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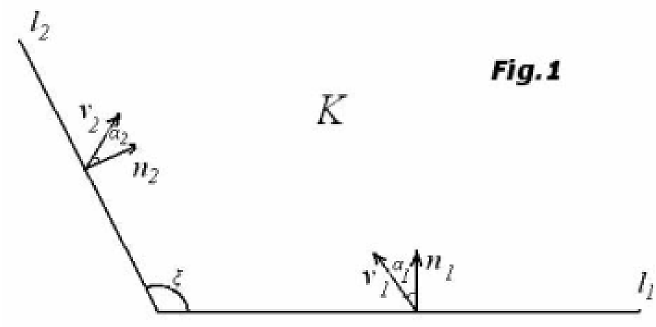
ON PROPERTIES OF BROWNIAN REFLECTING FLOW IN A WEDGE

Consider a planar Brownian flow in a wedge with oblique reflection on the sides. The necessary and sufficient conditions are obtained for the vertex to be reached by the flow.

Introduction

Consider a wedge $K \subset \mathbb{R}^2$ with a vertex at the origin. Assume that one side of the wedge belongs to the abscissa axis. Let ξ be the angle of the wedge, let l_1 and l_2 be sides of the wedge, let n_1 and n_2 be inner normal vectors to the sides, and let v_1 and v_2 be vectors such that $(n_1, v_1) = (n_2, v_2) = 1$. By $\alpha_1, \alpha_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we denote angles between n_1 and v_1, n_2 and v_2 (the angle α_i is referred to as a positive one if and only if v_i points toward the origin); see Fig. 1 with $\alpha_1 > 0,\ \alpha_2 < 0$. Set $K_0 = K \setminus \{0\}$.



Consider the Skorokhod SDE for a reflected Brownian motion $\varphi_t = \varphi_t(x)$ in the wedge K with oblique reflection on its sides:

(1) $$d\varphi_t(x) = dw(t) + v_1 L_1(dt, x) + v_2 L_2(dt, x),$$

(2) $$\varphi_0(x) = x, \varphi_t(x) \in K,\ x \in K,$$

where $\{w(t), t \geq 0\}$ is two-dimensional Wiener process, and, for any x, the processes

(3) $$L_1(t, x), L_2(t, x) \text{ are continuous and non-decreasing in } t,$$

(4) $$L_1(0, x) = L_2(0, x) = 0,$$

(5) $$L_i(t, x) = \int_0^t 1\!\!1_{\{\varphi_s(x) \in l_i\}} L_i(ds, x),\ i = 1, 2.$$

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Condition (5) means that $L_i(t,x)$ may increase in t only at instants, when a process $\varphi_t(x)$ hits l_i.

It is easy to construct a solution of (1)–(5) for fixed x on a time interval $[0,\tau(x))$, where

$$\tau(x) = \sup_n \inf_s \{s: \ |\varphi_s(x)| \le \frac{1}{n}\}. \tag{6}$$

We will say $\tau(x)$ is the vertex hitting moment.

The problem of existence of a strong solution to system (1)–(5) defined for all $t \ge 0$ and fixed x is non-trivial. Some sufficient conditions are given, for example, in [1, 2]. Varadhan and Williams [3] obtained necessary and sufficient conditions for the existence and the uniqueness of a weak solution satisfying $\int_0^\infty 1\!\!1_{\{\varphi_s(x)=0\}} ds = 0$. The general necessary and sufficient conditions ensuring the existence and the uniqueness of a strong solution to (1)–(5) are seemed to be absent.

This paper is the first step of the construction of a flow $\{\varphi_t(x), t \ge 0, x \in K_0\}$ and a study of a joint behavior of solutions to (1)–(5) started simultaneously from all $x \in K_0$. A sufficient condition that guarantees the existence of the flow $\{\varphi_t(x), t \ge 0, x \in K_0\}$ on the initial probability space is the simultaneous inaccessibility of 0 by solutions to (1)–(5) for any initial point $x \in K_0$, i.e.,

$$P(\forall\ x \in K_0: \ \tau(x) = +\infty) = 1. \tag{7}$$

The main aim of the paper is to calculate the probability of the vertex accessibility $p = P(\exists\ x \in K_0: \ \tau(x) < \infty)$ in terms of ξ, α_1, α_2. In particular, it will be proved that either $p = 0$ or $p = 1$. Moreover, if $\xi > \frac{\pi}{2}$, then $p = 1$ and there exists a random initial point $x = x(\omega) \in K_0$ such that $x + w(\cdot)$ hits the corner with probability 1 without hitting the sides of the wedge before this moment.

The problem on the vertex accessibility for a solution started from a fixed $x \in K_0$ was completely solved by Varadhan and Williams (see [3] and also some generalizations [4]-[8] and references therein). It was proved that

$$\forall\ x \in K_0: \ P(\tau(x) < \infty) = 0 \Leftrightarrow \alpha_1 + \alpha_2 \le 0, \tag{8}$$

$$P(\tau(x) < \infty) = 1 \Leftrightarrow \alpha_1 + \alpha_2 > 0. \tag{9}$$

It may be conjectured that condition (8) is a criterion of the vertex inaccessibility by the flow. However, it easy to show that this is not true. Really, let $\xi = \pi, v_1 = v_2 = (0,1)$. Then $\{\varphi_t(x), t \ge 0\}$ is a Brownian motion in the upper half-plane $\mathbb{R}^2_+$ with a normal reflection at the abscissa axis. Then, for any fixed $x \in \mathbb{R}^2_+ \setminus \{0\}$, the process $\varphi_t(x)$ does not hit the origin with probability 1. However,

$$P(\exists\ x \ne 0: \ \tau(x) < \infty) = 1.$$

Indeed, take $x = (x_1, x_2)$, where $x_2 > 0$. Let $w(t) = (w_1(t), w_2(t))$. Denote, by σ, the first instant of hitting the point $(-x_2)$ by the process w_2:

$$\sigma = \inf\{s : x_2 + \omega_2(s) = 0\}.$$

Let $\widetilde{x} = -w(\sigma)$. It can be easily checked that the process $\varphi_t(\widetilde{x})$ gets into the point 0 for $t = \sigma$, and this is the first instant, when the process $\varphi_t(\widetilde{x})$ hits the abscissa axis.

