2.7. The scheme (2.20) employs realizations of the increments of the Lévy jump process Z. The list of infinitely divisible distributions which can be simulated explicitly is rather short and includes α-stable laws, Gamma and variance Gamma distributions, as well as inverse Gaussian. We refer the reader to Protter and Talay [34, Section 3] and Cont and Tankov [4, Section II.6] for more information on this subject and the description of the corresponding numerical algorithms.

For the reader's convenience, in the following Sections 4–7 as well as in the Appendices A and B we assume that $d = m = 1$. In the proof we will not use any of the geometrical advantages of the one-dimensional setting and make this assumption just in order to simplify the notation significantly. The technical difficulties lie not in the higher dimensions of the state space but in the analysis of the interplay of the terms $\mathrm{d}t$, $\circ\,\mathrm{d}W$ and $\diamond\,\mathrm{d}Z$ with the corresponding terms in the approximation scheme (2.20). From this point of view, we are in a setting of a scalar equation driven by a three-dimensional Lévy process (t, W_t, Z_t).

3. Numerical illustration

In this Section we give a numerical illustration to Theorem 2.4. Consider a Marcus SDE

$$\mathrm{d}X_t = \mathrm{d}t + X_t\diamond\mathrm{d}Z_t \tag{3.1}$$

with the coefficients $a(x)\equiv 1$, $b(x)\equiv 0$ and $c(x)\equiv x$. The Lévy process Z is a compound Poisson process with the symmetric Lévy measure

$$\nu(\mathrm{d}z) = \frac{\lambda}{2\beta}\cdot \mathrm{e}^{-|z|/\beta}\,\mathrm{d}z,\quad \lambda>0,\quad \beta>0, \tag{3.2}$$

i.e. the jumps of Z are Laplace-distributed with the parameter β. To satisfy assumption $\mathbf{H}_\nu$ we assume that $\beta = 0.1$. We set the jump intensity $\lambda = 100$.

We calculate the expected value $\mathbf{E}_x f(X_1)$ for the function $f(x) = 10^3\sin(10^{-3}x)$ for different values of $x \in [-1, 1]$. Since $\max_{x\in[-20,20]}|f(x) - x| \leqslant 0.0015$, $\mathbf{E}_x f(X_1)$ can be seen as a good approximation of the *reference* mean value $\mathbf{E}_x X_1$. The generator of X is

$$Lf(x) = f'(x) + \int_{\mathbb{R}} \Big(f(x\mathrm{e}^z) - f(x) \Big) \nu(\mathrm{d}z), \quad f \in C^1(\mathbb{R}, \mathbb{R}), \tag{3.3}$$

and a straightforward application of the Dynkin formula yields the explicit formula for the mean value

$$\mathbf{E}_x X_t = x\mathrm{e}^{\rho t} + \frac{\mathrm{e}^{\rho t} - 1}{\rho}, \quad \rho = \frac{\lambda\beta^2}{1-\beta^2}, \quad t \geqslant 0. \tag{3.4}$$

Analogously one can calculate the second moment $\mathbf{E}_x X_t^2$ and the variance of X_t but we omit here the explicit cumbersome formulae.

Denoting $0 < \tau_1 < \cdots < \tau_N < 1$ the jump times of Z, and $J_1, \dots, J_N$ the iid Laplace-distributed jump sizes, we solve equation (3.1) explicitly as

$$\begin{aligned} X_t &= x + t, \quad t \in [0, \tau_1), \\ X_{\tau_1} &= X_{\tau_1-}\mathrm{e}^{J_1}, \\ X_t &= X_{\tau_1} + t - \tau_1, \quad t \in [\tau_1, \tau_2), \\ &\cdots \\ X_{\tau_N} &= X_{\tau_N-}\mathrm{e}^{J_N}, \\ X_t &= X_{\tau_N} + t - \tau_N, \quad t \in [\tau_N, 1]. \end{aligned} \tag{3.5}$$

The scheme (3.5) is exact and can be easily realized on the computer. To estimate $\mathbf{E}_x f(X_1)$ we simulate $n = 10^5$ independent samples $\{Z^{(k)}\}_{1 \leqslant k \leqslant n}$ of the paths of the Lévy process $Z = (Z_t)_{t\in[0,1]}$ and approximate the mean value by the empirical mean

$$\mathbf{E}_x f(X_1) \approx \langle f(X_1) \rangle := \frac{1}{n} \sum_{k=1}^{n} f(X_1^{(k)}), \tag{3.6}$$

where $X^{(k)}$ is the solution of (3.1) driven by the process $Z^{(k)}$.

Furthermore, for the step size $h > 0$, we employ the numerical scheme (2.20), which in our particular case has the form

$$\begin{aligned} \bar{X}_0^h &= x, \\ \bar{X}_{(k+1)h}^h &= \psi(\bar{X}_{kh}^h, h, Z_{(k+1)h} - Z_{kh}), \end{aligned} \tag{3.7}$$

where

$$\psi(x, h, z) = \begin{cases} \mathrm{e}^z x + \dfrac{\mathrm{e}^z - 1}{z} h, & z \neq 0, \\ x + h, & z = 0. \end{cases} \tag{3.8}$$

For the values $h = 0.1$ and $h = 0.01$ we also approximate

$$\mathbf{E}_x f(\bar{X}_1^h) \approx \langle f(\bar{X}_1^h) \rangle := \frac{1}{n} \sum_{k=1}^{n} f(\bar{X}_1^{h,(k)}), \tag{3.9}$$

where $\bar{X}^{h,(k)}$ is the Wong–Zakai approximation of (3.1) driven by the process $Z^{(k)}$.

The results of the reference values and the numerical simulations are presented in Table 1. Figure 1 contains sample paths of the Lévy process Z and its Wong–Zakai approximations Z^h as well as sample paths of the process X and its Wong–Zakai approximations $\bar{X}^h$. One can clearly observe that the approximation error increases mainly due to large jumps of Z.

x	$\mathbf{E}_x X_1$	$\sqrt{\mathrm{Var}_x X_1}$	$n^{-1/2}\sqrt{\mathrm{Var}_x X_1}$	$\langle f(X_1)\rangle$	$\langle f(\bar X_1^h)\rangle$, $h=0.1$	$\langle f(\bar X_1^h)\rangle$, $h=0.01$
-1.0	-1.0175	5.5349	0.0175	-0.9950	-1.0486	-1.0154
-0.5	0.3555	2.2352	0.0071	0.3550	0.3176	0.3578
0.0	1.7284	2.3608	0.0075	1.7160	1.7054	1.7206
0.5	3.1014	5.8848	0.0186	3.0935	3.0701	3.1157
1.0	4.4743	9.6005	0.0304	4.4769	4.4452	4.4931

TABLE 1. The results of the numerical simulations for the equation (3.1) for $\lambda = 100$, $\beta = 0.1$, $n = 10^5$.

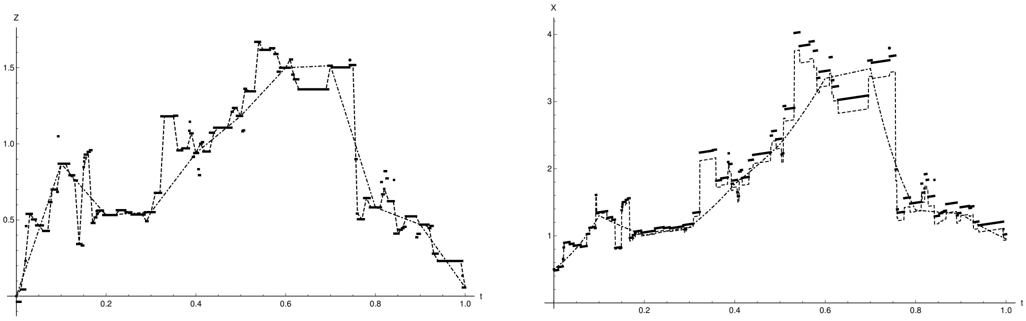

FIGURE 1. Left: a sample path of the Lévy process Z (bold line) and its Wong–Zakai approximations Z^h for $h = 0.1$ (dot-dashed line) and $h = 0.01$ (dashed line), see (2.9). Right: the corresponding sample path of the solution of the Marcus SDE X (bold line) and its Wong–Zakai approximations $\bar X^h$ for $h = 0.1$ (dot-dashed line) and $h = 0.01$ (dashed line), see (2.10).

4. PROOF OF THEOREM 2.1

Proof. 1. We denote

$$a^\diamond(x) = a(x) + \frac{1}{2}b'(x)b(x) + \int_{|z|\leqslant 1}\Big(\phi^z(x) - x - c(x)z\Big)\,\nu(\mathrm{d}z) + \int_{|z|>1}\Big(\phi^z(x) - x\Big)\,\nu(\mathrm{d}z) \tag{4.1}$$

and write (2.8) in dimension 1 as

$$X_t = X_0 + \int_0^t a^\diamond(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s + \int_0^t\int_{\mathbb{R}}\Big(\phi^z(X_{s-}) - X_{s-}\Big)\,\tilde N(\mathrm{d}s,\mathrm{d}z) \tag{4.2}$$

Due to Lemmas A.1 and A.2, the drift $a^\diamond$ is a Lipschitz continuous function, and since

$$|\phi^z(x) - x| \leqslant C(1+|x|)|z|\mathbb{I}(|z|\leqslant 1) + |x|(1+\mathrm{e}^{\|c'\|\cdot|z|})\mathbb{I}(|z|>1) \tag{4.3}$$

and

$$|\phi^z(x) - x - \phi^z(y) + y| \leqslant C|x-y|\cdot|z|\cdot\mathbb{I}(|z|\leqslant 1) + |x-y|(1+\mathrm{e}^{\|c'\|\cdot|z|})\mathbb{I}(|z|>1), \tag{4.4}$$

existence and uniqueness of the strong solution X with a finite fourth moment follows, e.g. from [16, Theorem 3.1].

2. The discrete time scheme $\bar X = (\bar X_{kh})_{k\geqslant 0}$ can be transformed to a continuous time process $\{\bar X_t\}_{t\geqslant 0}$ by taking

$$\bar X_t = \psi(\bar X_{kh}; h, W_t - W_{kh}, Z_t - Z_{kh}),\quad t\in[kh,(k+1)h). \tag{4.5}$$

Then, using the Itô formula on the time interval $[kh,(k+1)h]$ and taking into account condition $\mathbf{H}_\nu$ and the properties of the mapping ψ and its derivatives (see Lemma B.1), it is easy to show that

$$\mathbf{E}_x\bar X^4_{(k+1)h}-\mathbf{E}_x\bar X^4_{kh}\leqslant Ch\Big(1+\mathbf{E}_x\bar X^4_{kh}\Big),\quad k\geqslant 0 \tag{4.6}$$

with some constant C which does not depend on k. This gives

$$1+\mathbf{E}_x\bar X^4_{kh}\leqslant (1+Ch)^k(1+x^4),\quad k\geqslant 0, \tag{4.7}$$

which proves (2.24). □

5. One-step estimates

Theorem 5.1. *For any $f\in C^4_b$ there is a constant $C>0$ such that for any $h>0$ and $x\in\mathbb{R}$*

$$\Big|\mathbf{E}_x f(X_h)-\mathbf{E}_x f(\bar X_h)\Big|\leqslant Ch^2(1+x^4) \tag{5.1}$$

The proof of this Theorem will be given in Section 5.2 after necessary preparations made in the next Section.

