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THE NONLOCAL CONJUGATION PROBLEM FOR ONE-DIMENSIONAL PARABOLIC EQUATION WITH DISCONTINUOUS COEFFICIENTS AND ASSOCIATED FELLER SEMIGROUP

This paper is dedicated to the memory of our colleague and friend S. Ya. Makhno

By the boundary integral equations method we establish the classical solvability of the conjugation problem for one-dimensional linear parabolic equation of the second order (backward Kolmogorov equation) with nonlocal Feller-Wentzell conjugation condition. Using the solution of this problem, we construct the two-parameter Feller semigroup associated with the inhomogeneous diffusion process in bounded domain with moving membrane.

1. Introduction

The general form of boundary conditions for one-dimensional diffusion processes was established by W. Feller [1] and A.D. Wentzell [2]. They proved the assertions from which it follows that if $\{T_t,\, t \geq 0\}$ is Feller semigroup in $C[r_1, r_2]\,(-\infty < r_1 < r_2 < \infty)$ and its generator A is the restriction of $(L,\, C^2[r_1, r_2])$, where L is second order ordinary differential operator, then functions from $\mathcal{D}_A \subset C^2[r_1, r_2]$ must satisfy boundary conditions which, generally speaking, have nonlocal character. These boundary conditions contain the value of the function and its derivatives at boundary points $r_i\,(i = 1, 2)$ as well as the integral over $[r_1, r_2]$ with respect to some nonnegative measure which, furthermore, can be infinite.

There are many publications in which the problem on construction of Markov processes by given boundary conditions is formulated in different ways and is investigated by different approaches, see, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], and the references given there. One of such variants of the described problem is the so-called problem of pasting together two diffusion processes [9, 10], which is the object of research of our paper. More specifically, the present paper deals with the partial case of this problem where each of boundary conditions of Feller-Wentzell's type contains only the integral term. For our investigations, it is convenient to formulate the problem in terms of partial differential equations of parabolic type as follows.

Consider on a plane (s, x) two domains:

$$S_t^{(i)} = \{(s, x) : 0 \leq s < t \leq T,\, r_i(s) < x < r_{i+1}(s)\},$$

where $i = 1, 2$; T is a fixed positive number and $r_1,\ r_2,\ r_3$ are given functions defined on $[0, T]$ such that $r_1(s) < r_2(s) < r_3(s)$ for all $s \in [0, T]$. Let $\overline{S}_t^{(i)}$ denotes the closure of $S_t^{(i)}$ and let $S_t = S_t^{(1)} \cup S_t^{(2)}$. Denote also by D_{is} the interval $(r_i(s), r_{i+1}(s))$, $i = 1, 2$, and by D_s the union $D_{1s} \cup D_{2s}$.

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The problem is to find a function $u(s,x,t)$ defined on $(s,x)\in\overline{S}_t$, which satisfies the backward Kolmogorov equation

$$\frac{\partial u}{\partial s}+\frac{1}{2}b_i(s,x)\frac{\partial^2 u}{\partial x^2}+a_i(s,x)\frac{\partial u}{\partial x}=0,\quad (s,x)\in S_t^{(i)},\quad i=1,2, \tag{1}$$

the initial condition

$$\lim_{s\uparrow t}u(s,x,t)=\varphi(x),\quad x\in\overline{D}_t,\quad i=1,2, \tag{2}$$

two boundary conditions

$$\int\limits_{D_{is}}(u(s,r_{2i-1}(s),t)-u(s,y,t))\mu_{2i-1}(s,dy)=0,\quad 0\le s\le t\le T,\quad i=1,2, \tag{3}$$

and two conjugation conditions

$$u(s,r_2(s)-0,t)=u(s,r_2(s)+0,t),\quad 0\le s\le t\le T, \tag{4}$$

$$\int\limits_{D_s}(u(s,r_2(s),t)-u(s,y,t))\mu_2(s,dy)=0,\quad 0\le s\le t\le T. \tag{5}$$

The conditions (3) and (5) are nonlocal boundary and conjugation conditions of Feller-Wentzell's type respectively. The condition (4) is the condition of continuity of $u(s,x,t)$ at $(s,r_2(s))$, $0\le s\le t\le T$.

Throughout this paper, we will make the following assumptions:

I. The coefficients $a_i(s,x)$ and $b_i(s,x)$ $(i=1,2)$ are defined on

$$\Pi[0,T]\equiv\{(s,x):\,0\le s\le T,\,x\in\mathbb{R}\},$$

they are bounded and belong to Hölder class $H^{\frac{\alpha}{2},\alpha}(\Pi[0,T])$ for some $\alpha\in(0,1)$ (to recall the definitions of Hölder classes see [11, p.16]). Moreover, $b_i(s,x)$ is bounded away from zero.

II. The function φ is assumed to be defined on $\mathbb{R}$ and belongs to the space of bounded continuous functions on $\mathbb{R}$, which will be denoted by $C_b(\mathbb{R})$. The norm in this space is defined by the equality $\|\varphi\|=\sup_{x\in\mathbb{R}}|\varphi(x)|$. Furthermore, fitting conditions

$$\int\limits_{D_{it}}(\varphi(r_{2i-1}(t))-\varphi(y))\mu_{2i-1}(t,dy)=0,\quad i=1,2,$$

$$\varphi(r_2(t)-0)=\varphi(r_2(t)+0),\quad \int\limits_{D_t}(\varphi(r_2(t))-\varphi(y))\mu_2(t,dy)=0,\quad i=1,2,$$

hold.

III. μ_{2i-1} $(i=1,2)$ and μ_2 are probability measures on D_{is} and D_s respectively, such that $\mu_i(s,D_{is})=1$, $\mu_2(s,D_s)=1$, $s\in[0,T]$, and for all $f\in C_b(\mathbb{R})$ the integrals

$$F_f^{(2i-1)}(s)=\int\limits_{D_{is}}f(y)\mu_{2i-1}(s,dy),\quad F_f^{(2)}(s)=\int\limits_{D_s}f(y)\mu_2(s,dy),\quad i=1,2,$$

belong to $H^{\frac{1+\alpha}{2}}([0,T])$ as functions of s (α is the constant in I).

IV. The curves $r_i(s)$, $i=1,2,3$, are continuous and belong to $H^{\frac{1+\alpha}{2}}([0,T])$ (α is the constant in I).

Remark 1.1. For every $M>0$, there exists a constant C_M such that the Hölder inequality

$$|F_f^{(j)}(s)-F_f^{(j)}(s')|\le C_M|s-s'|^{\beta},\quad s,s'\in[0,T],\quad j=1,2,3, \tag{6}$$

holds for all $f\in C_b(\mathbb{R})$ which are bounded by M.