The paper is organized as follows. In Section 1, we construct and study a flow generated by the (deterministic) Skorokhod problem in the upper half-plane with constant reflection on the X-axis. Properties of the (deterministic) Skorokhod problem in a wedge are studied in §2. The probability $P(\exists\ x \ne 0: \ \tau(x) < \infty)$ is calculated in §3. In §4, we use the properties of a flow discussed in §2, §3 and give a new proof of the existence of the Brownian motion for a one-sided cone point for angles greater than $\pi/2$. The corresponding fact about cone points of the Brownian motion was discovered by Burdzy and Shimura [9, 10].

1. Skorokhod equation in a half-plane

In this Section, we consider an auxiliary problem on the behavior of a reflected Brownian flow in the upper half-plane with a constant oblique reflection at the abscissa axis.

Let $\mathbb{R}^2_+ = \mathbb{R} \times [0, \infty)$ be the upper half-plane, $v = (a, 1) \in \mathbb{R}^2$, and let $w \in C_0(\mathbb{R}^2, \mathbb{R}^2)$ be a continuous function, $w(0) = 0$. Consider the Skorokhod problem in $\mathbb{R}^2_+$ with an oblique reflection at the Ox axis:

$$\begin{aligned} d\psi_t(x) &= dw(t) + vL(dt, x),\ t \geq 0, \\ \psi_0(x) &= x,\ x = (x_1, x_2) \in \mathbb{R}^2_+. \end{aligned} \tag{10}$$

Here, $L(0, x) = 0$, L is non-decreasing and continuous in t for fixed x,

$$L(t, x) = \int_0^t 1\!\!1_{\psi_s(x) \in Ox} L(ds, x). \tag{11}$$

Let $\psi_t(x) = (\psi^1_t(x), \psi^2_t(x))$. Let us write system (10), (11) in the coordinate form

$$\begin{gathered} d\psi^1_t(x) = dw_1(t) + aL(dt, x),\ t \geq 0, \\ d\psi^2_t(x) = dw_2(t) + L(dt, x),\ t \geq 0, \\ \psi^1_t(x) = x_1,\ \psi^2_t(x) = x_2, \\ L(t, x) = \int_0^t 1\!\!1_{\psi^2_s(x) = 0} L(ds, x). \end{gathered}$$

Note that the process $\{\psi^2_t(x), t \geq 0\}$ is a solution of the one-dimensional Skorokhod problem with reflection at 0. It is well known that

$$\begin{cases} \psi^2_t(x) = x_2 + w_2(t) + \Gamma(x_2 + \omega_2(\cdot))(t), \\ L(t, x) = \Gamma(x_2 + \omega_2(\cdot))(t), \end{cases} \tag{12}$$

where $\Gamma f(t) := \sup_{s \in [0,t]} (-f(s) \vee 0)$.

Hence,

$$\psi^1_t(x) = x_1 + w_1(t) + a\Gamma(x_2 + \omega_2(\cdot))(t). \tag{13}$$

Fix $\xi \in (0, \pi)$. Let $r^x = \{x + s(\cos\xi, \sin\xi), s \geq 0\}$ be a ray in $\mathbb{R}^2_+$.

Denote, by $r^x_t = \psi_t(r^x)$, the image of the ray r_x under the mapping $\psi_t : \mathbb{R}^2_+ \to \mathbb{R}^2_+$. Let us describe r^x_t.

Set $m(t) = -\min_{s \in [0,t]} w_2(s)$. At first, we observe that if $x_2 - m(t) \geq 0$, then $r^x_t = r^x + w(t) = r^{x + w(t)}$ is a shift of the ray r^x by the vector $w(t)$. If $x_2 - m(t) < 0$, then the set r^x_t can be described as follows (see Fig. 2).

Let us take a point $c(t) \in r^x$ with ordinate $m(t)$,

$$c(t) = (c_1(t), m(t)) = (x_1 + (m(t) - x_2)\cot\xi, m(t)) = \tag{14}$$

$$= (x_1, x_2) + (m(t) - x_2)(\cot\xi, 1) = (x_1, x_2) + (m(t) - x_2)(\sin\xi)^{-1}(\cos\xi, \sin\xi).$$

Note that $r^{c(t)}$ is the infinite part of the ray r^x with the vertex in $c(t)$. Then