5.1. Bounded jumps estimates.

Consider the pure jump Lévy process

$$\tilde Z_t=\int_0^t\int_{|z|\leqslant 1} z\,\tilde N(\mathrm{d}z,\mathrm{d}s), \tag{5.2}$$

which is a zero mean Lévy process with $|\Delta\tilde Z_t|\leqslant 1$. We denote by $\tilde X$ the solution of the SDE

$$\begin{aligned}\tilde X_t&=x+\int_0^t a(\tilde X_s)\,\mathrm{d}s+\int_0^t b(\tilde X_s)\circ\mathrm{d}W_s+\int_0^t c(\tilde X_s)\diamond\mathrm{d}\tilde Z_s\\&=\int_0^t\tilde a(\tilde X_s)\,\mathrm{d}s+\int_0^t b(\tilde X_s)\,\mathrm{d}W_s+\int_0^t\int_{|z|\leqslant 1}\Big(\phi^z(\tilde X_{s-})-\tilde X_{s-}\Big)\,\tilde N(\mathrm{d}s,\mathrm{d}z)\end{aligned} \tag{5.3}$$

where we denote the *effective drift* by

$$\tilde a(x)=a(x)+\frac12 b'(x)b(x)+\int_{|z|\leqslant 1}\Big(\phi^z(x)-x-c(x)z\Big)\,\nu(\mathrm{d}z). \tag{5.4}$$

We also introduce for convenience the Stratonovich diffusion correction term

$$\mathring a(x)=a(x)+\frac12 b'(x)b(x). \tag{5.5}$$

Note that due to Lemma A.1, $|\tilde a(x)|,|\mathring a(x)|\leqslant C(1+|x|)$ and $\tilde a',\mathring a'\in C^3_b(\mathbb{R},\mathbb{R})$.

Lemma 5.2. *Assume that conditions $\mathbf{H}_{a,b,c}$ hold true. Then for any $T>0$, any $x\in\mathbb{R}$ there is a unique strong solution $\tilde X=(\tilde X_t)_{t\in[0,T]}$. Moreover for each $p\geqslant 1$ and $T>0$ there is a constant $K_{T,p}>0$ such that*

$$\mathbf{E}_x\sup_{t\in[0,T]}|\tilde X_t|^p\leqslant K_{T,p}(1+|x|^p),\quad x\in\mathbb{R}. \tag{5.6}$$

Proof. The proof is the same as in Theorem 2.1 with no conditions on big jumps $|z|>1$. □

The process $\tilde X$ is a strong Markov process with the generator

$$\tilde Lf(x)=\mathring a(x)f'(x)+\frac12 b^2(x)f''(x)+\int_{|z|\leqslant 1}\Big(f(\phi^z(x))-f(x)-f'(x)c(x)z\Big)\nu(\mathrm{d}z),\quad f\in C^2_c(\mathbb{R},\mathbb{R}). \tag{5.7}$$

Lemma 5.3. *There is a constant $C > 0$ such that for each $f \in C^2(\mathbb{R},\mathbb{R})$ with bounded first and second derivatives*

$$|\tilde{L}f(x)| \leqslant C\Big(\|f'\| + \|f''\|\Big)(1 + x^2), \quad x \in \mathbb{R}. \tag{5.8}$$

Proof. Taking into account the linear growth condition for $\mathring{a}$ and b we get for some $C > 0$

$$\Big|\mathring{a}(x)f'(x) + \frac{1}{2}b^2(x)f''(x)\Big| \leqslant C\|f'\|(1 + |x|) + C\|f''\|(1 + x^2). \tag{5.9}$$

To estimate the integral term in (5.7) we note that

$$f(\phi^z(x)) - f(x) - f'(x)c(x)z = z^2 \int_0^1 \int_0^s \Big(f''c^2 + f'cc'\Big)(\phi^z(u;x))\,\mathrm{d}u\,\mathrm{d}s, \tag{5.10}$$

and Lemma A.1 yields

$$\Big|\int_{|z|\leqslant 1} \Big(f(\phi^z(x)) - f(x) - f'(x)c(x)z\Big)\nu(\mathrm{d}z)\Big| \leqslant C(\|f''\| + \|f'\|)(1 + x^2). \tag{5.11}$$

□

Lemma 5.4. *Let $f \in C_b^4(\mathbb{R},\mathbb{R})$. Then there is a constant $C > 0$ such that for all $x \in \mathbb{R}$*

$$|\tilde{L}\tilde{L}f(x)| \leqslant C(1 + x^4). \tag{5.12}$$

Proof. Denote $G(x) := \tilde{L}f(x)$.
Then

$$\begin{aligned}(\tilde{L}\tilde{L}f)(x) = (\tilde{L}G)(x) &= \mathring{a}(x)G'(x) + \frac{1}{2}b^2(x)G''(x)\\ &\quad + \int_{|z|\leqslant 1} \Big(G(\phi^z(x)) - G(x) - G'(x)c(x)z\Big)\,\nu(\mathrm{d}z).\end{aligned} \tag{5.13}$$

We will show that $|G'(x)| \leqslant C(1 + x^2)$, $|G''(x)| \leqslant C(1 + x^2)$ and

$$\Big|\int_{|z|\leqslant 1} \Big(G(\phi^z(x)) - G(x) - G'(x)c(x)z\Big)\,\nu(\mathrm{d}z)\Big| \leqslant C(1 + x^4). \tag{5.14}$$

1. The first derivative G'. We have

$$\begin{aligned}G'(x) &= \mathring{a}'(x)f'(x) + \Big(\mathring{a}(x) + bb'(x)\Big)f''(x) + \frac{1}{2}b^2(x)f'''(x)\\ &\quad + \int_{|z|\leqslant 1} \Big(f'(\phi^z(x))\phi^z_x(x) - f''(x)c(x)z - f'(x)c'(x)z - f'(x)\Big)\,\nu(\mathrm{d}z)\\ &= (\tilde{L}f')(x) + \mathring{a}'(x)f'(x) + bb'(x)f''(x) + \int_{|z|\leqslant 1} \Big(f'(\phi^z(x))\big(\phi^z_x(x) - 1\big) - f'(x)c'(x)z\Big)\,\nu(\mathrm{d}z).\end{aligned} \tag{5.15}$$

The term $\tilde{L}f'$ is estimated by Lemma 5.3 by $C(1 + x^2)$, the term $\mathring{a}'(x)f'(x)$ by C and the term $bb'f''$ by $C(1 + |x|)$. To estimate the integral term, we use Lemma A.1 to get

$$\begin{aligned}f'(\phi^z(x))\big(\phi^z_x(x) - 1\big) - f'(x)c'(x)z &= f'(\phi^z(x))\big(c'(x)z + \varphi_x(1;x,z)\big) - f'(x)c'(x)z\\ &= c'(x)z^2 \int_0^1 (f'c)(\phi^z(s;x))\,\mathrm{d}s + f'(\phi^z(x))\varphi_x(1;x,z)\end{aligned} \tag{5.16}$$

Taking into account the bounds from Lemma A.1 we conclude that the integral term is estimated by $C(1 + |x|)$ and eventually

$$|G'(x)| \leqslant C(1 + x^2). \tag{5.17}$$

2. The second derivative G''. Straightforward differentiation yields

$$G''(x) = \mathring{a}''(x)f'(x) + \Big(2\mathring{a}'(x) + (b(x)b'(x))'\Big)f''(x) + \Big(\mathring{a}(x) + 2b(x)b'(x)\Big)f'''(x) + \frac{1}{2}b^2(x)f''''(x) + \int_{|z|\leqslant 1}\Big(f''(\phi^z(x))(\phi^z_x(x))^2 + f'(\phi^z(x))\phi^z_{xx}(x) - zf'(x)c''(x) - f''(x)(1+2zc'(x)) - zf'''(x)c(x)\Big)\,\nu(\mathrm{d}z). \tag{5.18}$$

Recalling that

$$(\tilde{L}f'')(x) = \mathring{a}(x)f'''(x) + \frac{1}{2}b^2(x)f^{(4)}(x) + \int_{|z|\leqslant 1}\Big(f''(\phi(x,z)) - f''(x) - f'''(x)c(x)z\Big)\,\nu(\mathrm{d}z) \tag{5.19}$$

we can rewrite

$$G''(x) = (\tilde{L}f'')(x) + \mathring{a}''(x)f'(x) + \Big(2\mathring{a}'(x) + (b(x)b'(x))'\Big)f''(x) + 2b(x)b'(x)f'''(x) + \int_{|z|\leqslant 1}\Big(f''(\phi^z(x))\Big(\phi^z_x(x)^2 - 1\Big) + f'(\phi^z(x))\phi^z_{xx}(x) - zf'(x)c''(x) - 2zf''(x)c'(x)\Big)\,\nu(\mathrm{d}z). \tag{5.20}$$

The first line of the previous formula is bounded by $C(1+x^2)$. We estimate the integrand in its second line similarly to (5.16) with the help of Lemma A.1. Denote for brevity $\varphi_x = \varphi_x(1;x,z)$, $\varphi_{xx} = \varphi_{xx}(1;x,z)$. Then

$$\begin{aligned}
f''(\phi^z(x))\Big(\phi^z_x(x)^2 - 1\Big) &+ f'(\phi^z(x))\phi^z_{xx}(x) - zf'(x)c''(x) - 2zf''(x)c'(x)\\
&= f''(\phi^z(x))\Big(c'(x)^2z^2 + \varphi_x^2 + 2c'(x)z + 2c'(x)z\varphi_x + 2\varphi_x\Big)\\
&\quad + f'(\phi^z(x))\Big(c''(x)z + \varphi_{xx}\Big) - zf'(x)c''(x) - 2zf''(x)c'(x)\\
&= 2zc'(x)\Big(f''(\phi^z(x)) - f''(x)\Big) + zc''(x)\Big(f'(\phi^z(x)) - f'(x)\Big)\\
&\quad + f''(\phi^z(x))\Big(c'(x)^2z^2 + \varphi_x^2 + 2c'(x)z\varphi_x + 2\varphi_x\Big) + f'(\phi^z(x))\varphi_{xx}\\
&= 2z^2c'(x)\int_0^1 (f'''c)(\phi^z(s;x))\,\mathrm{d}s + z^2c''(x)\int_0^1 (f'c)(\phi^z(s;x))\,\mathrm{d}s\\
&\quad + f''(\phi^z(x))\Big(c'(x)^2z^2 + \varphi_x^2 + 2c'(x)z\varphi_x + 2\varphi_x\Big) + f'(\phi^z(x))\varphi_{xx},
\end{aligned} \tag{5.21}$$

and hence the integral term in (5.20) is bounded by $C(1+|x|)$. Eventually

$$|G''(x)| \leqslant C(1+x^2). \tag{5.22}$$

3. The integral term of the generator. For $G(x) := \tilde{L}f(x)$ we recall (5.10), (5.17), (5.22), and the estimate $\sup_{|z|\leqslant 1}|\phi^z(x)| \leqslant C(1+|x|)$, to get

$$\Big|\int_{|z|\leqslant 1}\Big(G(\phi^z(x)) - G(x) - G'(x)c(x)z\Big)\,\nu(\mathrm{d}z)\Big| \leqslant C(1+x^4). \tag{5.23}$$

□

For the function $\psi = \psi(x;\tau,w,z)$ defined in (2.18) and (2.19), we introduce the process

$$Y_t = \psi(x;t,W_t,\tilde{Z}_t), \quad t\in[0,h]. \tag{5.24}$$

Since $\psi(\cdot;\cdot,\cdot,\cdot)\in C^4(\mathbb{R}^4,\mathbb{R})$, the Itô formula implies that Y is an Itô process and

$$\mathbf{E}_x f(Y_t) = f(x) + \int_0^t \mathbf{E}Qf(\psi(x;s,W_s,\tilde Z_s))\,\mathrm{d}s,\quad f\in C_c^2(\mathbb{R},\mathbb{R}), \tag{5.25}$$

with the generator

$$\begin{aligned}Qg(\tau,w,z) &= g_\tau(\tau,w,z)+\frac12 g_{ww}(\tau,w,z)\\ &\quad+\int_{|\xi|\leqslant 1}\Big(g(\tau,w,z+\xi)-g(\tau,w,z)-g_z(\tau,w,z)\cdot\xi\Big)\,\nu(\mathrm{d}\xi),\end{aligned} \tag{5.26}$$

defined on smooth real-valued functions $g(\tau,w,z)$.