The parabolic conjugation problem (1)-(5) arises, in particular, in the theory of diffusion processes in construction by analytical methods of a one-dimensional model of the diffusion phenomenon with a membrane. The Feller process associated with (1)-(5) (its Feller property is represented by (4)) coincides in D_{is} with the diffusion process with the drift coefficient $a_i(s,x)$ and the diffusion coefficient $b_i(s,x)\,(i=1,2)$. The behavior of this process at the points of the boundary is described by the boundary conditions (3) (at the points $r_{2i-1}(s)\,(i=1,2)$) and the conjugation condition (5) (at the point $r_2(s)$) which are the variants of the nonlocal conditions of Feller-Wentzell type corresponding to the jump-like exit of process from the points of boundary [1], [2], [12]. In the considered case, the boundary points $r_1(s)$, $r_2(s)$, $r_3(s)$ are supposed to be moving. The role of the membrane separating different (by their diffusion characteristics) media is being played by $r_2(s)$ which is the common boundary of domains D_{1s} and D_{2s}. The point $r_2(s)$ can be treated also as the point of "pasting together" two given diffusion processes.

Thus, the first purpose of this paper is to prove an existence and uniqueness theorem for the conjugation problem (1)-(5). The second purpose is to construct, using the solution of the problem (1)-(5), the two-parameter semigroup of operators associated with the Feller process which is a result of "pasting together" two diffusion processes.

Note that the scheme we will use to solve the problem (1)-(5) is partially presented in [13], where the similar problem is investigated in case of backward Kolmogotov equation given in $S_t = \cup_{i=1}^2 S_t^{(i)} = \cup_{i=1}^2\{(s,x):\ 0\le s<t\le T,\ (-1)^i(x-r(s))>0\}$ with Feller-Wentzell conjugation condition which is imposed at the common boundary $x=r(s)$ of curvilinear domains $S_t^{(1)}$ and $S_t^{(2)}$, and which contains, in addition to the integral term, the local one corresponding to the termination of process. Note also that similar problems (with different variants of Feller-Wentzell conjugation condition) were studied in our earlier papers for the cases where $S_t^{(i)}$ are finite [14, 15] or semi-infinite [16] rectangular domains. We would like to mention again papers [7, 8], which give the results concerning the construction of diffusion processes with nonlocal boundary conditions of the integral type by the methods of functional [7] and stochastic analysis [8].

The rest of this paper is organized as follows. Section 2 provides a brief review of auxiliary results on the fundamental solution of the backward Kolmogorov equation and the associated potentials which will be used in the subsequent sections. Section 3 is devoted to the proof of the existence and uniqueness theorem for the conjugation problem (1)-(5). In Section 3, using the solution of this problem, we construct the two-parameter Feller semigroup which describes the desired process.

2. Auxiliary Results

In this section we recall some auxiliary results concerning fundamental solution of equation (1) and the associated parabolic potentials.

Denote by $G_i(s,x,t,y)$ the fundamental solution of equation (1) in $\Pi[0,T]$. Its existence is assured by the condition I (see [11, Ch.IV, §11], [17, Ch.I], [9, Ch.II, §2]). Recall that the functions $G_i\,(i=1,2)$ are nonnegative, jointly continuous, continuously differentiable with respect to s, twice continuously differentiable with respect to x and satisfy the inequality

$$|D_s^r D_x^p G_i(s,x,t,y)| \le C(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\}, \tag{7}$$

for all $0\le s<t\le T$, $x,y\in\mathbb{R}$, where r and p are the nonnegative integers for which $2r+p\le 2$, D_s^r is the partial derivative with respect to s of order r, D_x^p is the partial derivative with respect to x of order p; symbols C and c denotes (here and in what follows) any one of various different positive constants.

Recall also that

$$G_i(s,x,t,y) = Z_{i0}(s,x,t,y) + Z_{i1}(s,x,t,y), \quad i=1,2, \tag{8}$$

where

$$Z_{i0}(s,x,t,y) = [2\pi b_i(t,y)(t-s)]^{-\frac{1}{2}} \exp\left\{-\frac{(y-x)^2}{2b_i(t,y)(t-s)}\right\}, \tag{9}$$

$$Z_{i1}(s,x,t,y) = \int\limits_s^t d\tau \int\limits_{\mathbb{R}} Z_{i0}(s,x,\tau,z)Q_i(\tau,z,t,y)dz, \tag{10}$$

and the function $Q_i(s,x,t,y)\,(i=1,2)$ is a solution of some singular Volterra integral equation of the second kind. Note that the functions Z_{i1} satisfy the inequality

$$|D_s^r D_x^p Z_{i1}(s,x,t,y)| \le C(t-s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\} \tag{11}$$

where $0 \le s < t \le T$, $x, y \in \mathbb{R}, 2r+p \le 2$, α is the constant in I.

Having the fundamental solutions G_i, we now define the parabolic potentials that will be used to solve the problem (1)-(5), namely the Poisson potential

$$u_{i0}(s,x,t) = \int\limits_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy,$$

where $0 \le s < t \le T$, $x \in \mathbb{R}$, φ is the function in (2), and the simple-layer potentials

$$u_{i1}^{(j)}(s,x,t) = \int\limits_s^t G_i(s,x,\tau,r_{i+j}(\tau))V_{2i-1+j}(\tau,t)d\tau, \quad i=1,2, \quad j=0,1,$$

where $0 \le s < t \le T,\ x \in \mathbb{R}$ and $V_{2i-1+j}\ (i=1,2,\ j=0,1)$ are some functions.

We recall some properties of functions u_{i0} and $u_{i1}^{(j)}\ (i=1,2,\ j=0,1)$ (see, for instance, [9, Ch.II, §3]). Let $\varphi \in C_b(\mathbb{R})$. From the properties of fundamental solutions G_i, $i=1,2$, it follows that the potentials u_{i0} exist and satisfy in $(s,x) \in [0,t) \times \mathbb{R}$ the equation (1) with the initial condition

$$\lim_{s \uparrow t} u_{i0}(s,x,t) = \varphi(x), \quad x \in \mathbb{R}, \quad i=1,2. \tag{12}$$

In addition, the functions $u_{i0}\ (i=1,2)$ satisfy the inequality

$$|D_s^r D_x^p u_{i0}(s,x,t)| \le C\|\varphi\|(t-s)^{-\frac{2r+p}{2}}, \tag{13}$$

where $0 \le s < t \le T,\ x \in \mathbb{R}$, r and p are positive integers for which $2r+p \le 2$.