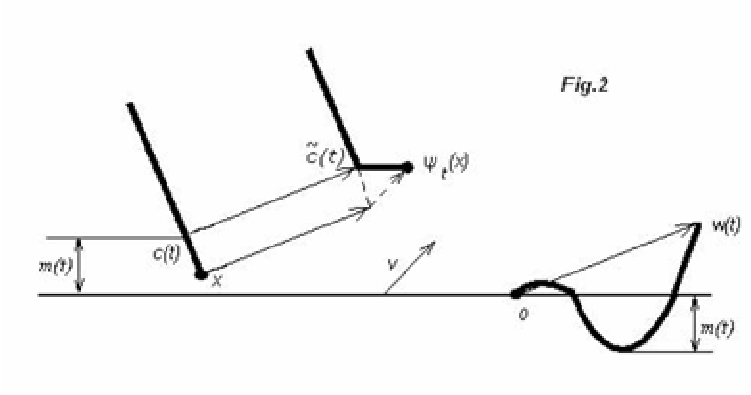
$$\psi_t(r^{c(t)}) = r^{c(t)} + w(t) = r^{c(t) + w(t)} \tag{15}$$

is a shift of $r^{c(t)}$ by a vector $w(t)$. From (12) and (13), we get that the image of $[x; c(t)]$ under the map ψ_t is a horizontal segment with one end-point at $\psi_t(x)$ and another end-point at $\widetilde{c}(t) = c(t) + w(t) = \psi_t(c(t))$; moreover,

$$\psi_t(x) = c(t) + w(t) + (x_1 - c_1(t))m(t)a(1, 0). \tag{16}$$

That is,

$$\psi_t(r^x) = r^{\widetilde{c}(t)} \cup [\psi_t(x); \widetilde{c}(t)]. \tag{17}$$



Remark 1. It follows from (12)–(14) that the point $\widetilde{c}(t)$ has the form $\widetilde{c}(t) = (\widetilde{c}_1(t), \widetilde{c}_2(t))$, where

$$\widetilde{c}_2(t) = x_2 + \omega_2(t) + \Gamma(x_2 + \omega_2(\cdot))(t),$$
$$\widetilde{c}_1(t) = x_1 + \omega_1(t) + \cot\xi\Gamma(x_2 + \omega_2(\cdot))(t).$$

In other words, $\widetilde{c}(t)$ satisfies the Skorokhod equation in $\mathbb{R}^2_+$ with reflection along the vector $(\cot\xi, 1)$ which is parallel to the ray r^x.

Remark 2. If the ray r^x is parallel to the X axis, then it is easy to check that r^x_t is a shift of r^x by the vector $(\psi_t(x) - x)$:

$$r^x_t = r^x + \psi_t(x) - x = r^{\psi_t(x)}, \tag{18}$$

and

$$\psi_t(x + s(1,0)) = \psi_t(x) + s(1,0),\ \ s \in \mathbb{R}. \tag{19}$$

Denote, by S_x, the wedge

$$S_x = \{x + s(\cos\xi, \sin\xi) + t(1,0),\ \ s \geq 0, t \geq 0\} \subset \mathbb{R}^2_+$$

with vertex in x and with angle ξ.

Let us introduce a partial order in $\mathbb{R}^2$ generated by S_0. We say that $x \leq y$ if

$$y - x \in S_0 = \{s(\cos\xi, \sin\xi) + t(1,0) :\ \ s \geq 0,\ \ t \geq 0\}.$$

Lemma 1. *Let $x, y \in \mathbb{R}^2_+, x \leq y$ and $v \notin S_0$. Then $\psi_t(x) \leq \psi_t(y)$ for any $t \geq 0$, i.e., the flow ψ_t is monotonous w.r.t. the partial order " $\leq$ ".*

Proof. Suppose at first that $y = x + s_1(\cos\xi, \sin\xi)$, where $s_1 \geq 0$. It follows from (15) and (16) that $\psi_t(x) \leq \psi_t(y)$. If $y = x + s_2(1,0)$, where $s_2 \geq 0$, then the inequality $\psi_t(x) \leq \psi_t(y)$ follows from (19). Combining these two cases, we obtain the general inequality

$$\forall\ s_1, s_2 \geq 0 :\ \ \psi_t(x) \leq \psi_t(x + s_1(\cos\xi, \sin\xi)) \leq \psi_t(x + s_1(\cos\xi, \sin\xi) + s_2(1,0)).$$

Combining all reasonings of this Section, we get the following statement:

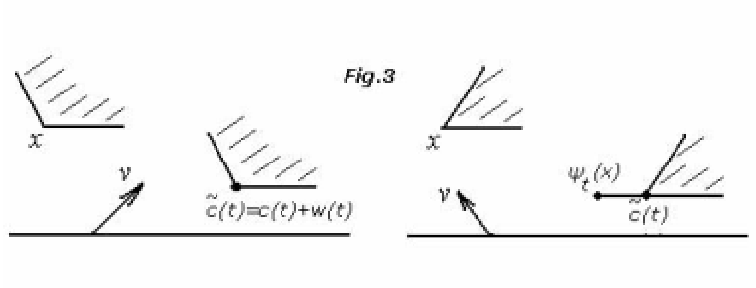
Lemma 2.

$$\psi_t(S_x) = \begin{cases} S_{\widetilde{c}(t)}, & \text{if } v \in S_0, \\ S_{\widetilde{c}(t)} \cup [\psi_t(x); \widetilde{c}(t)], & \text{if } v \notin S_0, \end{cases}$$

where $S_{\widetilde{c}(t)}$ is a wedge with vertex in $\widetilde{c}(t)$, and a function $\widetilde{c}(t)$ is a solution of the Skorokhod equation with reflection at Ox along the vector $(\cot\xi, 1)$.

There exists the minimal point of the set $\psi_t(S_x)$ w.r.t. the partial order " $\leq''$. It equals

a) $\widetilde{c}(t)$ if $v \in S_0$,



b) $\psi_t(x)$ *if* $v \notin S_0$.