Lemma 5.5. *Let $f\in C_b^2(\mathbb{R},\mathbb{R})$. Then*

$$\tilde L f(x) = Qf(\psi(x;0,0,0)). \tag{5.27}$$

Proof. For each $x\in\mathbb{R}$, applying (5.26) to $g(\tau,w,z):=f\circ\psi(x;\tau,w,z)$ we get

$$\begin{aligned}Qf(\psi(x;\tau,w,z)) &= f'(\psi(x;\tau,w,z))\psi_\tau(x;\tau,w,z)\\ &\quad+\frac12 f''(\psi(x;\tau,w,z))\cdot(\psi_w(x;\tau,w,z))^2+\frac12 f'(\psi(x;\tau,w,z))\psi_{ww}(x;\tau,w,z)))\\ &+\int_{|\xi|\leqslant 1}\Big(f(\psi(x;\tau,w,z+\xi))-f(\psi(x;\tau,w,z))-f'(\psi(x;\tau,w,z))\psi_z(x;\tau,w,z)\cdot\xi\Big)\,\nu(\mathrm{d}\xi).\end{aligned} \tag{5.28}$$

Recalling that $\psi(x;0,0,z)=\phi^z(x)$ and $\psi(x;0,0,0)=x$, and taking into account the formulae from Lemma B.1 we find that

$$\begin{aligned}\psi_\tau(x;0,0,0)&=a(x),\\ \psi_w(x;0,0,0)&=b(x),\\ \psi_{ww}(x;0,0,0)&=bb'(x),\\ \psi_z(x;0,0,0)&=c(x),\end{aligned} \tag{5.29}$$

and hence we get (5.27). □

Lemma 5.6. *Let $f\in C_b^4(\mathbb{R},\mathbb{R})$. Then there is a constant $C>0$ such that for any $\tau\geqslant 0$, $w\in\mathbb{R}$, $z\in\mathbb{R}$ and $x\in\mathbb{R}$*

$$|QQf(\psi(x;\tau,w,z))|\leqslant C(1+x^4)\cdot \mathrm{e}^{C(\tau+|w|+|z|)}. \tag{5.30}$$

Proof. Denoting for brevity where it is possible $\psi=\psi(x;\tau,w,z)=\psi(\tau,w,z)$ or adopting when necessary the notation $\psi(z):=\psi(x;\tau,w,z)$, we apply the formula (5.28) to a C_b^4-function f to get

$$\begin{aligned}Qf(\psi(\tau,w,z)) &= f'(\psi)\psi_\tau+\frac12 f''(\psi)\cdot\psi_w^2+\frac12 f'(\psi)\psi_{ww}\\ &\quad+\int_{|\xi|\leqslant 1}\Big(f(\psi(z+\xi))-f(\psi(z))-f'(\psi(z))\psi_z(z)\cdot\xi\Big)\,\nu(\mathrm{d}\xi)\\ &= f'(\psi)\psi_\tau+\frac12 f''(\psi)\cdot\psi_w^2+\frac12 f'(\psi)\psi_{ww}+\int_{|\xi|\leqslant 1}\xi^2\int_0^1\partial_{zz}f(\psi(z+\xi\theta))(1-\theta)\,\mathrm{d}\theta\,\nu(\mathrm{d}\xi).\end{aligned} \tag{5.31}$$

With the help of (5.26) we calculate

$$\begin{aligned}Q^2 f(\psi(\tau,w,z)) &= \partial_\tau Qf(\psi)+\frac12\partial^2_{ww}Qf(\psi)\\ &\quad+\int_{|\xi|\leqslant 1}\Big(Qf(\psi(z+\xi))-Qf(\psi(z))-\partial_z Qf(\psi(z))\cdot\xi\Big)\,\nu(\mathrm{d}\xi)\end{aligned} \tag{5.32}$$

$$= \partial_\tau Qf(\psi) + \frac{1}{2}\partial^2_{ww} Qf(\psi) + \int_{|\xi|\leqslant 1} \xi^2 \int_0^1 \partial_{zz} Qf(\psi(z+\theta\xi))(1-\theta)\,\mathrm{d}\theta\,\nu(\mathrm{d}\xi).$$

We estimate the summands in (5.32).

1. $\partial_\tau Qf$**.** First, we write

(5.33)
$$\begin{aligned}\partial_\tau Qf(\psi(\tau,w,z)) &= f''(\psi)\psi_\tau^2 + f'(\psi)\psi_{\tau\tau}\\ &+ \frac{1}{2}\Big(f'''(\psi)\cdot\psi_\tau\cdot(\psi_w)^2 + 2f''(\psi)\cdot\psi_w\cdot\psi_{\tau w} + f''(\psi)\cdot\psi_\tau\cdot\psi_{ww} + f'(\psi)\cdot\psi_{\tau ww}\Big)\\ &+ \int_{|\xi|\leqslant 1}\xi^2\int_0^1 \partial_{\tau zz} f(\psi(z+\xi\theta))(1-\theta)\,\mathrm{d}\theta\,\nu(\mathrm{d}\xi),\end{aligned}$$

where for the integral term we get

$$\partial_{\tau zz} f(\psi(\tau,w,z)) = f'''(\psi)\psi_\tau\psi_z^2 + f''(\psi)\psi_\tau\psi_{zz} + 2f''(\psi)\psi_{\tau z}\psi_\tau + f'(\psi)\psi_{\tau zz}. \tag{5.34}$$

Hence in view of Lemma B.1

$$|\partial_\tau Qf(\psi(\tau,w,z))| \leqslant C(1+|x|^3)\cdot(1+\tau+|w|+|z|)^2\cdot \mathrm{e}^{C(\tau+|w|+|z|)}. \tag{5.35}$$

2. $\partial_{ww} Qf$**.** Analogously

$$\begin{aligned}\partial_{ww} Qf(\psi) &= f'''(\psi)\psi_\tau\psi_w^2 + 2f''(\psi)\psi_{\tau w}\psi_w + f''(\psi)\psi_\tau\psi_{ww}\\ &+ f'(\psi)\psi_{\tau ww} + \frac{1}{2}f^{(4)}(\psi)\psi_w^4 + 3f'''(\psi)\psi_w^2\psi_{ww}\\ &+ \frac{3}{2}f''(\psi)\psi_{ww}^2 + 2f''(\psi)\psi_w\psi_{www} + \frac{1}{2}f'(\psi)\psi_{wwww}\\ &+ \int_{|\xi|\leqslant 1}\xi^2\int_0^1 \partial_{wwzz} f(\psi(z+\xi\theta))(1-\theta)\,\mathrm{d}\theta\,\nu(\mathrm{d}\xi),\end{aligned} \tag{5.36}$$

where for the integral term we calculate

(5.37)
$$\begin{aligned}\partial_{wwzz} f(\psi) &= f^{(4)}(\psi)\psi_w^2\psi_z^2\\ &+ f'''(\psi)\psi_{ww}\psi_z^2 + 4f'''(\psi)\psi_w\psi_{wz}\psi_z + f'''(\psi)\psi_w^2\psi_{zz}\\ &+ 2f''(\psi)\psi_w\psi_{wzz} + 2f''(\psi)\psi_{wz}^2 + (f'(\psi)+f''(\psi))\psi_{ww}\psi_{zz} + 2f''(\psi)\psi_{wwz}\psi_z,\end{aligned}$$

which yields

$$|\partial_{ww} Qf(\psi(\tau,w,z))| \leqslant C(1+x^4)\cdot(1+\tau+|w|+|z|)^2\cdot \mathrm{e}^{C(\tau+|w|+|z|)}. \tag{5.38}$$

3. $\partial_{zz} Qf$**.** We determine the derivatives

(5.39)
$$\begin{aligned}&\partial_{zz}\Big(f'(\psi)\psi_\tau + \frac{1}{2}f''(\psi)\cdot\psi_w^2 + \frac{1}{2}f'(\psi)\psi_{ww}\Big)\\ &= f'(\psi)\psi_\tau\psi_{zz} + f''(\psi)\psi_\tau\psi_z^2 + (f'(\psi)+f''(\psi))\psi_{\tau z}\psi_z + f'(\psi)\psi_{\tau zz}\\ &+ \frac{1}{2}f^{(4)}(\psi)\psi_w^2\psi_z^2 + 2f'''(\psi)\psi_w\psi_z\psi_{wz} + \frac{1}{2}f'''(\psi)\psi_w^2\psi_{zz} + f''(\psi)\psi_{wz}^2 + f''(\psi)\psi_w\psi_{wzz}\\ &+ \frac{1}{2}f'''(\psi)\psi_z^2\psi_{ww} + f''(\psi)\psi_z\psi_{wwz} + \frac{1}{2}\psi''(\psi)\psi_{ww}\psi_{zz} + \frac{1}{2}f'(\psi)\psi_{wwzz},\end{aligned}$$

and

(5.40)
$$\partial_{zzzz} f(\psi) = f^{(4)}(\psi)\psi_z^4 + 6f'''(\psi)\psi_z^2\psi_{zz} + 3f''(\psi)\psi_{zz}^2 + 4f''(\psi)\psi_z\psi_{zzz} + f'(\psi)\psi_{zzzz}$$

and apply Lemma B.1 to get

$$|\partial_{zz} Qf(\psi(x;t,w,z))| \leqslant C(1+x^4)\cdot(1+\tau+|w|+|z|)^3\cdot \mathrm{e}^{C(\tau+|w|+|z|)}. \tag{5.41}$$

□

Lemma 5.7. *For any $f \in C_b^4(\mathbb{R},\mathbb{R})$ there is a constant $C > 0$ such that for any $h \geqslant 0$ and any $x \in \mathbb{R}$*

$$|\mathbf{E}_x f(\tilde X_h) - \mathbf{E}_x f(\psi(x;h,W_h,\tilde Z_h))| \leqslant C(1+x^4)h^2. \tag{5.42}$$

Proof. Applying the Itô formula twice we get

$$\begin{aligned}\mathbf{E}_x f(\tilde X_h) - \mathbf{E}_x f(\psi(x;h,W_h,\tilde Z_h)) &= \int_0^h \mathbf{E}_x \tilde L f(\tilde X_s)\,\mathrm{d}s - \int_0^h \mathbf{E}_x Q f(\psi(x;s,W_s,\tilde Z_s))\,\mathrm{d}s\\ &= h\tilde L f(x) - hQf(\psi(x;0,0,0))\\ &+ \int_0^h\int_0^s \mathbf{E}_x \tilde L\tilde L f(\tilde X_r)\,\mathrm{d}r\,\mathrm{d}s - \int_0^h\int_0^s \mathbf{E} QQf(\psi(x;r,W_r,\tilde Z_r))\,\mathrm{d}r\,\mathrm{d}s,\end{aligned} \tag{5.43}$$

and hence by Lemma 5.5 and Hölder's inequality for any $p > 1$

$$\begin{aligned}&\Big|\mathbf{E}_x f(\tilde X_h) - \mathbf{E}_x f(\psi(x;h,W_h,\tilde Z_h))\Big|\\ &\leqslant h^2 \sup_{r\in[0,h]} \mathbf{E}_x|\tilde L\tilde L f(\tilde X_r)| + h^2 \sup_{r\in[0,h]} \mathbf{E}|QQf(\psi(x;r,W_r,\tilde Z_r))|\\ &\leqslant Ch^2\Big(1+\sup_{r\in[0,h]}\mathbf{E}_x|\tilde X_r|^4\Big) + Ch^2\sup_{r\in[0,h]}\mathbf{E}_x(1+|\tilde X_r|^4)\mathrm{e}^{C(r+|W_r|+1)}\\ &\leqslant Ch^2(1+|x|^4) + Ch^2\sup_{r\in[0,h]}\Big(\mathbf{E}_x(1+|\tilde X_r|^4)^p\Big)^{1/p}\Big(\mathbf{E}\mathrm{e}^{\frac{pC}{p-1}(r+|W_r|+1)}\Big)^{(p-1)/p}\\ &\leqslant Ch^2(1+|x|^4).\end{aligned} \tag{5.44}$$

□

5.2. **One-step estimate. Proof of Theorem 5.1.**

Proof. Decompose the jump process Z into a sum

$$Z_t = \tilde Z_t + \sum_{k=0}^{N_t} J_k. \tag{5.45}$$

Assume from the very beginning that $\lambda = \nu(|z|>1) > 0$. Denote $\sigma := \sigma_1$, the first jump time of $t \mapsto \int_0^t\int_{|z|>1} N(\mathrm{d}z,\mathrm{d}s)$, $J = J_1$ the size of the first large jump. First note, that $\mathbf{P}(\tau \leqslant t|N_h = 1) = t/h$, $t \in [0,h]$, and $\mathbf{P}(J \in A|N_h = 1) = \nu(A \cap \{|z|>1\})/\nu(|z|>1)$.