Concerning the potentials $u_{i1}^{(j)}\ (i=1,2,\ j=0,1)$, if we assume that the functions $V_{2i-1+j}(\tau,t)\,(i=1,2,\ j=0,1)$ are continuous for $\tau \in [s,t)$ and admit of a weak singularity with exponent greater than $-\frac{1}{2}$ when $\tau = t$, then each of the functions $u_{i1}^{(j)}(s,x,t)\,(i=1,2,\ j=0,1)$ is bounded continuous in $0 \le s \le t \le T$, $x \in \mathbb{R}$ and satisfies the equation (1) in $(s,x) \in [0,t) \times (\mathbb{R} \setminus r_{i+j}(s))$ with the initial condition: $u_{i1}^{(j)}(s,x,t) \to 0$ if $s \uparrow t\,(x \in \mathbb{R},\ i=1,2)$.

3. Solving the parabolic conjugation problem

The purpose of this section is to establish the classical solvability of parabolic conjugation problem (1)-(5).

Theorem 3.1. *(Existence and Uniqueness) Let the conditions I-IV hold. Then there exists a unique solution in* $C(\overline{S}_t)$ *of the nonlocal parabolic conjugation problem* (1)-(5).

Proof. We will find a solution of problem (1)-(5) in the form of a sum of Poisson potential u_{i0} and simple-layer potentials $u_{i1}^{(j)}$ $(i=1,2,\ j=0,1)$

$$(14)\qquad u(s,x,t)=u_i(s,x,t)=u_{i0}(s,x,t)+\sum_{j=0}^{1}u_{i1}^{(j)}(s,x,t),\quad (s,x)\in S_t^{(i)},\quad i=1,2$$

with the unknown densities V_{2i-1+j} (in the formula for $u_{i1}^{(j)}$ $(i=1,2,\ j=0,1)$) to be determined. In view of the properties of simple-layer potentials formulated in previous section we need to assume a priori that functions V_{2i-1+j} $(i=1,2,\ j=0,1)$ are continuous for $\tau\in[s,t)$ and admit of a weak singularity with exponent greater than $-\frac{1}{2}$ when $\tau=t$.

If we substitute instead of function u its expression from the right-hand side of (14) into each of equalities in (3), we get the following two Volterra integral equations of the first kind for V_{2i-1+j} $(i=1,2,\ j=0,1)$:

$$(15)\qquad \sum_{j=0}^{1}\int_s^t N_{ij}(s,\tau)V_{2i-1+j}(\tau,t)d\tau=\Phi_i(s,t),\quad i=1,2,$$

where

$$\begin{aligned}\Phi_i(s,t)&=\int_{D_{is}}[u_{i0}(s,y,t)-u_{i0}(s,r_{2i-1}(s),t)]\mu_{2i-1}(s,dy)=\\&=\int_{D_{is}}u_{i0}(s,y,t)\mu_{2i-1}(s,dy)-u_{i0}(s,r_{2i-1}(s),t),\\N_{ij}(s,\tau)&=\int_{D_{is}}[G_i(s,r_{2i-1}(s),\tau,r_{i+j}(\tau))-G_i(s,y,\tau,r_{i+j}(\tau))]\mu_{2i-1}(s,dy)=\\&=G_i(s,r_{2i-1}(s),\tau,r_{i+j}(\tau))-\int_{D_{is}}G_i(s,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(s,dy).\end{aligned}$$

The following lemma gives us properties of functions Φ_i $(i=1,2)$ which will be useful in the sequel.

Lemma 3.1. *Functions $\Phi_i(s,t)$ $(i=1,2)$ tend to zero as $s\uparrow t$ and for them the inequality*

$$(16)\qquad |\Phi_i(s,t)-\Phi_i(\widetilde{s},t)|\le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}$$

holds for all $0\le\widetilde{s}<s<t\le T$.

Proof. Passing to the limit $s\uparrow t$ in the expression for $\Phi_i(s,t)$ $(i=1,2)$ and recalling that the Poisson potentials u_{i0} satisfy the condition (12), we get the expression

$$\int_{D_{it}}\varphi(y)\mu_{2i-1}(t,dy)-\varphi(r_{2i-1}(t))$$

which, in view of II, is equal to zero.

In order to verify (16), we proceed as follows. Write the difference $\Phi_i(s,t)-\Phi_i(\widetilde{s},t)$ in the form

$$\begin{aligned}
&\Phi_i(s,t)-\Phi_i(\widetilde{s},t)=\int\limits_{D_{is}}[u_{i0}(s,y,t)-u_{i0}(\widetilde{s},y,t)]\mu_{2i-1}(s,dy)+\\
&+\Bigg[\int\limits_{D_{is}}u_{i0}(\widetilde{s},y,t)\mu_{2i-1}(s,dy)-\int\limits_{D_{i\widetilde{s}}}u_{i0}(\widetilde{s},y,t)\mu_{2i-1}(\widetilde{s},dy)\Bigg]+\\
&+[u_{i0}(\widetilde{s},r_{2i-1}(\widetilde{s}),t)-u_{i0}(s,r_{2i-1}(\widetilde{s}),t)]+[u_{i0}(s,r_{2i-1}(\widetilde{s}),t)-u_{i0}(s,r_{2i-1}(s),t)]
\end{aligned}\tag{17}$$

and note that for $\widetilde{s}<s$

$$\begin{aligned}
&|u_{i0}(s,y,t)-u_{i0}(\widetilde{s},y,t)|=|u_{i0}(s,y,t)-u_{i0}(\widetilde{s},y,t)|^{\frac{1+\alpha}{2}}|u_{i0}(s,y,t)-u_{i0}(\widetilde{s},y,t)|^{\frac{1-\alpha}{2}}\\
&\le\left|\frac{\partial u_{i0}(\hat{s},y,t)}{\partial\hat{s}}\bigg|_{\hat{s}=\widetilde{s}+\theta(s-\widetilde{s})}\cdot(s-\widetilde{s})\right|^{\frac{1+\alpha}{2}}(|u_{i0}(s,y,t)|+|u_{i0}(\widetilde{s},y,t)|)^{\frac{1-\alpha}{2}}\\
&\le C\|\varphi\|\left[(t-\widetilde{s}-\theta(s-\widetilde{s}))^{-1}(s-\widetilde{s})\right]^{\frac{1+\alpha}{2}}\le C\|\varphi\|\big[((t-s)\\
&+(s-\widetilde{s})(1-\theta))^{-1}(s-\widetilde{s})\big]^{\frac{1+\alpha}{2}}\le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}
\end{aligned}$$

$(0<\theta<1)$. Note also that the difference $u_{i0}(s,r_{2i-1}(\widetilde{s}),t)-u_{i0}(s,r_{2i-1}(s),t)$ in (17) can be easily estimated using the Lagrange finite-increments formula, the condition IV and the inequality (13) (with $r=0$, $p=1$). It satisfies the estimate

$$|u_{i0}(s,r_{2i-1}(\widetilde{s}),t)-u_{i0}(s,r_{2i-1}(s),t)|\le C\|\varphi\|(t-s)^{-\frac{1}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}.$$

Finally, the difference of integrals in the second line of the expression (17) satisfies the estimate with right-hand side $C\|\varphi\|(s-\widetilde{s})^{\frac{1+\alpha}{2}}$. This is the direct consequence of the condition III and the fact that the functions $f^{(i)}_{\widetilde{s}t}(y)=\frac{u_{i0}(\widetilde{s},y,t)}{\|\varphi\|}$ ($\|\varphi\|>0$, $i=1,2$), as functions of y, belong to $C_b(\mathbb{R})$ and are uniformly bounded.

