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B. CHIKVINIDZE

NEW PROOF OF THE NOVIKOV CRITERION USING BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

Using backward stochastic differential equations we give a new proof of well known Novikov's criterion.

1. The main result

Let us given a basic probability space $\big(\Omega, \mathcal{F}, P\big)$ with right continuous filtration $(\mathcal{F}_t)_{t<\infty}$ and let $\mathcal{F}_\infty$ be the smallest $\sigma-$Algebra containing all $\mathcal{F}_t$, $t>0$. With this let T be some deterministic time (which might be equal to ∞) and M be a continuous local martingale on the interval $[0;T]$ with $\langle M\rangle_T<\infty$ P a. s.

Denote by $\mathcal{E}(M)$ the stochastic exponential of a local martingale M:

$$\mathcal{E}_t(M)=\exp\{M_t-\frac{1}{2}\langle M\rangle_t\}.$$

Condition $\langle M\rangle_T<\infty$ P a. s. implies that $\mathcal{E}_t(M)>0$ a. s. for all $t\in[0;T]$, which allows us to define $\mathcal{E}_{t,T}(M)$ as $\mathcal{E}_{t,T}(M)=\mathcal{E}_T(M)/\mathcal{E}_t(M)$.

Now define the process $Y_t=E\big[\mathcal{E}_{t,T}(M)\big|\mathcal{F}_t\big]$. In our case $\mathcal{E}_t(M)$ is a positive local martingale which implies that $\mathcal{E}_t(M)$ is a supermartingale. So we have $E\big[\mathcal{E}_T(M)\big|\mathcal{F}_t\big]\le\mathcal{E}_t(M)$ which is equivalent to the inequality:

$$0<Y_t=E\big[\mathcal{E}_{t,T}(M)\big|\mathcal{F}_t\big]\le 1.$$

Since $Y_t\mathcal{E}_t(M)$ is a martingale and $\mathcal{E}_t(M)>0$, Y_t will be a semimartingale and let

$$Y_t=Y_0+A_t+\int_0^t Z_s dM_s+L_t$$

be the semimartingale decomposition of Y, where Z_s is a predictable process and L is a local martingale orthogonal to M.

Lemma 1.1. *The process $Y_t=E\big[\mathcal{E}_{t,T}(M)\big|\mathcal{F}_t\big]$ satisfies the following linear backward stochastic differential equation (BSDE):*

$$\begin{cases} Y_t=Y_0-\int_0^t Z_s d\langle M\rangle_s+\int_0^t Z_s dM_s+L_t,\\ Y_T=1.\end{cases}$$

Proof. Applying Ito's formula for $Y_t\mathcal{E}_t(M)$ we obtain:

$$Y_t\mathcal{E}_t(M)=Y_0+\int_0^t\mathcal{E}_s(M)dA_s+\int_0^t\mathcal{E}_s(M)Z_s dM_s+\int_0^t\mathcal{E}_s(M)dL_s+ \\ +\int_0^t Y_{s-}\mathcal{E}_s(M)dM_s+\int_0^t Z_s\mathcal{E}_s(M)d\langle M\rangle_s.$$

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Since $Y_t\mathcal{E}_t(M)$ is a local martingale we obtain

$$\int_0^t \mathcal{E}_s(M)dA_s + \int_0^t Z_s\mathcal{E}_s(M)d\langle M\rangle_s = \int_0^t \mathcal{E}_s(M)d\Big(\int_0^s Z_u d\langle M\rangle_u + A_s\Big) \equiv 0$$

and therefore $\int_0^t Z_s d\langle M\rangle_s + A_t \equiv 0$ and $A_t \equiv -\int_0^t Z_s\langle M\rangle_s$. So we need to insert this expression in semimartingale decomposition of Y:

$$Y_t = Y_0 - \int_0^t Z_s d\langle M\rangle_s + \int_0^t Z_s dM_s + L_t.$$

□

Now we are ready to prove Novikov's ([1]) criterion:

Theorem 1.1. *For continuous local martingale M, if $Ee^{\frac{1}{2}\langle M\rangle_T} < \infty$, then $\mathcal{E}(M)$ is a uniformly integrable martingale.*

Proof. For simplicity, we will prove theorem when the filtration $(\mathcal{F}_t)_{t\geq 0}$ is continuous, which means that every local martingale with respect to this filtration is continuous. Then we will make a remark for the case of right continuous filtration.