For each $x \in \mathbb{R}$ we get

$$\begin{aligned}|\mathbf{E}_x f(X_h) - \mathbf{E}_x f(\bar X_h)| &\leqslant |\mathbf{E}_x f(\tilde X_h) - \mathbf{E}_x f(\psi(x;h,W_h,\tilde Z_h))|\\ &+ \mathbf{E}_x\Big[|f(X_h) - f(\bar X_h)|\Big|N_h = 1\Big]\mathbf{P}(N_h = 1)\\ &+ 2\|f\|\mathbf{P}(N_h \geqslant 2).\end{aligned} \tag{5.46}$$

The first summand is estimated by Lemma 5.7 by $C(1+x^4)h^2$, the third has the order h^2. Let us estimate the second summand.

First note that $\mathbf{P}(N_h = 1) \leqslant Ch$. Then, on the event $\{N_h = 1\}$, the solution X_h can be represented as a composition

$$X_h(x) = \tilde X_{\sigma,h} \circ \phi^J \circ \tilde X_{0,\sigma-}(x) \tag{5.47}$$

and hence

$$\begin{aligned}f(\bar X_h(x)) - f(X_h(x)) &= f(\phi^J(x)) - f(\tilde X_{\sigma,h}\circ\phi^J(x))\\ &+ f(\tilde X_{\sigma,h}\circ\phi^J(x)) - f(\tilde X_{\sigma,h}\circ\phi^J\circ\tilde X_{0,\sigma-}(x))\\ &+ f(\bar X_h(x)) - f(\phi^J(x)).\end{aligned} \tag{5.48}$$

Step 1. Desintegrating the laws of σ, J and $\tilde{Z}$ we obtain from the Itô formula, Lemma 5.3 and Assumption $\mathbf{H}_\nu$ that

$$
\begin{aligned}
&|\mathbf{E} f(\tilde{X}_{\sigma,h}\circ\phi^J(x)) - \mathbf{E} f(\phi^J(x))| \\
&\leqslant \frac{1}{\lambda h}\int_0^h\int_{|z|>1} |\mathbf{E} f(\tilde{X}_{h-s}(\phi^z(x))) - f(\phi^z(x))|\,\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\leqslant \frac{1}{\lambda h}\int_0^h\int_{|z|>1}\int_0^{h-s} \mathbf{E}_{\phi^z(x)}|\tilde{L} f(\tilde{X}_r)|\,\mathrm{d}r\,\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\leqslant \frac{C}{\lambda h}\int_0^h\int_{|z|>1}\int_0^{h-s} \mathbf{E}_{\phi^z(x)}(1+|\tilde{X}_r|^2)\,\mathrm{d}r\,\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\leqslant \frac{C_1}{\lambda h}\int_0^h\int_{|z|>1} h\Big(1+|\phi^z(x)|^2\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\leqslant C_2 h\cdot\Big(1+\int_{|z|>1}|\phi^z(x)|^2\,\nu(\mathrm{d}z)\Big) \\
&\leqslant C_3 h(1+x^2).
\end{aligned}
\tag{5.49}
$$

Step 2. Acting similarly we estimate

$$
\begin{aligned}
\mathbf{E} f(\tilde{X}_{\sigma,h}\circ\phi^J(x)) &- \mathbf{E} f(\tilde{X}_{\sigma,h}\circ\phi^J\circ\tilde{X}_{\sigma-}(x)) \\
&= \mathbf{E}\Big[\mathbf{E} f(\tilde{X}_{\sigma,h}\circ\phi^J(x)) - \mathbf{E} f(\tilde{X}_{\sigma,h}\circ\phi^J\circ\tilde{X}_{\sigma-}(x))\Big|\mathscr{F}_\sigma\Big] \\
&= \mathbf{E}\Big[\mathbf{E}_{\phi^J(x)} f(\tilde{X}_{h-\sigma}) - \mathbf{E}_{\phi^J(\tilde{X}_{\sigma-}(x))} f(\tilde{X}_{h-\sigma})\Big] \\
&= \frac{1}{h}\int_0^h\int_{|z|>1}\mathbf{E}_x\Big[\mathbf{E}_{\phi^z(x)} f(\tilde{X}_{h-s}) - \mathbf{E}_{\phi^z(\tilde{X}_{s-}(x))} f(\tilde{X}_{h-s})\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}
\tag{5.50}
$$

Denote

$$
\tilde{f}^t(x) = \mathbf{E}_x f(\tilde{X}_t). \tag{5.51}
$$

Since by Theorem 2.2

$$
\sup_{t\in[0,T]}\Big(\|\tilde{f}^t_x\| + \|\tilde{f}^t_{xx}\|\Big) \leqslant C \tag{5.52}
$$

we can calculate

$$
\begin{aligned}
\|\partial_x \tilde{f}^t(\phi^z(x))\| &\leqslant C\cdot\|\phi^z_x\|, \\
\|\partial_{xx} \tilde{f}^t(\phi^z(x))\| &\leqslant C\Big(\|\phi^z_x\|^2 + \|\phi^z_{xx}\|\Big).
\end{aligned}
\tag{5.53}
$$

Then for each $s\in[0,h]$ the Itô formula and Lemma 5.3 imply

$$
\begin{aligned}
\Big|\mathbf{E}_x \tilde{f}^{h-s}(\phi^z(\tilde{X}_{s-})) - \tilde{f}^{h-s}(\phi^z(x))\Big| &\leqslant \int_0^s \mathbf{E}_x|\tilde{L}\tilde{f}^{h-s}(\phi^z(\tilde{X}_r))|\,\mathrm{d}r \\
&\leqslant C\cdot h\cdot\Big(\|\phi^z_x\|^2 + \|\phi^z_{xx}\|\Big)\cdot(1+\sup_{r\in[0,h]}\mathbf{E}_x|\tilde{X}_r|^2) \\
&\leqslant C\cdot h\cdot\Big(\|\phi^z_x\|^2 + \|\phi^z_{xx}\|\Big)\cdot(1+x^2).
\end{aligned}
\tag{5.54}
$$

Hence Assumption $\mathbf{H}_\nu$ yields

$$
|\mathbf{E} f(\tilde{X}_{\sigma,h}\circ\phi^J(x)) - \mathbf{E} f(\tilde{X}_{\sigma,h}\circ\phi^J\circ\tilde{X}_{\sigma-}(x))| \leqslant Ch(1+x^2). \tag{5.55}
$$

Step 3. Recall that $\bar{X}_h(x) = \psi(x;h,W_h,J+\tilde{Z}_h)$. The Taylor expansion of $\psi = \psi(x;\tau,w,J+\xi)$ at $(0,0,J)$ for a fixed x yields

$$\begin{aligned} f(\psi(x,h,w,J+\xi)) &= f(\psi(x,0,0,J)) \\ &\quad + f'(\psi(x,0,0,J))\Big(\psi_\tau(x;0,0,J)h + \psi_w(x;0,0,J)w + \psi_z(x;0,0,J)\xi\Big) \\ &\quad + R(x;h,w,J+\xi), \end{aligned} \tag{5.56}$$

with the remainder term

$$\begin{aligned} R(x;h,w,J+\xi) = \frac{1}{2}\int_0^1 f''(\psi(\theta))\Big(&\psi_{\tau\tau}(\theta)h^2 + 2\psi_{\tau w}(\theta)hw + 2\psi_{\tau z}(\theta)h\xi + \psi_{ww}(\theta)w^2 \\ &+ 2\psi_{wz}(\theta)w\xi + \psi_{zz}(\theta)\xi^2\Big)\,\mathrm{d}\theta = R_1 + \cdots + R_6. \end{aligned} \tag{5.57}$$

where we write $\psi(\theta) := \psi(x;\theta h,\theta w,\theta\xi + J)$.

Due to the independence of $\tilde{Z}$, J and W, and $\mathbf{E}W_h = \mathbf{E}\tilde{Z}_h = 0$, we get that the mean value of the second line in (5.56) vanishes.

To estimate the remainder term we have to estimate six terms with the help of (B.3). Thus

$$\begin{aligned} \mathbf{E}|R_1| &\leqslant h^2\|f''\|\int_0^1 \mathbf{E}|\psi_{\tau\tau}(x;\theta h,\theta W_h,\theta\tilde{Z}_h + J)|\,\mathrm{d}\theta \\ &\leqslant h^2\|f''\|C(1+x^2)\mathbf{E}(2+h+|W_h|+|J|)\mathrm{e}^{5(\|a'\|h+\|b'\||W_h|+\|c'\|(|J|+1))} \\ &\leqslant Ch^2(1+x^2). \end{aligned} \tag{5.58}$$

Analogously, the terms R_2 and R_3 are bounded by $Ch(1+x^2)$. Further,

$$\begin{aligned} \mathbf{E}|R_4| &\leqslant \|f''\|\int_0^1 \mathbf{E}|\psi_{wz}(x;\theta h,\theta W_h,\theta\tilde{Z}_h + J)|\cdot W_h^2\,\mathrm{d}\theta \\ &\leqslant \|f''\|C(1+x^2)\mathbf{E}\Big[W_h^2(2+h+|W_h|+|J|)\mathrm{e}^{5(\|a'\|h+\|b'\||W_h|+\|c'\|(|J|+1))}\Big] \\ &\leqslant Ch(1+x^2), \end{aligned} \tag{5.59}$$

where the factor h essentially comes from the term W_h^2. The R_2 and R_3 are bounded by $Ch(1+x^2)$ in a silmilar way. □

6. Main estimates and the proof of Theorem 2.4

According to Markov property of X, for each $t \in [0,T]$ and any bounded measurable f

$$\mathbf{E}_x f(X_T) = \mathbf{E}_x\mathbf{E}_{X_{T-t}} f(X_t) = \mathbf{E}_x f^t(X_{T-t}), \tag{6.1}$$

where

$$f^t(x) := \mathbf{E}_x f(X_t). \tag{6.2}$$

Let $h > 0$ and let for definiteness $T = nh$ for some $n > 0$. Denote

$$u_k(x) := \mathbf{E}_x f^{kh}(\tilde{X}_{T-kh}). \tag{6.3}$$

Then,

$$\begin{aligned} \mathbf{E}_x f(X_T) &= u_n, \\ \mathbf{E}_x f(\bar{X}_T) &= u_0, \end{aligned} \tag{6.4}$$

and we have the following chaining representation

$$\begin{aligned}\mathbf{E}_x f(X_{nh}) - \mathbf{E}_x f(\bar{X}_{nh}) &= \sum_{k=1}^{n}(u_k - u_{k-1}) \\ &= \sum_{k=1}^{n}\Big(\mathbf{E}_x f^{kh}(\bar{X}_{nh-kh}) - \mathbf{E}_x f^{(k-1)h}(\bar{X}_{nh-kh+h})\Big).\end{aligned} \tag{6.5}$$

Observe that

$$\mathbf{E}_x f^{(k-1)h}(\bar{X}_{(n-k+1)h}) = \mathbf{E}_x \mathbf{E}_{\bar{X}_{(n-k)h}} f^{(k-1)h}(\bar{X}_h), \tag{6.6}$$

and, using the property

$$f^{kh}(y) = \mathbf{E}_y f^{(k-1)h}(X_h), \tag{6.7}$$

we have that

$$\mathbf{E}_x f^{kh}(\bar{X}_{(n-k)h}) = \mathbf{E}_x \mathbf{E}_{\bar{X}_{(n-k)h}} f^{(k-1)h}(X_h). \tag{6.8}$$

Combining (6.5), (6.6) and (6.8), we finally have

$$\mathbf{E}_x f(X_{nh}) - \mathbf{E}_x f(\bar{X}_{nh}) = \sum_{k=1}^{n} \mathbf{E}_x \Big(\mathbf{E}_{\bar{X}_{(n-k)h}} f^{(k-1)h}(X_h) - \mathbf{E}_{\bar{X}_{(n-k)h}} f^{(k-1)h}(\bar{X}_h)\Big). \tag{6.9}$$

By Theorem 5.1 and the 4th moment bound (2.24) from Theorem 2.1,

$$\begin{aligned}\mathbf{E}_x \Big|\mathbf{E}_{\bar{X}_{(n-k)h}} f^{(k-1)h}(X_h) - \mathbf{E}_{\bar{X}_{(n-k)h}} f^{(k-1)h}(\bar{X}_h)\Big| \\ \leqslant C_1 h^2(1 + \mathbf{E}_x|\bar{X}_{(n-k)h}|^4) \leqslant C_2 h^2(1 + x^4),\end{aligned} \tag{6.10}$$

what together with (6.5) finishes the proof.