Combining the estimates on each component in (17), the inequality (16) follows. □

The condition (4) together with III allow us to rewrite (5) as

$$u_i(s,r_2(s),t)-\sum_{k=1}^{2}\int\limits_{D_{ks}}u_k(s,y,t)\mu_2(s,dy)=0,\quad 0\le s\le t\le T,\quad i=1,2.\tag{18}$$

After substituting (14) into (18), we obtain two more Volterra integral equations of the first kind for V_{2i-1+j} $(i=1,2,\ j=0,1)$:

$$\begin{aligned}
&\sum_{j=0}^{1}\Bigg(\int\limits_{s}^{t}G_i(s,r_2(s),\tau,r_{i+j}(\tau))V_{2i-1+j}(\tau,t)d\tau-\\
&-\sum_{k=1}^{2}\int\limits_{s}^{t}V_{2k-1+j}(\tau,t)d\tau\int\limits_{D_{ks}}G_k(s,y,\tau,r_{k+j}(\tau))\mu_2(s,dy)\Bigg)=\Psi_i(s,t),\quad i=1,2,
\end{aligned}\tag{19}$$

where

$$\Psi_i(s,t)=\sum_{k=1}^{2}\int\limits_{D_{ks}}u_{k0}(s,y,t)\mu_2(s,dy)-u_{i0}(s,r_2(s),t).$$

Remark 3.1. It is clear that the assertion of Lemma 3.1 remains valid when Φ_i is replaced by Ψ_i $(i=1,2)$.

Regularization of equations of system (15), (19) can be performed by Holmgren's method [18] (see also [19], [20]). For this purpose, we consider the integro-differential operator

$$\mathcal{E}(s,t)f = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^t (\rho - s)^{-\frac{1}{2}} f(\rho,t)d\rho, \quad 0 \le s < t \le T.$$

and apply it to the both sides of each equation in (15), (19). Our goal is to obtain the system of four Volterra integral equations of the second kind which is equivalent to the system (15), (19) and which can be solved by the method of successive approximations.

Consider the action of the operator $\mathcal{E}$ on the each side of (15).

Bearing in mind the assertion of Lemma 3.1, we get

$$\begin{aligned}\Upsilon_i(s,t) \equiv \mathcal{E}(s,t)\Phi_i = \frac{1}{\sqrt{2\pi}}\int\limits_s^t (\rho - s)^{-\frac{3}{2}}[\Phi_i(\rho,t) - \Phi_i(s,t)]d\rho - \\ - \sqrt{\frac{2}{\pi}}(t-s)^{-\frac{1}{2}}\Phi_i(s,t).\end{aligned} \tag{20}$$

Furthermore, the function Υ_i $(i = 1, 2)$ satisfies the inequality

$$|\Upsilon_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad 0 \le s < t \le T. \tag{21}$$

The application of the operator $\mathcal{E}$ to the left-hand side of (15) gives the expression which after interchanging the order of integration takes on the form

$$I_i(s,t) \equiv \sum_{j=0}^{1}\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^t V_{2i-1+j}(\tau,t)J_{ij}(s,\tau)d\tau, \quad i = 1, 2, \tag{22}$$

where

$$J_{ij}(s,\tau) = \int\limits_s^\tau (\rho - s)^{-\frac{1}{2}} N_{ij}(\rho,\tau)d\rho, \quad i = 1, 2, \quad j = 0, 1.$$

For further investigations, it is convenient to write N_{ij} $(i = 1, 2,\ j = 0, 1)$ as

$$N_{ij}(\rho,\tau) = N_{ij}^{(1)}(\rho,\tau) + N_{ij}^{(2)}(\rho,\tau) - N_{ij}^{(3)}(\rho,\tau),$$

where

$$\begin{aligned}
N_{ij}^{(1)}(\rho,\tau) &= Z_{i0}(\rho, r_{2i-1}(\tau), \tau, r_{i+j}(\tau)),\\
N_{ij}^{(2)}(\rho,\tau) &= Z_{i1}(\rho, r_{2i-1}(\tau), \tau, r_{i+j}(\tau)) + \\
&\quad + [G_i(\rho, r_{2i-1}(\rho), \tau, r_{i+j}(\tau)) - G_i(\rho, r_{2i-1}(\tau), \tau, r_{i+j}(\tau))],\\
N_{ij}^{(3)}(\rho,\tau) &= \int\limits_{D_{i\rho}} Z_{i0}(\rho, y, \tau, r_{i+j}(\tau))\mu_{2i-1}(\rho, dy) + \int\limits_{D_{i\rho}} Z_{i1}(\rho, y, \tau, r_{i+j}(\tau))\mu_{2i-1}(\rho, dy).
\end{aligned}$$

Set

$$J_{ij}^{(k)}(s,\tau) = \int\limits_s^\tau (\rho - s)^{-\frac{1}{2}} N_{ij}^{(k)}(\rho,\tau)d\rho, \quad i = 1, 2, \quad j = 0, 1, \quad k = 1, 2, 3.$$

Observe that for each $i = 1, 2$, the sum in (22) has two terms: one term with $i = j + 1$ and one term with $i \ne j + 1$. Let us show that

- the term with $i = j+1$ can be represented as:

$$-\frac{V_{3i-2}(s,t)}{\sqrt{b_i(s, r_{2i-1}(s))}} + \sqrt{\frac{2}{\pi}} \int_s^t V_{3i-2}(\tau,t) \frac{\partial}{\partial s} \left(J^{(2)}_{i,i-1}(s,\tau) - J^{(3)}_{i,i-1}(s,\tau) \right) d\tau; \tag{23}$$

- the term with $i \neq j+1$ equals

$$\sqrt{\frac{2}{\pi}} \int_s^t V_{2i-1+j}(\tau,t) \frac{\partial}{\partial s} J_{ij}(s,\tau) d\tau. \tag{24}$$

To prove the relations (23) and (24), we study separately $J^{(1)}_{ij}(s,\tau)$, $J^{(2)}_{ij}(s,\tau)$ and $J^{(3)}_{ij}(s,\tau)$. Concerning $J^{(1)}_{ij}(s,\tau)$ and $J^{(2)}_{ij}(s,\tau)$, let us make the following remarks.