Moreover, in case a), the ray $\{\widetilde{c}(t)+s(\cos\xi,\sin\xi):s\geq 0\}$ *belongs to the set* $\psi_t(\{x+s(\cos\xi,\sin\xi):s\geq 0\})$, *in particular,*

$$\min_{y\in S_x}\psi_t(y)=\widetilde{c}(t)\in\psi_t(\{x+s(\cos\xi,\sin\xi):s\geq 0\}).$$

In case b), the ray $\{\psi_t(x)+s(1,0):s\geq 0\}$ *is equal to* $\psi_t(\{x+s(1,0):s\geq 0\})$, *in particular,*

$$\min_{y\in S_x}\psi_t(y)=\psi_t(x)\in\psi_t(\{x+s(1,0):s\geq 0\}).$$

2. Construction of a reflecting flow up to the vertex hitting moment

In this Section, we construct a flow $\{\varphi_t(x)\}$ that satisfies (1)–(5) up to the vertex hitting moment.

Lemma 3. *Let* $w\in C_0(\mathbb{R}^2,\mathbb{R}^2)$ *be any continuous function,* $w(0)=0$. *Then there exist unique functions*

$$\tau:K_0\to(0,\infty),\ \ \varphi=\varphi_t(x):\ \{(t,x)\mid t\in[0,\tau(x)),x\in K_0\}\to K_0,$$
$$L_i=L_i(t,x):\ \{(t,x)\mid t\in[0,\tau(x)),x\in K_0\}\to\mathbb{R}_+,i=1,2,$$

such that (φ,L_1,L_2) *satisfies relations* (1) – (5) *for any* $x\in K_0,t\in[0,\tau(x))$, *where* $\tau(x)$ *is defined in* (6). *The functions* φ,L_1,L_2 *are continuous in* t,x *on the set* $\{(t,x)\mid t\in[0,\tau(x)),x\in K_0\}$.

Moreover, for any $x\in K_0$ *and* $t<\tau(x)$, *there exists a neighborhood* $U(x)$ *of the point* x *such that* $\tau(y)>t$ *for any* $y\in U(x)$, *and* φ *satisfies the following Lipschitz condition in* $U(x)$*:*

$$\exists\ L>0\ \forall\ y_1,y_2\in U(x):\ \ \sup_{s\in[0,t]}|\varphi_s(y_1)-\varphi_s(y_2)|\leq L|y_1-y_2|.$$

Remark 3. In this Lemma, w is an arbitrary non-random function (it is not a Wiener process), and Eqs. (1) – (5) are non-random ones (and not stochastic equations).

The proof of the Lemma can be easily done, by using the localization technique. Let us sketch the main steps only.

Denote, by $r=r(x),\phi=\phi(x)$, the polar coordinates of a point $x\in\mathbb{R}^2,r=\sqrt{x_1^2+x_2^2}$, $\tan\phi=\frac{x_2}{x_1}$. Represent K_0 as a union $K_0=B_1\cup B_2$, where $B_1=\left\{x\in K_0:\phi\in\left[0,\frac{2\xi}{3}\right)\right\}$, $B_2=\left\{x\in K_0:\phi\in\left(\frac{\xi}{3},\xi\right]\right\}$.

Let $x\in K_0$. For the sake of definiteness, we assume that $x\in B_1$. Note that, until the exit from B_1, the process $\varphi_t(x)$ is a solution of the Skorokhod problem considered in the upper half-plane with a constant reflection direction at Ox.

It is well known that the solution exists, and it is unique. Moreover, the explicit formula for $\varphi_t(x)$ can be written (see §1). Really, let $x=(x_1,x_2)$, $\varphi_t(x)=(\varphi_t^1(x),\varphi_t^2(x))$,

$w(t) = (w_1(t), w_2(t)), v_1 = (a_1, 1)$. Then the function $\varphi_t^2(x)$ is a solution of the one-dimensional Skorokhod problem with reflection at zero, and

$$\varphi_t^2(x_1, x_2) = x_2 + w_2(t) + \Gamma(x_2 + w_2(\cdot))(t),\ t \le \sigma_1,$$
$$L_1(t, x) = \Gamma(x_2 + w_2(\cdot))(t),\ t \le \sigma_1, \tag{20}$$

where $\Gamma f(t) := \sup_{s \in [0,t]}(-f(s) \vee 0)$, $\sigma_1 = \sigma_1(x)$ is the exit moment of $\varphi.(x)$ from B_1. Hence,

$$\varphi_t^1(x) = x_1 + w_1(t) + a_1 L_1(t, x),\ t \le \sigma_1.$$

The process $\varphi_t(x)$ does not hit l_2 until σ_1; thus,

$$L_2(t, x) = 0,\ t \le \sigma_1.$$

Assume that $\varphi_{\sigma_1}(x) \ne 0$. It is easy to check that there exists a constant C_1 independent of x and σ_1, and there exists a neighborhood $U_1(x)$ of a point x such that, for any $y \in U_1(x)$, the processes $\{\varphi_t(y), t \in [0, \sigma_1]\}$ had not hit l_2, $\varphi_{\sigma_1}(y) \in B_2$, $\varphi_t(y)$ is continuous in (t, y) on a set $[0, \sigma_1] \times U(x)$, and

$$\forall\ y_1, y_2 \in U_1(x) : \sup_{s \in [0,\sigma_1]} |\varphi_s(y_1) - \varphi_s(y_2)| \le C_1 |y_1 - y_2|. \tag{21}$$

Arguing as above, we can extend a solution $\varphi_t(x)$ to a time interval $[\sigma_1, \sigma_2]$, where σ_2 is the exit moment from B_2. Moreover, there is a neighborhood $U_2(x) \subset U_1(x)$ such that $\varphi_t(y), y \in U_2(x)$ is defined for all $t \le \sigma_2$, φ is continuous in (t, y) and Lipschitzian in y (cf. (21)) with some constant C_2.