7. C^4-smoothness of the Marcus semigroup. Proof of Theorem 2.2

We separate the proof in two parts. First, we prove the required statement in the case $\nu(|z| > 1) = 0$; that is, for $X = \tilde{X}$. We consider all the derivatives of f^t till the order 4:

$$\partial_x f^t(x) = \mathbf{E}_x \Big(f'(X_t)\partial_x X_t\Big), \tag{7.1}$$

$$\partial_{xx} f^t(x) = \mathbf{E}_x \left(f''(X_t)(\partial_x X_t)^2\right) + \mathbf{E}_x \left(f'(X_t)\partial_{xx} X_t\right), \tag{7.2}$$

$$\partial_{xxx} f^t(x) = \mathbf{E}_x \left(f'''(X_t)(\partial_x X_t)^3\right) + 3\mathbf{E}_x \left(f''(X_t)(\partial_x X_t)(\partial_{xx} X_t)\right) + \mathbf{E}_x \left(f'(X_t)\partial_{xxx} X_t\right), \tag{7.3}$$

$$\begin{aligned}\partial_{xxxx} f^t(x) = \mathbf{E}_x \Big(f^{(4)}(X_t)(\partial_x X_t)^4\Big) &+ 6\mathbf{E}_x \left(f'''(X_t)(\partial_x X_t)^2(\partial_{xx} X_t)\right) \\ + 3\mathbf{E}_x \left(f''(X_t)(\partial_{xx} X)^2\right) &+ 7\mathbf{E}_x \left(f''(X_t)(\partial_x X_t)(\partial_{xxx} X_t)\right) \\ + \mathbf{E}_x \left(f'(X_t)\partial_{xxxx} X_t\right).&\end{aligned} \tag{7.4}$$

Then the required statement follows from the following Proposition.

Proposition 7.1. *Let $\nu(|z| > 1) = 0$ and $\mathbf{H}_{a,b,c}$ holds. Then for any $p > 1$ and $T > 0$*

$$\sup_{t \leqslant T, x \in \mathbb{R}} \mathbf{E}|\partial_x^k X_t(x)|^p < \infty, \quad k = 1, \dots, 4. \tag{7.5}$$

Proposition 7.1 has the same spirit as Lemma 4.2 in Protter and Talay [34]. However, their result is not applicable here directly, because the Itô form of the Marcus SDE

$$\mathrm{d}X_t = \tilde a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t + \int_{|z|\leqslant 1}\Big(\phi^z(X_{t-}) - X_{t-}\Big)\,\tilde N(\mathrm{d}t,\mathrm{d}z), \tag{7.6}$$

contains the integral w.r.t. the compensated Poisson random measure, while [34] deals with the Itô-SDEs w.r.t. $\mathrm{d}Z$ with a Lévy process Z. Because of that, we outline the proof, mainly in order to make it visible how the non-linear structure of the jump part effects on the assumptions required.

Proof. Without loss of generality we can assume $p\geqslant 2$, which will allow us to apply the Itô formula with the C^2-function $|x|^p$.

1. The first derivative. Denote $X'_t := \partial_x X_t$, then

$$\mathrm{d}X'_t = \tilde a'(X_t)X'_t\,\mathrm{d}t + b'(X_t)X'_t\,\mathrm{d}W_t + \int_{|z|\leqslant 1}\Big(\phi^z_x(X_{t-}) - 1\Big)X'_{t-}\,\tilde N(\mathrm{d}t,\mathrm{d}z), \tag{7.7}$$

and the Itô formula yields

$$\begin{aligned}|X'_t|^p &= 1 + p\int_0^t |X'_s|^p\tilde a'(X_s)\,\mathrm{d}s + \frac{p(p-1)}{2}\int_0^t |X'_s|^p b'(X_s)^2\,\mathrm{d}s\\ &\quad+\int_0^t\int_{|z|\leqslant 1}\Big(|\phi^z_x(X_s)|^p - 1 - p(\phi^z_x(X_s)-1)\Big)|X'_s|^p\,\nu(\mathrm{d}z)\,\mathrm{d}s\\ &\quad+p\int_0^t |X'_s|^p b'(X_s)\,\mathrm{d}W_s + \int_0^t\int_{|z|\leqslant 1}\Big(|\phi^z_x(X_{s-})|^p - 1\Big)|X'_{s-}|^p\,\tilde N(\mathrm{d}t,\mathrm{d}z),\end{aligned} \tag{7.8}$$

where the last two terms are local martingales. Then the standard argument, based on the martingale localization and the Fatou lemma, yields

$$\begin{aligned}\mathbf{E}|X'_t|^p &\leqslant 1 + p\int_0^t \mathbf{E}|X'_s|^p|\tilde a'(X_s)|\,\mathrm{d}s + \frac{p(p-1)}{2}\int_0^t \mathbf{E}|X'_s|^p b'(X_s)^2\,\mathrm{d}s\\ &\quad+\int_0^t\int_{|z|\leqslant 1}\mathbf{E}\Big(|\phi^z_x(X_s)|^p - 1 - p(\phi^z_x(X_s)-1)\Big)|X'_s|^p\,\nu(\mathrm{d}z)\,\mathrm{d}s\end{aligned} \tag{7.9}$$

We have the following elementary inequality: for any $p\geqslant 2$ there exists C_p such that for $a,\,\delta\in\mathbb{R}$

$$|a+\delta|^p \leqslant |a|^p + p|a|^{p-1}(\operatorname{sgn} a)\delta + C_p\Big(|a|^{p-2}\delta^2 + |\delta|^p\Big). \tag{7.10}$$

In addition, we have $\tilde a', b'$ bounded and, by Lemma A.1,

$$|\phi^z_x(x) - 1| \leqslant C|z|,\quad |z|\leqslant 1. \tag{7.11}$$

Then, applying (7.10) with $a=1$, $\delta = \phi^z_x(x) - 1$ we get from (7.8)

$$\mathbf{E}_x|X'_t|^p \leqslant 1 + C_{p,T}\int_0^t \mathbf{E}_x|X'_s|^p\,\mathrm{d}s,\quad t\leqslant T, \tag{7.12}$$

which yields (7.5) for $k=1$ by the Gronwall lemma.

2. The second derivative. Denote $X''_t := \partial_{xx}X_t = \partial_x X'_t$. Then

$$\begin{aligned}\mathrm{d}X''_t &= \Big(\tilde a''(X_t)(X'_t)^2 + a'(X_t)X''_t\Big)\,\mathrm{d}t\\ &\quad+\Big(b''(X_t)(X'_t)^2 + b'(X_t)X''_t\Big)\,\mathrm{d}W_t\\ &\quad+\int_{|z|\leqslant 1}\Big[\phi^z_{xx}(X_{t-})(X'_{t-})^2 + \Big(\phi^z_x(X_{t-}) - 1\Big)X''_{t-}\Big]\,\tilde N(\mathrm{d}t,\mathrm{d}z),\quad X''_0 = 0.\end{aligned} \tag{7.13}$$

By the Itô formula, localization, and the Fatou lemma,
(7.14)

$$
\begin{aligned}
\mathbf{E}|X''_t|^p \leqslant p\mathbf{E}\int_0^t \Big(|\tilde a''(X_s)|(X'_s)^2 + |\tilde a'(X_s)||X''_s|\Big)|X''_s|^{p-1}\,\mathrm{d}s \\
+ \frac{p(p-1)}{2}\mathbf{E}\int_0^t \Big(|b''(X_s)||X'_s|^2 + |b'(X_s)||X''_s|\Big)^2|X''_s|^{p-2}\,\mathrm{d}s \\
+ \mathbf{E}\int_0^t\int_{|z|\leqslant 1}\Big[\Big|\phi^z_{xx}(X_s)(X'_s)^2 + \phi^z_x(X_s)X''_s\Big|^p - |X''_s|^p \\
- p|X''_s|^{p-1}\operatorname{sgn}(X''_s)\Big(\phi^z_{xx}(X_s)(X'_s)^2 + \Big(\phi^z_x(X_s)-1\Big)X''_s\Big)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}
$$

We apply (7.10) with $a = A(X'') = X''$, $\delta = \delta(X,X',X'',z) = \phi^z_{xx}(X)(X')^2 + \Big(\phi^z_x(X) - 1\Big)X''$. By Lemma A.1, we have for $|z|\leqslant 1$

$$
|\phi^z_{xx}(x)| \leqslant C|z|, \tag{7.15}
$$

which together with (7.11) gives
(7.16)

$$
|\delta(X,X',X'',z)|^2 \leqslant C(|X'|^4 + |X''|^2)|z|^2,\quad |\delta(X,X',X'',z)|^p \leqslant C(|X'|^{2p} + |X''|^p)|z|^p.
$$

Since $\tilde a', \tilde a'', b', b''$ are bounded and $|z|^p\leqslant |z|^2$ for $|z|\leqslant 1$, this yields the inequality

$$
\mathbf{E}|X''_t|^p \leqslant C\mathbf{E}\int_0^t \Big(|X''_s|^p + |X''_s|^{p-1}|X'_s|^2 + |X''_s|^{p-2}|X'_s|^4 + |X'_s|^{2p}\Big)\,\mathrm{d}s. \tag{7.17}
$$

By the Young inequality

$$
ab \leqslant \frac{a^{p'}}{p'} + \frac{b^{q'}}{q'},\quad a,b\geqslant 0,\quad \frac{1}{p'}+\frac{1}{q'} = 1, \tag{7.18}
$$

we have

$$
|X''_s|^{p-1}|X'_s|^2 \leqslant \frac{1}{p}|X'_s|^{2p} + \frac{p-1}{p}|X''_s|^p,\quad |X''_s|^{p-2}|X'_s|^4 \leqslant \frac{2}{p}|X'_s|^{2p} + \frac{p-2}{p}|X''_s|^p. \tag{7.19}
$$

Then (7.5) with $2p$ and $k=1$, (7.17), and the Gronwall inequality yield (7.5) with p and $k=2$.