Remark 3.2. Consider $J^{(1)}_{ij}(s,\tau)$. If $i = j+1$, then

$$J^{(1)}_{ij}(s,\tau) = \frac{1}{\sqrt{2\pi b_i(\tau, r_{2i-1}(\tau))}} \int_s^\tau (\tau-\rho)^{-\frac{1}{2}} (\rho - s)^{-\frac{1}{2}} d\rho = \sqrt{\frac{\pi}{2 b_i(\tau, r_{2i-1}(\tau))}}.$$

If $i \neq j+1$, then $J^{(1)}_{ij}(s,\tau)$ tends to zero as $s \uparrow \tau$.

Remark 3.3. Consider $J^{(2)}_{ij}(s,\tau)$. Note that $\lim_{s\uparrow\tau} J^{(2)}_{ij}(s,\tau) = 0$. This relation follows from the estimate

$$\begin{aligned} |N^{(2)}_{ij}(\rho,\tau)| &\le |Z_{i1}(\rho, r_{2i-1}(\tau), \tau, r_{i+j}(\tau))| + \\ &+ |D^1_x G_i(\rho, x_0, \tau, r_{i+j}(\tau))| \cdot |r_{2i-1}(\tau) - r_{2i-1}(\rho)| \le C(\tau-\rho)^{-\frac{1}{2}+\frac{\alpha}{2}} \end{aligned}$$

(x_0 is a point in the open interval with endpoints $r_{2i-1}(\tau)$ and $r_{2i-1}(\rho)$), which can be obtained by applying the mean value theorem to difference $G_i(\rho, r_{2i-1}(\rho), \tau, r_{i+j}(\tau)) - G_i(\rho, r_{2i-1}(\tau), \tau, r_{i+j}(\tau))$ and using the condition IV and the estimates (7), (11).

From Remarks 3.2 and 3.3 it follows that in the case $k=1$ and $i = j+1$,

$$I^{(1)}_{ij}(s,t) = \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^t V_{3i-2}(\tau,t) \sqrt{\frac{\pi}{2b_i(\tau, r_{2i-1}(\tau))}} d\tau = -\frac{V_{3i-2}(s,t)}{\sqrt{b_i(s, r_{2i-1}(s))}}. \tag{25}$$

Furthermore

$$\begin{aligned} I^{(k)}_{ij}(s,t) &\equiv \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^t V_{2i-1+j}(\tau,t) J^{(k)}_{ij}(s,\tau) d\tau = \\ &= \sqrt{\frac{2}{\pi}} \int_s^t V_{2i-1+j}(\tau,t) \frac{\partial}{\partial s} J^{(k)}_{ij}(s,\tau) d\tau \end{aligned} \tag{26}$$

in each of two cases: 1) $k=1$, $i \neq j+1$; 2) $k=2$.

Now, let us consider the integral $J^{(3)}_{ij}(s,\tau)$ and show that it also satisfies the relation (26). For this it suffices to prove that

$$\lim_{s\uparrow\tau} J^{(3)}_{ij}(s,\tau) = 0. \tag{27}$$

Proof of (27). Denote by $N^{(31)}_{ij}$ the first term in the expression for $N^{(3)}_{ij}$ and by $J^{(31)}_{ij}$ the integral $J^{(3)}_{ij}$ with $N^{(3)}_{ij}$ replaced by $N^{(31)}_{ij}$. In view of (7) and (11), we may verify (27) only for $J^{(31)}_{ij}$.

Taking into account (9) and that

$$\frac{1}{\tau-\rho}=\frac{1}{\tau-s}+\frac{\rho-s}{(\tau-s)(\rho-s)},$$

we write $J_{ij}^{(31)}$ in the form

$$J_{ij}^{(31)}(s,\tau)=L_{ij}^{(1)}(s,\tau)+L_{ij}^{(2)}(s,\tau)+L_{ij}^{(3)}(s,\tau),\quad i=1,2,\quad j=0,1,$$

where

$$\begin{aligned}L_{ij}^{(1)}(s,\tau)=&\frac{1}{\sqrt{2\pi b_i(\tau,r_{i+j}(\tau))}}\int\limits_s^\tau(\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}d\rho\\&\times\Bigg[\int\limits_{D_{i\rho}}\exp\left\{-\frac{(y-r_{i+j}(\tau))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-\rho)}\right\}\mu_{2i-1}(\rho,dy)\\&-\int\limits_{D_{is}}\exp\left\{-\frac{(y-r_{i+j}(\tau))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-\rho)}\right\}\mu_{2i-1}(s,dy)\Bigg],\end{aligned}$$

$$\begin{aligned}L_{ij}^{(2)}(s,\tau)=&\frac{1}{\sqrt{2\pi b_i(\tau,r_{i+j}(\tau))}}\int\limits_{D_{is}}\Bigg[\exp\left\{-\frac{(y-r_{i+j}(\tau))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-s)}\right\}\\&-\exp\left\{-\frac{(y-r_{i+j}(s))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-s)}\right\}\Bigg]R_{ij}(s,\tau,y)\mu_{2i-1}(s,dy),\end{aligned}$$

$$\begin{aligned}L_{ij}^{(3)}(s,\tau)=&\frac{1}{\sqrt{2\pi b_i(\tau,r_{i+j}(\tau))}}\times\\&\times\int\limits_{D_{is}}\exp\left\{-\frac{(y-r_{i+j}(s))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-s)}\right\}R_{ij}(s,\tau,y)\mu_{2i-1}(s,dy),\end{aligned}$$

and $R_{ij}(s,\tau,y)$ denotes the integral

$$R_{ij}(s,\tau,y)=\int\limits_s^\tau(\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}\exp\left\{-\frac{(y-r_{i+j}(\tau))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-s)}\cdot\frac{\rho-s}{\tau-\rho}\right\}d\rho,$$

which after the change of variables $z=\frac{\rho-s}{\tau-\rho}$ takes on the form

$$R_{ij}(s,\tau,y)=\int\limits_0^\infty z^{-\frac{1}{2}}(z+1)^{-1}\exp\left\{-\frac{(y-r_{i+j}(\tau))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-s)}\cdot z\right\}dz,$$

and so

$$|R_{ij}(s,\tau,y)|\leq C. \tag{28}$$

From this and III it follows immediately that

$$|L_{ij}^{(1)}(s,\tau)|\leq C(\tau-s)^{\frac{1+\alpha}{2}}, \tag{29}$$

$$|L_{ij}^{(3)}(s,\tau)|\leq C\left(\mu_{2i-1}(s,U_{i\delta}(r_{i+j}(s)))+\exp\left\{-c\frac{\delta^2}{(\tau-s)}\right\}\right), \tag{30}$$

where $U_{i\delta}(r_{i+j}(s))=\{y\in D_{is}:\,|y-r_{i+j}(s)|<\delta\}$, δ is any positive constant.