Similarly, we may define a solution $\varphi_t(x)$ on a set $\{(t, x) | x \in K_0, t < \sup_n \sigma_n(x)\}$.

Note that the function $\varphi.(x)$ obviously cannot reach the infinity in a finite moment of time staying in one of the sets B_1 or B_2 (see representation (20)).

Remark 4. Actually, we have also considered a case where there exists n such that $\varphi_{\sigma_n}(x) = 0$. This situation can be treated similarly, and we omit the corresponding consideration.

To conclude the proof, it is sufficient to verify that a function $\varphi_s(x), s \in [0, \sup_n \sigma_n(x))$ does not visit, in turn, the sets B_1 and B_2 the infinite number of times if

$$\inf_n \inf_{s \in [0, \sup_n \sigma_n(x))} |\varphi_s(x)| > 0$$

and $\sup_n \sigma_n(x) < \infty$. Assume the converse. Then there exists the infinite number of disjoint segments $[s_k, t_k] \subset [0, \sup_n \sigma_n(x))$ such that $\varphi_{s_k}(x) \in B_1, \varphi_{t_k}(x) \in B_2$, $\varphi_s(x) \notin l_1 \cup l_2$ for $s \in [s_k, t_k]$. Therefore,

$$\varphi_{t_k}(x) - \varphi_{s_k}(x) = w(t_k) - w(s_k). \tag{22}$$

Put

$$r := \inf_{s \in [0, \sup_n \sigma_n(x))} |\varphi_s(x)|,$$
$$C := \inf_{x \in B_1,\ y \in B_2,\ \|x\| \ge r,\ \|y\| \ge r} \|x - y\|.$$

Let $r > 0$. Then $C > 0$. So,

$$\inf_k |\varphi_{t_k}(x) - \varphi_{s_k}(x)| \ge C > 0.$$

Since the intervals $[s_k, t_k]$ are disjoint, we have $\inf_k |t_k - s_k| = 0$. This and (22) imply that the function $w(t), t \in [0, \sup_n \sigma_n(x)]$ is not uniformly continuous. This contradiction concludes the proof.

Remark 5. It follows from the above reasoning that $\lim_{t \to \tau(x)-} \varphi_t(x) = 0$ if $\tau(x) < \infty$. If $v_1 \ne v_2$, then $L_1(\tau(x)-, x) < \infty$, $L_2(\tau(x)-, x) < \infty$, and we may extend (1) for $t = \tau(x)$, where $\varphi_{\tau(x)}(x) := 0,\ L_1(\tau(x), x) := L_1(\tau(x)-, x),\ L_2(\tau(x), x) := L_2(\tau(x)-, x)$.

3. Main Result

Let $\varphi_t(x), x \in K_0,\ t \in [0, \tau(x))$ be a solution to SDE (1)–(5), where $\tau(x)$ is defined in (6).

By

$$p = P(\exists\ x \in K_0 :\ \tau(x) < \infty), \tag{23}$$

we denote the probability of hitting 0 by the flow $\{\varphi_t(x)\}$.

Remark 6. The process $\varphi_t(x)$ is continuous in (t, x). So a set under the probability sign on the right-hand side of (23) is measurable.

Theorem 1. *The probability of hitting zero by the flow $\{\varphi_t(x)\}$ equals either 0 or 1. Moreover, $p = 1$ iff at least one of the following conditions holds:*
a) $\alpha_1 + \alpha_2 > 0$;
b) $\xi > \frac{\pi}{2}$;
c) $\xi \in \left(0, \frac{\pi}{2}\right], \xi + \alpha_1 > \frac{\pi}{2}$;
d) $\xi \in \left(0, \frac{\pi}{2}\right], \xi + \alpha_2 > \frac{\pi}{2}$.

Proof. The case $\alpha_1 + \alpha_2 > 0$ is trivial. Really, in this case for any $x \in K$, we have $P(\tau(x) < \infty) = 1$ (see (8) and (9)).

If $\xi \geq \pi$, then the probability of reaching zero is also equal to 1. This can be proved as for $\xi = \pi$ (see Introduction). So, it will be assumed further that $\xi \in (0; \pi)$.

By

$$K_{\{x\}} = K + x = \{y :\ y - x \in K\},$$

we denote a shift of the wedge K by a vector x.

To prove the theorem, it is sufficient to check that the probability p_x of hitting zero by the set $\varphi_t(K_{\{x\}})$,

$$p_x = P(\exists\ y \in K_{\{x\}} :\ \tau(y) < \infty),$$

has the same form as that in the formulation of the theorem.

The idea of a proof is the following. We will verify that a set $\varphi_t(K_{\{x\}})$ has a minimal point $\widetilde{\varphi}_t(x)$ w.r.t. the partial order generated by K; in addition, it will be shown that $\widetilde{\varphi}_t(x)$ satisfies the Skorokhod SDE in K with constant reflection at each side of the wedge l_1 and l_2. Therefore, the probability of hitting zero by the set $\varphi_t(K_{\{x\}})$ is equal to the probability of hitting zero by the process $\widetilde{\varphi}_t(x)$; and we will apply results of work [3] for the study of the last probability.

Let us introduce a sequence of stopping times $\{\tau_n\}_{n\geq 1}$. Denote, by τ_1, the first instant, when $\varphi_t(K_{\{x\}})$ hits l_1 or l_2 (to be definite, assume that it hits l_1 at first). Put

$$\tau_{2n} = \inf\{t > \tau_{2n-1} :\ \varphi_t(K_{\{x\}}) \cap l_2 \neq \emptyset\},$$
$$\tau_{2n+1} = \inf\{t > \tau_{2n} :\ \varphi_t(K_{\{x\}}) \cap l_1 \neq \emptyset\}.$$

Note that if we prove the existence of the minimal point $\widetilde{\varphi}_t(x)$ of the set $\varphi_t(K_{\{x\}})$, then the equality $\tau_n = \tau_{n+1}$ means $\widetilde{\varphi}_{\tau_n}(x) = 0$. In this case, the proof is trivial (however, it can be verified that the corresponding probability equals zero).