3. The third derivative. Denote $X'''_t := \partial_{xxx}X_t = \partial_{xx}X'_t = \partial_x X''_t$, then
(7.20)

$$
\begin{aligned}
\mathrm{d}X'''_t &= \Big(\tilde a'''(X_t)(X'_t)^3 + 3\tilde a''(X_t)X'_tX''_t + \tilde a'(X_t)X'''_t\Big)\,\mathrm{d}t \\
&+ \Big(b'''(X_t)(X'_t)^3 + 3b''(X_t)X'_tX''_t + b'(X_t)X'''_t\Big)\,\mathrm{d}W_t \\
&+ \int_{|z|\leqslant 1}\Big[\phi^z_{xxx}(X_{t-})(X'_{t-})^3 + 3\phi^z_{xx}(X_{t-})X'_{t-}X''_{t-} + \Big(\phi^z_x(X_{t-})-1\Big)X'''_{t-}\Big]\,\tilde N(\mathrm{d}t,\mathrm{d}z)
\end{aligned}
$$

By the Itô formula, localization, and the Fatou lemma,

(7.21)

$$
\begin{aligned}
\mathbf{E}|X'''_t|^p \leqslant p\mathbf{E}\int_0^t \Big(|\tilde a'''(X_s)||X'_s|^3 + 3|\tilde a''(X_s)||X'_s||X''_s| + |\tilde a'(X_s)||X'''_s|\Big)|X'''_s|^{p-1}\,\mathrm{d}s \\
+ \frac{p(p-1)}{2}\mathbf{E}\int_0^t \Big(|b'''(X_s)||X'_s|^3 + 3|b''(X_s)||X'_s||X''_s| + |b'(X_s)||X'''_s|\Big)^2|X'''_s|^{p-2}\,\mathrm{d}s \\
+ \mathbf{E}\int_0^t\int_{|z|\leqslant 1}\Big[\Big|X'''_s + \phi^z_{xxx}(X_s)(X'_s)^3 + 3\phi^z_{xx}(X_s)X'_sX''_s + \Big(\phi^z_x(X_s)-1\Big)X'''_s\Big|^p - |X'''_s|^p \\
-p(X'''_s)^{p-1}\operatorname{sgn}(X'''_s)\Big(\phi^z_{xxx}(X_s)(X'_s)^3 + 3\phi^z_{xx}(X_s)X'_sX''_s + \Big(\phi^z_x(X_s)-1\Big)X'''_s\Big)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}
$$

We apply (7.10) with

$$a = X''', \delta = \delta(X,X',X'',X''',z) = \phi^z_{xxx}(X)(X')^3 + 3\phi^z_{xx}(X)X'X'' + \Big(\phi^z_x(X)-1\Big).$$

We have for $|z|\leqslant 1$ by Lemma A.1

$$|\phi^z_{xxx}(x)|\leqslant C|z|, \tag{7.22}$$

which together with (7.11), (7.15) and the Young inequality gives

$$\begin{aligned}|\delta(X,X',X'',X''',z)|^2&\leqslant C(|X'|^6+|X''|^4+|X'''|^2)|z|^2,\\ |\delta(X,X',X'',X''',z)|^p&\leqslant C(|X'|^{3p}+|X''|^{2p}+|X'''|^p)|z|^p.\end{aligned} \tag{7.23}$$

Since the derivatives of $\tilde a, b$ are bounded and $|X'||X''|\leqslant C(|X'|^3+|X''|^{3/2})$ by the Young inequality, we get

$$\begin{aligned}\mathbf{E}|X'''_t|^p\leqslant C\mathbf{E}\int_0^t\Big(&|X'''_s|^p+|X'''_s|^{p-1}|X'_s|^3+|X'''_s|^{p-1}|X''_s|^{3/2}\\ &+|X'''_s|^{p-2}|X'_s|^6+|X'''_s|^{p-2}|X''_s|^3+|X'_s|^{3p}+|X''_s|^{3p/2}\Big)\,\mathrm{d}s.\end{aligned} \tag{7.24}$$

Then (7.5) for $k=3$ with given p follows from the same bounds with $k=1,3p$ and $k=2,3p/2$, the Young inequality, and the Gronwall inequality.

4. The fourth derivative. Denote $X''''_t:=\partial_{xxxx}X_t$, then

$$\begin{aligned}\mathrm{d}X''''_t=&\\ &\Big(a'(X_t)X''''_t+4\tilde a''(X_t)X'_tX'''_t+6\tilde a'''(X_t)(X'_t)^2X''_t+3\tilde a''(X_t)(X''_t)^2+\tilde a''''(X_t)(X'_t)^4\Big)\,\mathrm{d}t\\ &+\Big(3b''(X_t)(X''_t)^2+6(X'_t)^2X''_tb'''(X_t)+4X'_tb''(X_t)X'''_t+(X'_t)^4b''''(X_t)+b'(X_t)X''''_t\Big)\,\mathrm{d}W_t\\ &+\int_{|z|\leqslant 1}\Big[3(X''_{t-})^2\phi^z_{xx}(X_{t-})+4X'_{t-}\phi^z_{xx}X'''_{t-}+6(X'_{t-})^2X''_{t-}\phi^z_{xxx}(X_{t-})+(\phi^z_x(X_{t-})-1)X''''_{t-}\\ &\qquad+(X'_{t-})^4\phi^z_{xxxx}(X_{t-})\Big]\,\tilde N(\mathrm{d}t,\mathrm{d}z).\end{aligned} \tag{7.25}$$

By the Itô formula, localization, and the Fatou lemma,

$$\begin{aligned}\mathbf{E}|X''''_t|^p\leqslant p\mathbf{E}\int_0^t&\Big(|a'(X_s)||X''''_s|+4|\tilde a''(X_s)||X'_s||X'''_s|\\ &+6|\tilde a'''(X_s)||X'_s|^2|X''_s|+3|\tilde a''(X_s)||X''_s|^2+|\tilde a''''(X_s)|X'_s|^4\Big)|X''''_s|^{p-1}\,\mathrm{d}s\\ +\frac{p(p-1)}{2}\mathbf{E}\int_0^t&\Big(3|b''(X_s)||X''_s|^2+6|X'_s|^2|X''_s||b'''(X_s)|+4|X'_s|b''(X_s)||X'''_s|\\ &+|X'_s|^4|b''''(X_s)|+|b'(X_s)||X''''_s|\Big)^2|X''''_t|^{p-2}\,\mathrm{d}s\\ +\mathbf{E}\int_{|z|\leqslant 1}&\Big[\Big(3(X''_s)^2\phi^z_{xx}(X_s)+4X'_s\phi^z_{xx}(X_s)X'''_s+6(X'_s)^2X''_s\phi^z_{xxx}(X_s)+\phi^z_x(X_s)X''''_s\\ +(X'_s)^4\phi^z_{xxxx}(X_s)\Big)^2&-(X''''_s)^2-p\Big(3(X''_s)^2\phi^z_{xx}(X_s)+4X'_s\phi^z_{xx}X'''_s+6(X'_s)^2X''_s\phi^z_{xxx}(X_s)\\ &+(\phi^z_x(X_s)-1)X''''+(X'_s)^4\phi^z_{xxxx}(X_s)\Big)(X''''_s)^{p-1}\operatorname{sgn}(X''''_s)\Big]\,\nu(\mathrm{d}z)\mathrm{d}s.\end{aligned} \tag{7.26}$$

We apply (7.10) with $a=X''''$ and

$$\begin{aligned}\delta=\delta(X,X',X'',X''',X'''',z)=&\,3(X'')^2\phi^z_{xx}(X)+4X'\phi^z_{xx}X'''+6(X')^2X''\phi^z_{xxx}(X)\\ &+(\phi^z_x(X)-1)X''''+(X')^4\phi^z_{xxxx}(X).\end{aligned} \tag{7.27}$$

We have for $|z|\leqslant 1$ by Lemma A.1

$$|\phi^z_{xxxz}(x)|\leqslant C|z|, \tag{7.28}$$

which together with (7.11), (7.15), (7.22) and the Young inequality gives

$$\begin{aligned}&|\delta(X,X',X'',X''',z)|^2\leqslant C(|X'|^8+|X''|^4+|X'''|^{8/3}+|X''''|^2)|z|^2,\\&|\delta(X,X',X'',X''',z)|^p\leqslant C(|X'|^{4p}+|X''|^{2p}+|X'''|^{4p/3}+|X''''|^p)|z|^p.\end{aligned} \tag{7.29}$$

Since the derivatives of $\tilde a, b$ are bounded, applying the Young inequality once again we get

$$\begin{aligned}\mathbf{E}|X''''_t|^p\leqslant C\mathbf{E}\int_0^t\Big(&|X''''_s|^p+|X''''_s|^{p-1}|X'_s|^4+|X''''_s|^{p-1}|X''_s|^2+|X''''_s|^{p-1}|X'''_s|^{4/3}\\+|X''''_s|^{p-2}|X'_s|^8&+|X''''_s|^{p-2}|X''_s|^4+|X''''_s|^{p-2}|X'''_s|^{8/3}+|X'_s|^{4p}+|X''_s|^{2p}+|X'''_s|^{4p/3}\Big)\,\mathrm{d}s.\end{aligned} \tag{7.30}$$

Then (7.5) for $k=4$ with given p follows from the Young inequality, the Gronwall inequality, and the bounds (7.5) with $k=1$, 2, 3 and p' equal $4p$, $2p$, $4p/3$, respectively. $\square$

Now, let us consider the general case of non-trivial large jump part. The semigroup P_t of the solution to (2.5) admits the following representation. Consider the SDE (7.6), which corresponds to the driving noise with large jumps (i.e. $|z|>1$) truncated away. Denote the corresponding semigroup $\tilde P_t$, $t\geqslant 0$. Denote by $\mathcal{Q}$ the operator which corresponds to a single large jump of the driving noise:

$$\mathcal{Q}f(x)=\int_{|z|>1}\Big(f(\phi^z(x))-f(x)\Big)\,\nu(\mathrm{d}z). \tag{7.31}$$

Then we have

$$P_t=\mathrm{e}^{-\lambda t}\tilde P_t+\sum_{k=1}^\infty \mathrm{e}^{-\lambda t}\int_{0\leqslant s_1\leqslant\cdots\leqslant s_k\leqslant t}\tilde P_{t-s_k}\,\mathcal{Q}\tilde P_{s_k-s_{k-1}}\mathcal{Q}\dots\mathcal{Q}P_{s_1}\,\mathrm{d}s_1\dots\mathrm{d}s_k, \tag{7.32}$$

where $\lambda=\nu(|z|>1)$ is the intensity of large jumps. The above representation follows easily by independence of the processes

$$\tilde Z_t=\int_0^t\int_{|z|\leqslant 1}z\,\tilde N(\mathrm{d}s,\mathrm{d}z),\quad\text{and}\quad Z_t-\tilde Z_t=\int_0^t\int_{|z|>1}z\,N(\mathrm{d}s,\mathrm{d}z) \tag{7.33}$$

and the compound Poisson structure of $Z-\tilde Z$.

We have shown in the first part of the proof that

$$\|\tilde P_t\|_{C^4\to C^4}\leqslant C_T,\quad t\leqslant T. \tag{7.34}$$

On the other hand, for the function $\mathcal{Q}f$ given by the integral formula (7.31) its derivatives of the orders $1,\dots 4$ admit integral representations similar to (7.1)–(7.4), and then it is a direct calculation to see that

$$\|\mathcal{Q}\|_{C^4\to C^4}\leqslant C_{\mathcal{Q}}. \tag{7.35}$$

Then for the semigroup P_t we have for $t\leqslant T$

$$\|P_t\|_{C^4\to C^4}\leqslant \mathrm{e}^{-\lambda t}C_T+\sum_{k=1}^\infty \mathrm{e}^{-\lambda t}\frac{t^k}{k!}(C_T)^{k+1}(C_{\mathcal{Q}})^k=C_T\mathrm{e}^{t(C_{\mathcal{Q}}C_T-\lambda)}\leqslant C_T\mathrm{e}^{T(C_{\mathcal{Q}}C_T-\lambda)_+}, \tag{7.36}$$

which completes the proof.