Applying the mean value theorem to the difference of exponents within square brackets in the expression for $L_{ij}^{(2)}$, we get, after using the condition IV as well as the estimate (28) and the inequality $\sigma^{\nu}\exp\{-c\sigma\} \le C\,(0 \le \sigma < \infty,\ 0 \le \nu < \infty)$,

$$|L_{ij}^{(2)}(s,\tau)| \le C(\tau - s)^{\frac{\alpha}{2}}. \tag{31}$$

The estimates (29)-(31) imply that $J_{ij}^{(31)}(s,\tau) \to 0$ as $s \uparrow \tau$. This completes the proof of (27). Thus, the relation (26) holds also for $k = 3$.

In view of (25) ($k = 1$, $i = j+1$) and the fact that (26) holds in each of three cases: 1) $k = 1$, $i \ne j+1$; 2) $k = 2$; 3) $k = 3$, we obtain the relations (23) and (24).

Now, equating the sum of expressions in (23) and (24) to $\Upsilon_i(s,t)\,(i = 1,2)$, we obtain two Volterra integral equations of the second kind which are equivalent to (15). These two equations can be written in the form

$$V_{3i-2}(s,t) = \sum_{j=0}^{1}\int_s^t K_{ij}(s,\tau)V_{2i-1+j}(\tau,t)d\tau + \Lambda_{3i-2}(s,t), \tag{32}$$

$(0 \le s < t \le T,\ i = 1,2)$, where

$$\Lambda_{3i-2}(s,t) = -\sqrt{b_i(s,r_{2i-1}(s))}\,\Upsilon_i(s,t),$$
$$K_{i,i-1}(s,\tau) = \sqrt{\frac{2b_i(s,r_{2i-1}(s))}{\pi}}\frac{\partial}{\partial s}\big(J_{i,i-1}^{(2)}(s,\tau) - J_{i,i-1}^{(3)}(s,\tau)\big),$$
$$K_{ij}(s,\tau) = \sqrt{\frac{2b_i(s,r_{2i-1}(s))}{\pi}}\frac{\partial}{\partial s}J_{ij}(s,\tau), \quad i \ne j+1.$$

Consider the functions $\Lambda_{3i-2}(s,t)$ and K_{ij} in (32) ($i = 1,2$, $j = 0,1$). The function $\Lambda_{3i-2}(s,t)$ satisfy the same estimate as $\Upsilon_i(s,t)$, i.e., the estimate (21).

Let us estimate the kernels $K_{ij}(s,\tau)$. To do this, we have to estimate $\frac{\partial}{\partial s}J_{ij}(s,\tau)$. Note that $\frac{\partial}{\partial s}J_{ij}^{(31)}(s,\tau)$ which is a term in the expression for $\frac{\partial}{\partial s}J_{ij}(s,\tau)$ does not have a weak singularity. For it we can only obtain the estimate

$$\left|\frac{\partial}{\partial s}J_{ij}^{(31)}(s,\tau)\right| \le C(\tau - s)^{-1}, \quad 0 \le s < \tau < t \le T. \tag{33}$$

The estimate (33) is caused by the integral

$$\int_{U_{i\delta}(r_{i+j}(s))}\frac{\partial}{\partial y}Z_{i0}(s,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(s,dy) \tag{34}$$

which appears after writing $\frac{\partial}{\partial s}J_{ij}^{(31)}(s,\tau)$ as

$$\frac{\partial}{\partial s}J_{ij}^{(31)}(s,\tau) = \frac{\partial}{\partial s}\int_s^{\tau}(\rho - s)^{-\frac{1}{2}}\Bigg(\int_{D_{i\rho}} Z_{i0}(\rho,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(\rho,dy) -$$
$$- \int_{D_{is_0}} Z_{i0}(\rho,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(s_0,dy)\Bigg)\Bigg|_{s_0=s} +$$
$$+ \frac{\partial}{\partial s}\int_s^{\tau}(\rho - s)^{-\frac{1}{2}}d\rho\int_{D_{is_0}} Z_{i0}(\rho,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(s_0,dy)\Bigg|_{s_0=s}$$

and taking the derivative of the last term of this expression. Namely,

$$\frac{\partial}{\partial s}\int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho \int\limits_{D_{is_0}} Z_{i0}(\rho,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(s_0,dy)\Big|_{s_0=s} = \frac{1}{\sqrt{2\pi b_i(\tau,r_{i+j}(\tau))}}\times$$

$$\times\frac{\partial}{\partial s}\int\limits_{D_{is_0}} \exp\left\{-\frac{(y-r_{i+j}(\tau))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-s)}\right\}R_{ij}(s,\tau,y)\mu_{2i-1}(s_0,dy)\Big|_{s_0=s} =$$

$$= \frac{1}{\sqrt{2\pi b_i(\tau,r_{i+j}(\tau))}}\frac{\partial}{\partial s}\int\limits_{D_{is_0}}\mu_{2i-1}(s_0,dy)\int\limits_0^\infty z^{-\frac{1}{2}}(z+1)^{-1}\times$$

$$\times\exp\left\{-\frac{(y-r_{i+j}(\tau))^2}{2b_i(\tau,r_{i+j}(\tau))(\tau-s)}\cdot(z+1)\right\}dz\Big|_{s_0=s} = \sqrt{\frac{\pi b_i(\tau,r_{i+j}(\tau))}{2}}\times$$

$$\times\int\limits_{D_{is}}\frac{\partial}{\partial y}Z_{i0}(s,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(s,dy) = \sqrt{\frac{\pi b_i(\tau,r_{i+j}(\tau))}{2}}\times$$

$$\times\Bigg(\int\limits_{U_{i\delta}(r_{i+j}(s))}\frac{\partial}{\partial y}Z_{i0}(s,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(s,dy)+$$

$$+\int\limits_{D_{is}\setminus U_{i\delta}(r_{i+j}(s))}\frac{\partial}{\partial y}Z_{i0}(s,y,\tau,r_{i+j}(\tau))\mu_{2i-1}(s,dy)\Bigg).$$

One can easily verify that all terms in the expression for $\frac{\partial}{\partial s}J_{ij}$ except for the integral term (34) can be estimated by $C(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}$, where $C(\delta)$ is the positive constant depending on δ.