Observe also that, for all $t \in [0, \tau_1)$, we have the equality $\varphi_t(K_{\{x\}}) = K_{\{x+w(t)\}}$, because all points $\varphi_t(K_{\{x\}})$ have not reached sides of the wedge K; moreover, $L_1(t) = L_2(t) = 0$.

Consider the following cases of arrangement of the vectors v_1 and v_2:
1) $v_1, v_2 \in K$;
2) $v_1, v_2 \notin K$;
3a) $v_1 \notin K,\ v_2 \in K$;
3b) $v_1 \in K,\ v_2 \notin K$.

Case 1. Denote, by $\widetilde{\varphi}_t(x)$, a solution to the SDE

$$d\widetilde{\varphi}_t(x) = dw(t) + \widetilde{v}_1 d\widetilde{L}_1(t) + \widetilde{v}_2 d\widetilde{L}_2(t),$$

where $\widetilde{L}_i$ are non-decreasing and continuous, $\widetilde{L}_i(0) = 0, i = 1, 2,$

$$\widetilde{L}_i(t) = \int_0^t 1\!\!\mathrm{I}_{\widetilde{\varphi}_z(x)\in l_i} d\widetilde{L}_i(z),$$
$$\widetilde{\varphi}_0(x) = x,$$

and the vectors $\widetilde{v}_1$ and $\widetilde{v}_2$ are parallel to l_2 and l_1, respectively, $(\widetilde{v}_i, n_i) = 1, i = 1, 2.$

A process $\widetilde{\varphi}_t(x)$ is uniquely defined up to the moment of hitting 0 (see Section 2).

Let us verify that

$$\varphi_t(K_{\{x\}}) = K_{\{\widetilde{\varphi}_t(x)\}} \tag{24}$$

for all $t < \sup_n \tau_n$.

Let $t \in [\tau_1, \tau_2)$. Without loss of generality, we assume that the image of $K_{\{x\}}$ hits l_1 at the instant τ_1.

Note that the set $\varphi_t(K_{\{x\}}),\ t \in [\tau_1, \tau_2)$ does not have common points with l_2, so it reflects only at l_1. Hence, we may apply the reasoning of § 1 about the motion of a wedge in the half-plane with reflection at the X-axis. Therefore, $\varphi_t(K_{\{x\}}) = \psi_t(K_{\{x\}})$, where $\psi_t(x)$ is a solution of (10) with $v = v_1$. It follows from Lemma 2 that $\varphi_t(K_{\{x\}}) = K_{\{\widetilde{\varphi}_t(x)\}}, t \in [\tau_1, \tau_2)$.

A similar equality also holds for $t \in [\tau_2, \tau_3)$. However, in this case, we have to consider a generalization of Lemma 2 to the case of reflection at l_2, rather than at the X-axis.

Arguing as above, we see that relation (24) is satisfied. Therefore, the set $\varphi_t(K_{\{x\}})$ reaches 0 in a finite time if and only if the process $\widetilde{\varphi}_t(x)$ reaches 0 in a finite time. It follows from the result in [3] (see (8) and (9)) that the probability of the last event equals either 0 or 1, if $\xi \le \frac{\pi}{2}$ or $\xi > \frac{\pi}{2}$, respectively.

Note that neither of cases a)-d) of the theorem is satisfied if $\xi \le \frac{\pi}{2}$.

Case 2. It follows from Lemma 1 that

$$\forall\ y \in K_{\{x\}}\ \forall\ t \in [0, \sup_n \tau_n)\ \forall\ \omega :\ \varphi_t(x) \le \varphi_t(y),$$

where the partial order $\le$ is generated by K $(y_1 \le y_2 \Leftrightarrow y_2 - y_1 \in K)$. So, $\varphi_t(K_{\{x\}})$ reaches 0 in a finite time iff $\varphi_t(x)$ reaches zero in a finite time. It follows from (8) and (9) that this is true iff $\alpha_1 + \alpha_2 > 0$.

Note that neither of cases a)-d) of the theorem is satisfied if $\alpha_1 + \alpha_2 \le 0,\ v_1 \notin K, v_2 \notin K$.

Case 3a. Let $\widetilde{\varphi}_t(x)$ be a solution of (1)– (5), where we take $\widetilde{v}_2$ in place of v_2 so that $\widetilde{v}_2$ is parallel to l_1 and $(\widetilde{v}_2, n_2) = 1$.

Let us check that, for any $t \in [0, \sup_n \tau_n)$,

1) $\widetilde{\varphi}_t(x) = \min_{y\in K_{\{x\}}} \varphi_t(y)$,

2) a ray $\{\widetilde{\varphi}_t(x) + s(1, 0) : s \ge 0\}$ is contained in $\varphi_t(K_{\{x\}})$.

Let $t \in [0, \tau_2)$. Recall that $\varphi_t(K_{\{x\}})$ hits l_1 for the first time at an instant $t = \tau_1$, and it does not hit l_2 for all $t \in [0, \tau_2)$.