Appendix A. Properties of $\phi^z(u;x)$ and its derivatives

Lemma A.1. *Let $\mathbf{H}_{a,b,c}$ holds true and let*

$$\varphi(u;x,z)=\phi^z(u;x)-x-c(x)zu,\quad u\in[0,1]. \tag{A.1}$$

Then there is a constant $C>0$ such that for all $|z|\leqslant 1$ and all $x\in\mathbb{R}$

$$\begin{aligned}|\varphi(u;x,z)|&\leqslant C\cdot z^2\cdot|c(x)|,\\ |\nabla_x^k\varphi(u;x,z)|&\leqslant C\cdot z^2,\quad 1\leqslant k\leqslant 4.\end{aligned} \tag{A.2}$$

In particular, the effective drift $\tilde a\in C^4(\mathbb{R},\mathbb{R})$ and $\|\nabla^k\tilde a\|<\infty$, $k=1,\dots,4$, and for $|z|\leqslant 1$

$$\begin{aligned}|\phi^z(u;x)-x|&\leqslant C(1+|x|),\\ |\phi^z_x(u;x)-1|&\leqslant C|z|,\\ |\nabla^k_x\phi^z(u;x)|&\leqslant C|z|,\quad k=2,3,4.\end{aligned} \tag{A.3}$$

Proof. Estimate the integral term.

1. We write

$$\phi^z(u;x)=x+c(x)zu+\varphi(u;x,z),\quad u\in[0,1] \tag{A.4}$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\phi^z(u;x)&=c(x)z+\dot\varphi(u;x,z)=c(x+c(x)zu+\varphi(u;x,z))z\\ &=c(x)z+c'(\xi)\Big(c(x)zu+\varphi(u;x,z)\Big)z,\quad \xi=\xi(u,x,z)\end{aligned} \tag{A.5}$$

Hence

$$\begin{aligned}\dot\varphi(u;x,z)&=c'(\xi)c(x)z^2u+\varphi(u;x,z)c'(\xi)z,\\ |\varphi(u;x,z)|&\leqslant\int_0^u\Big(\|c'\||c(x)|z^2+\|c'\||z||\varphi(r;x,z)|\Big)\,\mathrm{d}r,\\ |\varphi(u;x,z)|&\leqslant z^2\|c'\||c(x)|\cdot \mathrm{e}^{\|c'\|}.\end{aligned} \tag{A.6}$$

Hence

$$|\phi^z(x)-x-c(x)z|\leqslant z^2\|c'\||c(x)|\cdot\mathrm{e}^{\|c'\|} \tag{A.7}$$

and $\tilde a$ is of linear growth.

2. Analogously,

$$\phi^z_x(u;x)=1+c'(x)zu+\varphi_x(u;x,z),\quad u\in[0,1] \tag{A.8}$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\phi^z_x(u;x)&=c'(x)z+\dot\varphi_x(u;x,z)=c'(\phi^z(u;x))\phi^z_x(u;x)z\\ &=c'(\phi^z(u;x))\Big(1+c'(x)zu+\varphi_x(u;x,z)\Big)z\end{aligned} \tag{A.9}$$

Hence

$$\begin{aligned}\dot\varphi_x(u;x,z)&=\Big(c'(\phi^z(u;x))-c'(x)\Big)z+c'(\phi^z(u;x))c'(x)z^2u+c'(\phi^z(u;x))\varphi_x(u;x,z)z\\ &=z^2\int_0^u c''(\phi^z(r;x))c(\phi^z(r;x))\,\mathrm{d}r+c'(\phi^z(u;x))c'(x)z^2u+c'(\phi^z(u;x))\varphi_x(u;x,z)z\end{aligned} \tag{A.10}$$

Hence

$$
\begin{aligned}
&|\varphi_x(u;x,z)| \leqslant (\|c''c\| + \|c'\|^2)z^2 + \|c'\||z| \int_0^u |\varphi(r;x,z)|\,\mathrm{d}r, \\
&|\varphi_x(u;x,z)| \leqslant z^2(\|c''c\| + \|c'\|^2)\cdot \mathrm{e}^{\|c'\|}.
\end{aligned} \tag{A.11}
$$

3. Analogously,

$$
\phi^z_{xx}(u;x) = c''(x)zu + \varphi_{xx}(u;x,z), \quad u \in [0,1] \tag{A.12}
$$

Then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}\phi^z_{xx}(u;x) &= c''(x)z + \dot\varphi_{xx}(u;x,z) \\
&= c''(\phi^z(u;x))\Big(1 + c'(x)zu + \varphi^z_x(u;x)\Big)^2 z + c'(\phi^z(u;x))\Big(c''(x)zu + \varphi_{xx}(u;x,z)\Big)z \\
&= c''(\phi^z(u;x))\Big(1 + 2(c'(x)zu + \varphi^z_x(u;x)) + (c'(x)zu + \varphi^z_x(u;x))^2\Big)z \\
&\quad + c'(\phi^z(u;x))\Big(c''(x)zu + \varphi_{xx}(u;x,z)\Big)z
\end{aligned} \tag{A.13}
$$

Taking into account that $\|c'''c\| < \infty$ and

$$
c''(\phi^z(u;x)) - c''(x) = z\int_0^u c'''(\phi^z(r;x))c(\phi^z(r;x))\,\mathrm{d}r \tag{A.14}
$$

we get that

$$
|\varphi_x(u;x,z)| \leqslant z^2 \cdot C_2 \cdot \mathrm{e}^{\|c'\|}. \tag{A.15}
$$

4. The higher derivatives are checked analogously.

□

We have the following formulae for the derivatives of the Marcus flow $x \mapsto \phi^z(x)$. These derivatives are hence solutions of non-autonomous non-homogeneous linear differential equations.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}\phi^z_x &= zc'(\phi^z)\phi^z_x, \quad \phi^z(0;x) = 1, \\
\frac{\mathrm{d}}{\mathrm{d}u}\phi^z_{xx} &= zc''(\phi^z)\phi^2_x + zc'(\phi^z)\phi^z_{xx}, \\
\frac{\mathrm{d}}{\mathrm{d}u}\phi^z_{xxx} &= z\Big(c'''(\phi^z)\phi^3_x + 3c''(\phi^z)\phi^z_x\phi^z_{xx}\Big) + zc'(\phi^z)\phi^z_{xxx}, \\
\frac{\mathrm{d}}{\mathrm{d}u}\phi^z_{xxxx} &= z\Big(c''''(\phi^z)(\phi^z_x)^4 + 6c'''(\phi^z)\phi^z_x(\phi^z_{xx})^2 + 3c''(\phi^z)(\phi^z_{xx})^2 \\
&\quad + 4c'(\phi^z)\phi^z_x(\phi^z_{xxx})^3\Big) + zc'(\phi^z)\phi^z_{xxxx},
\end{aligned} \tag{A.16}
$$

Lemma A.2. *Under assumption $\mathbf{H}_{a,b,c}$ we have for all $|z| > 1$ and $x \in \mathbb{R}$*

$$
\begin{aligned}
|\phi^z_x(u;x)| &\leqslant \mathrm{e}^{\|c'\||z|}, \\
|\phi^z_{xx}(u;x)| &\leqslant |z|\mathrm{e}^{3\|c'\||z|}, \\
|\phi^z_{xxx}(u;x)| &\leqslant |z|^2\mathrm{e}^{5\|c'\||z|}, \\
|\phi^z_{xxxx}(u;x)| &\leqslant |z|^3\mathrm{e}^{8\|c'\||z|}, \quad u \in [0,1].
\end{aligned} \tag{A.17}
$$

In particular,

$$
\begin{aligned}
|\phi^z(x) - x| &\leqslant |x|(1 + \mathrm{e}^{\|c'\||z|}), \\
|\phi^z_x(x) - 1| &\leqslant 1 + \mathrm{e}^{\|c'\||z|}.
\end{aligned} \tag{A.18}
$$

Proof. Indeed, solving the linear equations (A.16) we get

$$
\begin{aligned}
\phi^z_x(u) &= \mathrm{e}^{\int_0^t c'(\phi^z) z\,\mathrm{d}r},\\
\phi^z_{xx}(u) &= \int_0^u z c''(\phi^z)\phi_x^2\cdot \mathrm{e}^{\int_s^u c'(\phi^z) z\,\mathrm{d}r}\,\mathrm{d}s,\\
\phi^z_{xxx}(u) &= \int_0^u z\Big(c'''(\phi^z)\phi_x^3 + 3c''(\phi^z)\phi^z_x\phi^z_{xx}\Big)\mathrm{e}^{\int_s^u c'(\phi^z) z\,\mathrm{d}r}\,\mathrm{d}s,\\
\phi^z_{xxxx}(u) &= \int_0^u z\Big(c''''(\phi^z)(\phi^z_x)^4 + 6c'''(\phi^z)\phi^z_x(\phi^z_{xx})^2 + 3c''(\phi^z)(\phi^z_{xx})^2\\
&\quad + 4c''(\phi^z)\phi^z_x\phi^z_{xxx}\Big)\mathrm{e}^{\int_s^u c'(\phi^z) z\,\mathrm{d}r}\,\mathrm{d}s.
\end{aligned}
\tag{A.19}
$$

and hence the estimates follow.

By the Gronwall lemma, $|\phi^z(x)| \leqslant |x|\mathrm{e}^{\|c'\|\cdot|z|}$, and

$$
|\phi^z(x) - x| \leqslant 1 + \mathrm{e}^{\|c'\|\cdot|z|}. \tag{A.20}
$$

□

In the multidimensional setting, solutions should be written in terms of the fundamental solution of the linear differential equation with the matrix $Dc(\phi^z(u;x))z$ and the estimates (A.19) follow, for example from Hartman [8, Section IV.4]).

Appendix B. Properties of $\psi(u;x;\tau,w,z)$ and its derivatives

For the estimates of the Lemma 5.6 we need the following elementary inequalities.

Lemma B.1. *Let* $\mathbf{H}_{a,b,c}$ *hold true. Then there is a constant* $C > 0$ *such that for all* $\tau \geqslant 0$, $w \in \mathbb{R}$, $z \in \mathbb{R}$, *and* $x \in \mathbb{R}$

$$
\sup_{u\in[0,1]} |\psi(u;x;\tau,w,z)| \leqslant C(1+|x|)\cdot \mathrm{e}^{\|a'\|\tau + \|b'\||w| + \|c'\||z|}, \tag{B.1}
$$

$$
\sup_{u\in[0,1]} |\partial_i\psi(u;x,\tau,w,z)| \leqslant C(1+|x|)\cdot \mathrm{e}^{2(\|a'\|\tau + \|b'\||w| + \|c'\||z|)},\ i \in \{\tau,w,z\}, \tag{B.2}
$$

$$
\begin{aligned}
\sup_{u\in[0,1]} |\partial_{ij}\psi(u;x;\tau,w,z)| &\leqslant C(1+x^2)\cdot(1+\tau+|w|+|z|)\mathrm{e}^{5(\|a'\|\tau + \|b'\||w| + \|c'\||z|)},\\
&\quad i,j \in \{\tau,w,z\},
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
\sup_{u\in[0,1]} |\psi_{ijk}(u;x;\tau,w,z)| &\leqslant C(1+|x|^3)\cdot(1+\tau+|z|+|w|)^2\mathrm{e}^{8(\|a'\|\tau + \|b'\||w| + \|c'\||z|)},\\
&\quad i,j,k \in \{\tau,w,z\},
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
\sup_{u\in[0,1]} |\psi_{ijkl}(u;x;\tau,w,z)| &\leqslant C(1+|x|^4)\cdot(1+\tau+|z|+|w|)^3\mathrm{e}^{11(\|a'\|\tau + \|b'\||w| + \|c'\||z|)},\\
&\quad i,j,k,l \in \{\tau,w,z\}.
\end{aligned}
\tag{B.5}
$$

Proof. These estimates are obtained directly.