We now get down to studying the action of the operator $\mathcal{E}$ on the both sides of (19). Using the considerations similar to those leading to (32), we obtain the following two Volterra integral equations of the second kind which are equivalent to (19):

$$V_{i+1}(s,t) = \sum_{j=0}^{1}\int\limits_s^t\Big[Q_{ij}(s,\tau)V_{2i-1+j}(\tau,t)- -\sum_{k=1}^{2}P_{kj}(s,\tau)V_{2k-1+j}(\tau,t)\Big]d\tau+\Lambda_{i+1}(s,t), \tag{35}$$

$(0\le s<t\le T,\ i=1,2)$, where

$$\Lambda_{i+1}(s,t) = -\sqrt{b_i(s,r_2(s))}\Theta_i(s,t)$$

(the function Θ_i is defined by formula (20) with Φ_i replaced by Ψ_i),

$$Q_{i,2-i}(s,\tau) = \sqrt{\frac{2b_i(s,r_2(s))}{\pi}}\frac{\partial}{\partial s}\int\limits_s^\tau(\rho-s)^{-\frac{1}{2}}\Big(Z_{i1}(\rho,r_2(\tau),\tau,r_2(\tau))+$$

$$+\big[G_i(\rho,r_2(\rho),\tau,r_2(\tau))-G_i(\rho,r_2(\tau),\tau,r_2(\tau))\big]\Big)d\rho,$$

$$Q_{ij}(s,\tau) = \sqrt{\frac{2b_i(s,r_2(s))}{\pi}}\frac{\partial}{\partial s}\int\limits_s^\tau(\rho-s)^{-\frac{1}{2}}G_i(\rho,r_2(\rho),\tau,r_{i+j}(\tau))d\rho,\quad i+j\ne 2,$$

$$P_{kj}(s,\tau)=\sqrt{\frac{2b_i(s,r_2(s))}{\pi}}\frac{\partial}{\partial s}\int\limits_s^\tau(\rho-s)^{-\frac{1}{2}}d\rho\int\limits_{D_{k\rho}}G_k(\rho,y,\tau,r_{k+j}(\tau))\mu_2(\rho,dy).$$

We have thus obtained the system of four Volterra integral equations of the second kind (32), (35) which is equivalent to the system (15), (19). Rewrite the system (32), (35) in the form

$$V_i(s,t)=\sum_{j=1}^4\int\limits_s^t H_{ij}(s,\tau)V_j(\tau,t)d\tau+\Lambda_i(s,t),\quad i=1,2,3,4, \tag{36}$$

where

$$H_{ij}(s,\tau)=Q_{i-1,j-1}(s,\tau)-P_{i-1,j-1}(s,\tau) \text{ if } i=2,3,\ j=1,2,$$
$$H_{3i-2,j}(s,\tau)=K_{i,j-2i+1}(s,\tau) \text{ if } i=1,\ j=1,2 \text{ or if } i=2,\ j=3,4,$$
$$H_{i+1,j}(s,\tau)=-P_{3-i,j+2i-5}(s,\tau) \text{ if } i=1,\ j=3,4 \text{ or if } i=2,\ j=1,2,$$

and all other H_{ij} equal to zero.

From what we have proved so far it is clear that the function Λ_i $(i=1,2,3,4)$ satisfies the inequality (21) and that some of the kernels H_{ij} $(i=1,2,3,4,\ j=1,2,3,4)$, namely ones containing the integral of the type (34), do not have a weak singularity. Concerning other components of the expression for H_{ij}, they admit the estimate with the right-hand side $C(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}$.

Despite the strong singularity of kernels H_{ij} of the type (34), one can prove that the ordinary method of successive approximations can still be applied to the system of equations (36) (for details, see, for instance, [14, 15]).

The solution of (36) $V_i(s,t)$ $(0\le s<t\le T,\ i=1,2,3,4)$ has the form

$$V_i(s,t)=\sum_{n=0}^\infty V_i^{(n)}(s,t), \tag{37}$$

where

$$V_i^{(0)}(s,t)=\Lambda_i(s,t),$$
$$V_i^{(n)}(s,t)=\sum_{j=1}^4\int\limits_s^t H_{ij}(s,\tau)V_j^{(n-1)}(\tau,t)d\tau,\quad n=1,2,\dots$$

The convergence of series (37) and so the existence of the function V_i is the consequence of the following inequality:

$$|V_i^{(n)}(s,t)|\le C\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^n C_n^k a^{(n-k)}b^k,\quad 0\le s<t\le T,\ i=1,2,3,4, \tag{38}$$

where

$$a^{(k)}=\frac{\left(4C(\delta_0)T^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})\right)^k\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1+k\alpha}{2}\right)},\quad k=0,1,\dots,n,$$
$$b=\max_{s\in[0,T]}\sum_{i=1}^2\sum_{j=0}^1\Big(\mu_{2i-1}(s,U_{i\delta_0}(r_{i+j}(s)))+\mu_2(s,U_{i\delta_0}(r_{i+j}(s)))\Big)$$

and the constant $\delta=\delta_0$ is chosen to be sufficiently small so that $b<1$. One can prove the estimate (38) by induction basing on considerations leading to the estimates (29), (30) and (31) (see [14, 15] where the proof of similar estimate is given).

Furthermore, the estimate (38) implies that the function V_i $(0\le s<t\le T,\ i=1,2,3,4)$ satisfies the inequality (21). From this and (7) (with $r=p=0$) it follows that

there exist simple-layer potentials $u_{i1}^{(j)}(s,x,t)\,(i=1,2,\ j=0,1)$ in (14), and for them the inequality

$$|u_{i1}^{(j)}(s,x,t)| \le C\|\varphi\|, \quad (s,x)\in \overline{S}_t,\ i=1,2,\ j=0,1, \tag{39}$$

holds. It is clear (see (13)) that the same inequality is also true for the Poisson potential $u_{i0}(s,x,t)\,(i=1,2)$ in (14). Thus, the function $u(s,x,t)$ given by (14), (37) is the desired classical solution of problem (1)-(5) and the assertion on existence is proved.

To prove uniqueness of solution of problem (1)-(5), suppose

$$u^{(1)}(s,x,t)=u_i^{(1)}(s,x,t), \quad u^{(2)}(s,x,t)=u_i^{(2)}(s,x,t), \quad (s,x)\in \overline{S}_t^{(i)},\ i=1,2,$$

are two solutions, continuous in $\overline{S}_t$. Set

$$v_i(s,x,t)=u_i^{(1)}(s,x,t)-u_i^{(2)}(s,x,t), \quad (s,x)\in \overline{S}_t^{(i)},\ i=1,2. \tag{40}$$

and note that the function

$$v(s,x,t)=v_i(s,x,t), \quad (s,x)\in \overline{S}_t^{(i)},\ i=1,2,$$

is the solution of conjugation problem (1)-(5) with $\varphi\equiv 0$ in (2), which is continuous in $\overline{S}_t$. At the same time, each of functions v_i, $i=1,2$, can be treated as the solution of the following parabolic first boundary value problem:

$$\frac{\partial v_i}{\partial s}+\frac{1}{2}b_i(s,x)\frac{\partial^2 v_i}{\partial x^2}+a_i(s,x)\frac{\partial v_i}{\partial x}=0, \quad (s,x)\in S_t^{(i)}, \quad i=1,2, \tag{41}$$

$$\lim_{s\uparrow t} v_i(s,x,t)=0, \quad x\in \overline{D}_{it},\ i=1,2, \tag{42}$$

$$\begin{aligned} &v_i(s,r_{2i-1}(s),t)=f_{2i-1}(s,t), \quad 0\le s<t\le T,\ i=1,2,\\ &v_i(s,r_2(s),t)=f_2(s,t), \quad 0\le s<t\le T, \end{aligned} \tag{43}$$

where

$$f_{2i-1}(s,t)=\int\limits_{D_{is}} v(s,y,t)\mu_{2i-1}(s,dy), \quad f_2(s,t)=\int\limits_{D_s} v(s,y,t)\mu_2(s,dy).$$