It follows from Lemma 2 (case b)) that

$$\min_{y\in K_{\{x\}}} \varphi_t(y) = \varphi_t(x) = \widetilde{\varphi}_t(x)$$

and

$$\{\widetilde{\varphi}_t(x) + s(1, 0) : s \ge 0\} \subset \varphi_t(K_{\{x\}}),\ t \in [0, \tau_2).$$

Since $\varphi_t(y) \ge \varphi_t(x) = \widetilde{\varphi}_t(x), y \in K_{\{x\}}$, we have

$$\varphi_t(K_{\{x\}}) \subset K_{\{\widetilde{\varphi}_t(x)\}}. \tag{25}$$

Let now $t \in [\tau_2, \tau_3)$. Recall that $\varphi_t(K_{\{x\}}) \cap l_1 = \emptyset,\ t \in [\tau_2, \tau_3)$. Denote, by $\{\varphi_{st}(y), t \geq s\}$, the solution of (1)–(5) with initial data $\varphi_{ss}(y) = y$. Then

$$\varphi_t(K_{\{x\}}) = \varphi_{\tau_2 t}(\varphi_{\tau_2}(K_{\{x\}})).$$

As was mentioned above, the following inclusions hold:

$$\{\widetilde{\varphi}_{\tau_2}(x) + s(1,0), s \geq 0\} \subset \varphi_{\tau_2}(K_{\{x\}}) \subset K_{\{\widetilde{\varphi}_{\tau_2}(x)\}}. \tag{26}$$

Let us apply Lemma 2 (case a)) to the equation with reflection at l_2. Then

$$\min \varphi_{\tau_2 t}(K_{\{\widetilde{\varphi}_{\tau_2}(x)\}}) = \widetilde{\varphi}_{\tau_2 t}(\widetilde{\varphi}_{\tau_2}(x)) = \widetilde{\varphi}_t(x) =$$
$$= \min \varphi_{\tau_2 t}(\{\widetilde{\varphi}_{\tau_2}(x) + s(1,0), s \geq 0\}).$$

This and (26) yield

$$\widetilde{\varphi}_t(x) = \min \varphi_{\tau_2 t}(\varphi_{\tau_2}(K_{\{x\}})) = \min \varphi_t(K_{\{x\}}).$$

Moreover (see Lemma 2 again), a ray $\{\widetilde{\varphi}_t(x) + s(1,0) : s \geq 0\}$ is contained in $\varphi_t(K_{\{x\}})$.

Continuing this line of reasoning for $t \in [\tau_3, \tau_4), t \in [\tau_4, \tau_5)$, etc., we obtain

$$\widetilde{\varphi}_t(x) = \min_{y \in K_{\{x\}}} \varphi_t(y),\ \ t \in [0, \sup_n \tau_n).$$

So, $\varphi_t(K_{\{x\}})$ reaches 0 in a finite time iff $\widetilde{\varphi}_t(x)$ reaches 0 in a finite time.

Apply (8), (9). The angle between $\widetilde{v}_2$ and n_2 is equal to $\left(\xi - \frac{\pi}{2}\right)$ (in agreement with Introduction). Hence,

$$p_x = 1,\ \text{if}\ \ \alpha + \xi - \frac{\pi}{2} > 0,$$
$$p_x = 0,\ \text{if}\ \ \alpha + \xi - \frac{\pi}{2} \leq 0.$$

It can be easily checked that if $v_1 \notin K,\ v_2 \in K$, then the inequality $\alpha + \xi - \frac{\pi}{2} \leq 0$ yields neither of cases a)-d) from the formulation of the theorem.

The theorem is proved. □

4. Accessibility of the vertex without hitting sides of the wedge

Let us find the probability ρ that there exists a random point $x \in K_0 = K \setminus \{0\}$ such that a Wiener trajectory started from x reaches the corner without hitting the sides of the wedge, i.e.,

$$\rho = P(\exists x \in K_0\ \exists t > 0 :\ \ x + w(t) = 0 \text{ and } x + w(s) \notin l_1 \cup l_2, s \in [0; t)). \tag{27}$$

This problem is equivalent to the existence of a one-sided cone point of the Brownian motion. We now recall the corresponding definition.

Definition 1. Let $t > 0$. A point $z = w(t)$ is a one-sided cone point with angle $\alpha \in (0; \pi]$ if a set $\{w(t) - w(s),\ s \in [0; t]\}$ is included in a wedge $\{(x_1, x_2) :\ x_1 \geq 0, |\frac{x_2}{x_1}| \leq \tan \frac{\alpha}{2}\}$.

The main result of this section is the following.

Theorem 2. $\rho = 1$ *if and only if* $\xi > \pi/2$.
Otherwise, $\rho = 0$.

Remark 7. This result was proved originally by Burdzy and Shimura [9, 10]. We give another proof based on a geometric approach.

Proof. Consider a reflecting flow $\{\varphi_t(x)\}$ in K, where the directions of reflections v_1, v_2 are parallel to l_2 and l_1, respectively. Observe that the probability in (27) equals

$$P(\exists x \in K_0 :\ \varphi_{\tau(x)}(x) = 0 \text{ and } \varphi_s(x) \notin l_1 \cup l_2, s \in [0; \tau(x))).$$

If $\xi \leq \pi/2$, then the flow φ_t does not hit 0 (see §3). Therefore, $\rho = 0$.

Let $\xi > \pi/2$.