0. Estimate of ψ. For $\tau,w,z \in \mathbb{R}$, denote $\psi(u) = \psi(u;x;\tau,w,z)$ the solution to the Cauchy problem

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}\psi(u) = a(\psi(u))\tau + b(\psi(u))w + c(\psi(u))z,\\
&\psi(0) = x, \quad u \in [0,1].
\end{aligned}
\tag{B.6}
$$

Since

$$|a(x)| \leqslant |a(0)| + \|a'\||x|, \quad |b(x)| \leqslant |b(0)| + \|b'\||x|, \quad |c(x)| \leqslant |c(0)| + \|c'\||x|, \tag{B.7}$$

the Gronwall inequality yields (B.1) for some $C > 0$.

1. Estimates of ψ_τ, ψ_w, ψ_z. The derivative w.r.t. τ satisfies the lienar non-autonomous ODE

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}\psi_\tau &= a(\psi) + (a'(\psi)\tau + b'(\psi)w + c'(\psi)z)\psi_\tau \\ \psi_\tau(0; x; \tau, w, z) &= 0 \end{aligned} \tag{B.8}$$

which can be solved explicitly

$$\psi_\tau(u) = \int_0^u a(\psi(s))\mathrm{e}^{\int_s^u (\tau a'(\psi(r)) + wb'(\psi(r)) + zc'(\psi(r)))\,\mathrm{d}r}\,\mathrm{d}s, \tag{B.9}$$

Applying the estimate (B.1) we get (for a different constant $C > 0$)

$$\sup_{u\in[0,1]} |\psi_\tau(u; x; \tau, w, z)| \leqslant C(1 + |x|) \cdot \mathrm{e}^{2(\|a'\||\tau| + \|b'\||w| + \|c'\||z|)}. \tag{B.10}$$

Due to the symmetry of the ODE for ψ w.r.t. τ, w, and z the same estimate holds for ψ_w and ψ_z.

2. Estimates of $\psi_{\tau\tau}$, $\psi_{\tau w}$, $\psi_{\tau z}$, ψ_{ww}, ψ_{wz}, ψ_{zz}. We consider derivatives $\psi_{\tau\tau}$ and $\psi_{\tau w}$,

(B.11)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau\tau} &= 2a'(\psi)\psi_\tau + \Big(a''(\psi)\tau + b''(\psi)w + c''(\psi)z\Big)\psi_\tau^2 + \Big(a'(\psi)\tau + b'(\psi)w + c'(\psi)z\Big)\psi_{\tau\tau}, \\ &\psi_{\tau\tau}(0; x; t, w, z) = 0, \\ \frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau w} &= a'(\psi)\psi_w + b'(\psi)\psi_\tau + \Big(a''(\psi)\tau + b''(\psi)w + c''(\psi)z\Big)\psi_\tau \cdot \psi_w \\ &\quad + \Big(a'(\psi)\tau + b'(\psi)w + c'(\psi)z\Big)\psi_{\tau w}, \psi_{\tau w}(0; x; \tau, w, z) = 0. \end{aligned}$$

Writing down the solution explicitly and using the estimates from the previous steps yields the result.

3. Estimates of $\psi_{\tau\tau\tau}$, $\psi_{\tau\tau w}$, $\psi_{\tau\tau z}$, $\psi_{\tau ww}$, ... We consider derivatives $\psi_{\tau\tau\tau}$ and $\psi_{\tau\tau w}$, and $\psi_{\tau wz}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau\tau\tau} &= 3a''(\psi)\psi_\tau^2 + 3a'(\psi)\psi_{\tau\tau} + 3\Big(a''(\psi)\tau + b''(\psi)w + c''(\psi)z\Big)\psi_\tau\psi_{\tau\tau} \\ &\quad + \Big(a'''(\psi)\tau + b'''(\psi)w + c'''(\psi)z\Big)\psi_\tau^3 + \Big(a'(\psi)\tau + b'(\psi)w + c'(\psi)z\Big)\psi_{\tau\tau\tau}, \\ &\psi_{\tau\tau\tau}(0; x; t, w, z) = 0, \end{aligned} \tag{B.12}$$

(B.13)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau\tau w} &= b''(\psi)\psi_\tau^2 + b'(\psi)\psi_{\tau\tau} + 2a'(\psi)\psi_{\tau w} + 2\Big(a''(\psi)(1+\tau) + b''(\psi)w + c''(\psi)z\Big)\psi_\tau\psi_w \\ &\quad + \Big(a'''(\psi)\tau + b'''(\psi)w + c'''(\psi)z\Big)\psi_\tau^2\psi_w + \Big(a''(\psi)\tau + b''(\psi)w + c''(\psi)z\Big)\psi_{\tau\tau}\psi_w \\ &\quad + \Big(a'(\psi)\tau + b'(\psi)w + c'(\psi)z\Big)\psi_{\tau\tau w}, \end{aligned}$$

(B.14)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau w z} &= c''(\psi)\psi_\tau\psi_w + b''(\psi)\psi_\tau\psi_z + a''(\psi)\psi_w\psi_z + c'(\psi)\psi_{\tau w} + b'(\psi)\psi_{\tau z} + a'(\psi)\psi_{wz}\\ &+ \Big(a''(\psi)\tau + b''(\psi)w + c''(\psi)z\Big)\Big(\psi_\tau\psi_{wz} + \psi_{\tau w}\psi_z + \psi_{\tau z}\psi_w\Big)\\ &+ \Big(a'''(\psi)\tau + b'''(\psi)w + c'''(\psi)z\Big)\psi_\tau\psi_w\psi_z + \Big(a'(\psi)\tau + b'(\psi)w + c'(\psi)z\Big)\psi_{\tau w z}\\ &\psi_{\tau w z}(0;x;\tau,w,z) = 0.\end{aligned}$$

4. Estimates of $\psi_{\tau\tau\tau\tau}$, $\psi_{\tau\tau\tau w}$, $\psi_{\tau\tau\tau z}$, $\psi_{\tau\tau ww}$, ...

We consider derivatives $\psi_{\tau\tau\tau\tau}$ and $\psi_{\tau\tau\tau w}$, and $\psi_{\tau\tau ww}$, and $\psi_{\tau\tau wz}$:

(B.15)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau\tau\tau\tau} &= \Big(\tau a^{(4)}(\psi) + wb^{(4)}(\psi) + zc^{(4)}(\psi)\Big)\psi_\tau^4 + 6\Big(\tau a'''(\psi) + wb'''(\psi) + zc'''(\psi)\Big)\psi_\tau^2\psi_{\tau\tau}\\ &+ \Big(\tau a''(\psi) + wb''(\psi) + zc''(\psi)\Big)\Big(3\psi_{\tau\tau}^2 + 4\psi_\tau\psi_{\tau\tau\tau}\Big)\\ &+ 4\Big(a'''(\psi)\psi_\tau^3 + 3a''(\psi)\psi_\tau\psi_{\tau\tau} + a'(\psi)\psi_{\tau\tau\tau}\Big) + \Big(\tau a'(\psi) + wb'(\psi) + zc'(\psi)\Big)\psi_{\tau\tau\tau\tau}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau\tau\tau w} &= b'''(\psi)\psi_\tau^3 + b'(\psi)\psi_{\tau\tau\tau} + 3b''(\psi)\psi_\tau\psi_{\tau\tau}\\ &+ 3\Big(a'''(\psi)\psi_w\psi_\tau^2 + 2a''(\psi)\psi_\tau\psi_{\tau w} + a''(\psi)\psi_w\psi_{\tau\tau} + a'(\psi)\psi_{\tau\tau w}\Big)\\ &+ \Big(\tau a^{(4)}(\psi) + wb^{(4)}(\psi) + zc^{(4)}(\psi)\Big)\psi_\tau^3\psi_w\\ &+ 3\Big(\tau a'''(\psi) + wb'''(\psi) + zc'''(\psi)\Big)\Big(\psi_\tau^2\psi_{\tau w} + \psi_w\psi_\tau\psi_{\tau\tau}\Big)\\ &+ \Big(\tau a''(\psi) + wb''(\psi) + zc''(\psi)\Big)\Big(3\psi_{\tau w}\psi_{\tau\tau} + 3\psi_\tau\psi_{\tau\tau w} + \psi_w\psi_{\tau\tau\tau}\Big)\\ &+ \Big(\tau a'\psi + wb'\psi + zc'\psi\Big)\psi_{\tau\tau\tau w}\end{aligned}\tag{B.16}$$

(B.17)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau\tau ww} &= \tau a^{(4)}(\psi)\psi_\tau^2\psi_w^2 + a'''(\psi)\psi_\tau\psi_w^2 + 4a''(\psi)\psi_w\psi_{\tau w} + 2a''(\psi)\psi_\tau\psi_{ww} + 2a'(\psi)\psi_{\tau ww}\\ &+ 2b'''(\psi)\psi_\tau^2\psi_w + 4b''(\psi)\psi_\tau\psi_{\tau w} + 2b''(\psi)\psi_w\psi_{\tau\tau} + 2b'(\psi)\psi_{\tau\tau w}\\ &+ \Big(\tau a^{(4)}(\psi) + wb^{(4)}(\psi) + zc^{(4)}(\psi)\Big)\psi_\tau^2\psi_w^2\\ &+ \Big(\tau a'''(\psi) + wb'''(\psi) + zc'''(\psi)\Big)\Big(4\psi_w\psi_\tau\psi_{\tau w} + \psi_\tau^2\psi_{ww} + \psi_{\tau\tau}\psi_w^2\Big)\\ &+ \Big(\tau a''(\psi) + wb''(\psi) + zc''(\psi)\Big)\Big(2\psi_{\tau w}^2 + 2\psi_\tau\psi_{\tau ww} + 2\psi_w\psi_{\tau\tau w} + \psi_{\tau\tau}\psi_{ww}\Big)\\ &+ \Big(\tau a'(\psi) + wb'(\psi) + zc'(\psi)\Big)\psi_{\tau\tau ww}\end{aligned}$$

(B.18)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\psi_{\tau\tau ww} &= 2a'(\psi)\psi_{\tau wz} + b'(\psi)\psi_{\tau\tau z} + c'(\psi)\psi_{\tau\tau w}\\ &+ 2a''(\psi)\Big(\psi_{\tau z}\psi_w + \psi_{\tau w}\psi_z + \psi_\tau\psi_{wz}\Big) + b''(\psi)\Big(2\psi_\tau\psi_{\tau z} + \psi_{\tau\tau}\psi_z\Big) + c''(\psi)\Big(2\psi_\tau\psi_{\tau w} + \psi_{\tau\tau}\psi_w\Big)\\ &+ 2a'''(\psi)\psi_\tau\psi_w\psi_z + b'''(\psi)\psi_\tau^2\psi_z + c'''(\psi)\psi_\tau^2\psi_w\\ &+ \Big(\tau a^{(4)}(\psi) + wb^{(4)}(\psi) + zc^{(4)}(\psi)\Big)\psi_\tau^2\psi_w\psi_z\\ &+ \Big(\tau a'''(\psi) + wb'''(\psi) + zc'''(\psi)\Big)\Big(\psi_\tau^2\psi_{wz} + 2\psi_w\psi_\tau\psi_{\tau z} + 2\psi_\tau\psi_{\tau w}\psi_z + \psi_{\tau\tau}\psi_w\psi_z\Big)\end{aligned}$$

$$+ \Big(\tau a''(\psi) + wb''(\psi) + zc''(\psi)\Big)\Big(2\psi_{\tau w}\psi_{\tau z} + 2\psi_\tau\psi_{\tau wz} + \psi_{\tau\tau}\psi_{wz} + \psi_{\tau\tau z}\psi_w + \psi_{\tau\tau w}\psi_z\Big) + \Big(\tau a'(\psi) + wb'(\psi) + zc'(\psi)\Big)\psi_{\tau\tau wz}.$$

□

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