Consider the problem (41)-(43). From III, it follows that $f_k\in H^{\frac{1+\alpha}{2}}([0,T])$, $k=1,2,3$. Hence for each $i=1,2$ the first boundary value problem (41)-(43) can be solved by the boundary integral equations method and it has the unique solution, continuous in $\overline{S}_t^{(i)}$ $(i=1,2)$, which, furthermore, can be determined by formula (14), where there is no Poisson potential (see [21], [11, Ch.IV]). Thus, v_i in (40) is the unique solution of problem (41)-(43) $(i=1,2)$ and hence it can be represented in the form (14) with $u_{i0}\equiv 0\,(i=1,2)$ and some densities $V_k\,(k=1,2,3,4)$ to be determined.

Repeating the considerations of the present section leading to the system of Volterra integral equations of the second kind (36), we find that (V_1,V_2,V_3,V_4) is the solution of the same system of equations, but with $\Lambda_i\equiv 0$. This means that $V_k\equiv 0\,(k=1,2,3,4)$. Then $v_i\equiv 0\,(i=1,2)$ and hence $v\equiv 0$. This completes the proof of uniqueness. □

We close this section with the next assertion which follows directly from the proof of Theorem 3.1.

Theorem 3.2. *Let the conditions I-IV hold. Then the classical solution of problem* (1)-(5) *has the form* (14), (37).

4. Construction of Feller semigroup

In this section, using the solution of problem (1)-(5), we define the two-parameter semigroup associated with the Feller process which is the result of "pasting together" two diffusion processes.

Let $C_0(\overline{D}_t)$ be the space all functions $\varphi \in C(\overline{D}_t)$ satisfying fitting conditions in II. From Theorems 3.1 and 3.2 it follows that there exists a unique solution $u(s,x,t)$ of the problem (1)-(5) in the domain $(s,x) \in \overline{S}_t = [0,t] \times \overline{D}_s$ (see (14), (37)) and the solution satisfies the condition that $u(t,x,t) = \varphi(x)$, where φ is a function in $C_0(\overline{D}_t)$ which is assumed to be extended to $\mathbb{R}$ in such a way that φ is bounded continuous.

Denote by $T_{st}\varphi(x)$ the value of $u(s,x,t)$ at point (s,x), $s \le t$, $x \in \overline{D}_s = [r_1(s), r_3(s)]$. If $s = t$, $T_{ss}\varphi(x) = \varphi(x)$, i.e., the operator $T_{ss} = E$, where E is the identity operator. If $s \le t$, the function $T_{st}\varphi(x)$ is continuous (provided fitting conditions in II holds). Thus, the operator T_{st} maps $C_0(\overline{D}_t)$ into $C(\overline{D}_s)$. It is also obvious that this operator is linear.

We get down to studying properties of operators T_{st} in $C_0(\overline{D}_t)$.

First we prove that the operators T_{st} $(0 \le s < t \le T)$ remain the cone of nonnegative functions invariant.

Lemma 4.1. *If $\varphi \in C_0(\overline{D}_t)$ and $\varphi(x) \ge 0$ for all $x \in \overline{D}_t$, then $T_{st}\varphi(x) \ge 0$ for all $0 \le s \le t$, $x \in \overline{D}_s$.*

Proof. Let $\varphi \in C_0(\overline{D}_t)$ be nonnegative. Denote by γ the minimum of $T_{st}\varphi(x)$ in $\overline{S}_t$. Suppose that $\gamma < 0$. Then, from the minimum principle [17, Ch.II] it follows that the value γ may be attained only when $s \in (0,t)$ and $x = r_i(s)\,(i = 1,2,3)$. In case $T_{s_0t}\varphi(r_2(s_0)) = \gamma$, $s_0 \in (0,t)$, we get

$$\int\limits_{D_{s_0}} (T_{s_0t}\varphi(r_2(s_0)) - T_{s_0t}\varphi(y)\mu_2(s_0,dy) < 0$$

which contradicts (5). Analogously, we derive a contradiction for the case $T_{s_0t}\varphi(r_i(s_0)) = \gamma$, $s_0 \in (0,t)$, $i \in \{1,3\}$. Therefore $\gamma \ge 0$ and the assertion of the lemma follows. $\square$

Note also that $T_{st}1 \equiv 1$. This property together with the assertion of Lemma 4.1 allow us to assert that operators T_{st} are contractive, i.e.,

$$\|T_{st}\varphi\| \le \|\varphi\|$$

for all $0 \le s < t \le T$.

Finally, we show that the operator family T_{st} has the semigroup property

$$T_{st} = T_{s\tau}T_{\tau t}, \quad 0 \le s < \tau < t \le T.$$

This property is a consequence of the assertion of uniqueness of the solution of the problem (1)-(5). Indeed, to find $u(s,x,t) = T_{st}\varphi(x)$, when it is given that $u(s,x,t) \to \varphi(x)$ as $s \uparrow t$, one can solve the problem first in time interval $[\tau,t]$ and then solve it in the time interval $[s,\tau]$ with that initial function $u(\tau,x,t) = T_{\tau t}\varphi(x)$ which was obtained; in other words, $T_{st}\varphi(x) = T_{s\tau}(T_{\tau t}\varphi)(x)$, $\varphi \in C(\overline{D}_t)$ or $T_{st} = T_{s\tau}T_{\tau t}$.

The above properties of operators T_{st} imply the following assertion (see [22, Ch.II, §1]).

Theorem 4.1. *Let the conditions I-IV hold. Then the two-parameter family of operators T_{st}, $0 \le s < t \le T$, defined by the solution of* (1)-(5) *is the semigroup associated with the inhomogeneous Feller process on $\overline{D}_s$ which coincides in each of domains D_{is} $(i = 1,2)$ with the diffusion process with the drift coefficient $a_i(s,x)$ and the diffusion coefficient $b_i(s,x)$. The behavior of this process at the points $r_{2i-1}(s)$ $(i = 1,2)$ and $r_2(s)$ is described by the boundary conditions in* (3) *and the conjugation condition* (5) *respectively.*

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