The flow φ_t can be constructed for all $t \geq 0$; moreover, the precise formula for φ_t can be written. Really, let A be a linear operator in $\mathbb{R}^2$ such that $Av_1 = n_1,\ Av_2 = n_2$, where $n_1 = (0,1),\ n_2 = (1,0)$. Then $A(K) = [0;\infty)^2$. Put $\widetilde{x} = Ax,\ \widetilde{w}(t) = Aw(t),\ \widetilde{w}(t) = (\widetilde{w}_1(t), \widetilde{w}_2(t))$, and

$$\widetilde{\varphi}_t(\widetilde{x}) = A\varphi_t(x). \tag{28}$$

It is easy to see that $\varphi_t(x)$ satisfies (1)–(5) iff $\widetilde{\varphi}_t(x)$ satisfies the following Skorokhod SDE in the quadrant $[0;\infty)^2$:

$$d\widetilde{\varphi}_t(\widetilde{x}) = d\widetilde{w}(t) + n_1\widetilde{L}_1(dt,\widetilde{x}) + n_2\widetilde{L}_2(dt,\widetilde{x}), \tag{29}$$

$$\widetilde{\varphi}_0(\widetilde{x}) = \widetilde{x},\ \widetilde{\varphi}_t(\widetilde{x}) \in [0;\infty)^2,\ \widetilde{x} \in [0;\infty)^2, \tag{30}$$

where $\widetilde{L}_i(t,\widetilde{x})$ satisfy conditions similar to (3)–(5). It is not difficult to check that $\widetilde{L}_i(t,\widetilde{x}) = L_i(t,x)$.

If we write Eq. (29) in the coordinate-wise form, then we see that each coordinate $\widetilde{\varphi}^i_t(\widetilde{x}),\ i \in \{1;2\}$ satisfies the one-dimensional Skorokhod SDE

$$d\widetilde{\varphi}^i_t(\widetilde{x}) = d\widetilde{w}_i(t) + \widetilde{L}_i(dt,\widetilde{x}) \tag{31}$$

(with the rest needed relations on $\widetilde{L}_i(t,\widetilde{x})$).

Hence,

$$\widetilde{\varphi}^i_t(\widetilde{x}) = \widetilde{x}_i + \widetilde{w}_i(t) + \Gamma(\widetilde{x}_i + \widetilde{w}_i(\cdot))(t),\ i \in \{1;2\}. \tag{32}$$

Introduce a partial order generated by K. We will say that $x \leq y$ if $y - x \in K$ and $x < y$ if $y - x \in K \setminus \partial K$.

It follows from §1 and §2 (or (28)-(32)) that the flow φ_t is monotonous in the following sense. If $x \leq y$, then $\varphi_t(x) \leq \varphi_t(y), t \in [0,\tau(x))$, and $\tau(x) \leq \tau(y)$. Recall that $\tau(x) < \infty$ a.s. (see (9)). Formulas (28)-(32) yield $L_i(\tau(x),x) < \infty, i \in \{1;2\}$ a.s. and $x + w(\tau(x)) + v_1L_1(\tau(x),x) + v_2L_2(\tau(x),x) = 0$. □

Lemma 4. *For any $x \in K_0$, the point $\tau(x)$ is a.s. a point of growth of the processes $L_1(t,x),\ L_2(t,x),\ t \in [0,\tau(x))$, i.e.,*

$$P(\forall t \in [0,\tau(x)) :\ L_i(\tau(x),x) > L_i(t,x)) = 1, i \in \{1;2\}.$$

Proof of Lemma 4. It is well known that a.s. all points of hitting zero by the one-dimensional reflected Brownian motion are points of growth of a local time at zero. Therefore, all points t such that $\varphi_t(x) \in l_1$ or $\varphi_t(x) \in l_2$ are points of growth of $L_1(\cdot,x)$ or $L_2(\cdot,x)$, respectively, with probability 1 (see (31) and relation between $\varphi_t(x)$ and $\widetilde{\varphi}_t(\widetilde{x})$).

Assume the converse to the statement of the Lemma. Then there exists $x \in K_0$ such that

$$P(\varphi_{\tau(x)-}(x) = 0 \text{ and } \exists\varepsilon > 0 :\ \varphi_s(x) \notin l_1, s \in [\tau(x) - \varepsilon;\tau(x))) > 0$$

or

$$P(\varphi_{\tau(x)-}(x) = 0 \text{ and } \exists\varepsilon > 0 :\ \varphi_s(x) \notin l_2, s \in [\tau(x) - \varepsilon;\tau(x))) > 0.$$

Suppose, for instance, that the second inequality is satisfied. Let $\bar{\varphi}_t(x)$ be the reflected Brownian motion in the upper half-plane with reflection at Ox along v_1, i.e., $\bar{\varphi}_t(x)$ is a solution of (10), (11) with $v = v_1$. Observe that $\bar{\varphi}_t(x) = \varphi_t(x)$, if $\varphi_s(x) \notin l_2, s \in [0;t]$.

The process $\bar{\varphi}_t(x)$ can be considered as the reflected Brownian motion in a wedge with $\xi = \pi$, where $v_2 = v_1$. In this case, the angles of reflection are opposite in sign, $\alpha_1 = -\alpha_2$; so $\alpha_1 + \alpha_2 = 0$. It follows from (9) that

$$0 = P(\bar{\tau}(x) < \infty) \geq P(\varphi_{\tau(x)-}(x) = 0 \text{ and } \varphi_s(x) \notin l_2, s \in [0;\tau(x))).$$

This contradiction proves the lemma.

Now we can prove Theorem 2. Let $x \in K_0$ be fixed. Put $\hat{x} = x + v_1L_1(\tau(x)-,x) + v_2L_2(\tau(x)-,x)$. Then $\hat{x} + w(\tau(x)) = 0$. The monotonicity of the flow and Lemma 4 imply that $\hat{x} + w(t) > 0, t \in [0, \tau(x))$ a.s., i.e., $\hat{x} + w(t) \notin l_1 \cup l_2,\ t \in [0, \tau(x))$. Theorem 2 is proved.

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