Notice that $EY_0 = E\mathcal{E}_T(M)$ and the stochastic exponential $\mathcal{E}(M)$ is uniformly integrable, if and only if, $EY_0 = 1$. So using the BSDE tool and Lemma 1 we only need to show that $EY_0 = 1$. Let $\beta > 0$ be any constant. According to the Ito formula for the process $e^{-\beta Y_t + \frac{1}{2}\langle M\rangle_t}$ we obtain the following chain of equalities:

$$e^{-\beta Y_t + \frac{1}{2}\langle M\rangle_t} = e^{-\beta Y_0} + \int_0^t e^{-\beta Y_s + \frac{1}{2}\langle M\rangle_s}\big(\beta Z_s + \frac{1}{2}\big)d\langle M\rangle_s +$$
$$+\frac{\beta^2}{2}\int_0^t e^{-\beta Y_s + \frac{1}{2}\langle M\rangle_s} Z_s^2 d\langle M\rangle_s + \frac{\beta^2}{2}\int_0^t e^{-\beta Y_s + \frac{1}{2}\langle M\rangle_s} d\langle L\rangle_s + \textit{ local martingale} =$$
$$= e^{-\beta Y_0} + \frac{1}{2}\int_0^t e^{-\beta Y_s + \frac{1}{2}\langle M\rangle_s}(\beta Z_s + 1)^2 d\langle M\rangle_s + \frac{\beta^2}{2}\int_0^t e^{-\beta Y_s + \frac{1}{2}\langle M\rangle_s} d\langle L\rangle_s +$$
$$+ \textit{ local martingale}.$$

From this we deduce that for any constant $\beta > 0$, $e^{-\beta Y_t + \frac{1}{2}\langle M\rangle_t}$ is a local submartingale, but since it is majorized by integrable random variable $e^{\frac{1}{2}\langle M\rangle_T}$, it is a submartingale. So we can write the submartingale inequality:

$$Ee^{-\beta Y_0} \leq e^{-\beta} E\big[e^{\frac{1}{2}\langle M\rangle_T}\big]. \tag{1}$$

According to Jensen's inequality $e^{-\beta EY_0} \leq Ee^{-\beta Y_0}$. So using this, from inequality (1) we obtain:

$$e^{\beta(1-EY_0)} \leq Ee^{\frac{1}{2}\langle M\rangle_T}.$$

Taking limit as $\beta \to \infty$ we get that $EY_0 \geq 1$, which in our case is equivalent to the $E\mathcal{E}_T(M) = EY_0 = 1$. This means that $\mathcal{E}(M)$ is a uniformly integrable martingale.

Remark 1.1. In case of right continuous filtration the Ito formula representation of $e^{-\beta Y_t + \frac{1}{2}\langle M\rangle_t}$ requires an additional term

$$\sum_{0<s\leq t} e^{-\beta Y_{s-} + \frac{1}{2}\langle M\rangle_s}\Big(e^{-\beta\triangle Y_s} + \beta \triangle Y_s - 1\Big) \tag{2}$$

where $\triangle Y_s$ denotes the jumps of the process Y. Since $e^{-\beta\triangle Y_s} + \beta \triangle Y_s - 1 \geq 0$, the process in (2) will be increasing, so $e^{-\beta Y_t + \frac{1}{2}\langle M\rangle_t}$ remains to be submartingale. After that the proof continues by the exactly same way as it was done in case of continuous filtration.

$\square$

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References

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Institute of Cybernetics of Georgian Technical University, Georgian-American University, Business school, 8 M. Aleksidze Srt., Tbilisi 0160, Georgia

E-mail address: beso.chiqvinidze@gmail.com